Moment matrices and SL(n) equivariant valuations on polytopes

Dan Ma

Department of Mathematics, Shanghai Normal University, Shanghai 200234, China, madan@shnu.edu.cn

Abstract

All SL(n) equivariant symmetric matrix valued valuations on convex polytopes in $\mathbb{R}^n$ are completely classified without any continuity assumptions. The unique ones turn out to be the moment matrices corresponding to the classical Legendre ellipsoid and the isotropic position.

1 Introduction

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$. We write $x = (x_1, \ldots, x_n)$ for the corresponding coordinates. For a convex body (compact convex set) $K \subset \mathbb{R}^n$, the *moment matrix* $\mathrm{M}K$ of K is the $(n \times n)$-matrix with coefficients

$$\int_K x_i x_j dx.$$

For a convex body K with nonempty interior, $\mathrm{M}K$ is positive definite. With the standard inner product $x \cdot y$ for $x, y \in \mathbb{R}^n$, it generates the *Legendre ellipsoid* ΓK of K by

$$\Gamma K = \sqrt{\frac{n+2}{V(K)}} E_{(\mathrm{M}K)^{-1}},$$

2010 *Mathematics subject classification.* 52B45, 52A20

Key words and phrases. moment matrix, valuation, convex polytope, SL(n) equivariance.

where $V(K)$ denotes the n-dimensional volume of K and $E_A = \{x \in \mathbb{R}^n : x \cdot Ax \leq 1\}$ for a matrix A. The Legendre ellipsoid ΓK is a classical concept from mechanics. When it is centered at the barycenter of K, the Legendre ellipsoid is the unique ellipsoid that shares the same moment of inertia with K about every axis passing through its barycenter (see [22, 35–37]).

The moment matrix also provides a connection to the asymptotic theory of convex bodies. We call a convex body K *isotropic* if its moment matrix is a multiple of the identity matrix. In this case, the *isotropic constant* L_K is the constant defined by

$$L_K^2 = \int_K (x \cdot \theta)^2 dx$$

for every unit vector θ. One of the main open problem in the asymptotic theory asks for a universal upper bound of L_K for all isotropic convex body K with volume 1. The best result so far is Klartag's improvement on Bourgain's estimate (see [4, 18]).

Moment matrices define a class of matrix valuations. A function μ defined on a lattice $(\mathcal{L}, \cap, \cup)$ and taking values in an abelian semigroup is called a *valuation* if

$$\mu(P) + \mu(Q) = \mu(P \cup Q) + \mu(P \cap Q) \tag{1.1}$$

for every $P, Q \in \mathcal{L}$. A function μ defined on some subset $\mathcal{L}_0$ of $\mathcal{L}$ is called a valuation on $\mathcal{L}_0$ if (1.1) holds whenever $P, Q, P \cup Q, P \cap Q \in \mathcal{L}_0$.

The study and classification of geometric notions which are compatible with transformation groups are important tasks in geometry as proposed in Felix Klein's Erlangen program in 1872. As many functions defined on geometric objects satisfy the inclusion-exclusion principle, the property of being a valuation is natural to consider in the classification of those functions. Valuations also have their origins in Dehn's solution of Hilbert's Third Problem in 1901. The most famous result is Hadwiger's characterization theorem which classifies all continuous and rigid motion invariant real valuations on the space of convex bodies in $\mathbb{R}^n$. This celebrated result initiated a systematic study on the classification of valuations using compatibility with certain linear maps and the topology induced by the Hausdorff metric.

These studies are also a classical part of geometry with important applications in integral geometry (see [6, Chap. 7], [17], [40, Chap. 6]). They turned out to be extremely fruitful and useful especially in the affine geometry of convex bodies. Examples are intrinsic volumes, affine surface areas, the projection body operator and the intersection body operator (see [1, 2, 7–12, 16, 19–21, 23–25, 27–30, 32, 38, 39]).

The aim of this paper is to obtain a complete classification of $\mathrm{SL}(n)$ equivariant matrix valuation on $\mathcal{P}^n$, the space of convex polytopes in $\mathbb{R}^n$, without any continuity assumptions (see Section 2 for definitions). In 2003, Ludwig [26] established the first characterization of the moment matrix and the Lutwak-Yang-Zhang (LYZ) matrix. Let $\mathcal{P}^n_{(0)}$ denote the space of convex polytopes containing the origin in their interiors and $\mathbb{M}^n$ denote the set of symmetric $(n \times n)$-matrices over $\mathbb{R}^n$. Here, a function with values in an Euclidean space is called *measurable* if the preimage of every open set is a Borel set with respect to the corresponding topology. A function $\mu : \mathcal{P}^n_{(0)} \to \mathbb{M}^n$ is called $\mathrm{GL}(n)$ *equivariant* if

$$\mu(\phi P) = |\det \phi|^q \, \phi \mu(P) \phi^t$$

for every $P \in \mathcal{P}^n_{(0)}$, $\phi \in \mathrm{GL}(n)$ and some $q \in \mathbb{R}$, where $\det \phi$ denotes the determinant of ϕ and ϕ^t denotes the transpose of ϕ. For $P \in \mathcal{P}^n_{(0)}$, the *LYZ matrix* of P is defined as the matrix $\mathrm{M}_{-2}(P)$ with coefficients

$$\sum_u \frac{a(u)}{h(u)} u_i u_j,$$

where we sum over all unit normals u of facets of P and where $a(u)$ is the $(n-1)$-dimensional volume of the facet with normal u and $h(u)$ is the distance from the origin to the hyperplane containing this facet.

Theorem 1.1. *Let $n \geq 3$. A function $\mu : \mathcal{P}^n_{(0)} \to \mathbb{M}^n$ is a measurable $\mathrm{GL}(n)$ equivariant valuation if and only if there are constants $c_1, c_2 \in \mathbb{R}$ such that*

$$\mu(P) = c_1 \mathrm{M} P + c_2 \mathrm{M}_{-2} P^*$$

for every $P \in \mathcal{P}^n_{(0)}$, where P^* denotes the polar body of P.

An improvement is included in Haberl and Parapatits's classification of tensor valuations [13] without any homogeneity assumptions (see [3, 5, 14, 15, 30, 31, 34] for more information on matrix and tensor valuations). Note that Ludwig's definition of $\mathrm{GL}(n)$ equivariance turns out to coincide with those of tensors (see Section 2 for details).

Recently, Ludwig and Reitzner [33] established a characterization of $\mathrm{SL}(n)$ invariant valuations on $\mathcal{P}^n$ without any continuity assumptions.

Theorem 1.2. *A function $\mu : \mathcal{P}^n \to \mathbb{R}$ is an $\mathrm{SL}(n)$ invariant valuation if and only if there are constants $c_0, c_0', d_0 \in \mathbb{R}$ and solutions $\alpha, \beta : [0, \infty) \to \mathbb{R}$ of Cauchy's functional equation such that*

$$z(P) = c_0 V_0(P) + c_0'(-1)^{\dim P} \chi_{\mathrm{relint}\, P}(0) + \alpha(V(P)) + d_0 \chi_P(0) + \beta(V([0, P]))$$

for every $P \in \mathcal{P}^n$, where V_0 denotes the Euler characteristic, $[0, P]$ denotes the convex hull of P and the origin and χ denotes the indicator function.

Afterwards, Zeng and the author [43] obtained a classification of the moment vector on $\mathcal{P}^n$ again without any continuity assumptions. Here, a function $\nu : \mathcal{P}^n \to \mathbb{R}^n$ is called $\mathrm{SL}(n)$ *covariant* if $\nu(\phi P) = \phi \nu(P)$ for every $P \in \mathcal{P}^n$ and $\phi \in \mathrm{SL}(n)$.

Theorem 1.3. *Let $n \geq 3$. A function $\nu : \mathcal{P}^n \to \mathbb{R}^n$ is an* $\mathrm{SL}(n)$ *covariant valuation if and only if there are constants $c_1, c_2 \in \mathbb{R}$ such that*

$$\nu(P) = c_1 m(P) + c_2 m([0, P])$$

for every $P \in \mathcal{P}^n$, where $m(P)$ denotes the moment vector of P.

Let $\mathcal{P}_0^n$ denote the space of convex polytopes containing the origin. We are able to show that the moment matrix is essentially the unique $\mathrm{SL}(n)$ equivariant matrix valuation on $\mathcal{P}_0^n$. Let $\mathcal{Q}^n$ be either $\mathcal{P}_0^n$ or $\mathcal{P}^n$. A function $\mu : \mathcal{Q}^n \to \mathbb{M}^n$ is called $\mathrm{SL}(n)$ *equivariant* if $\mu(\phi P) = \phi \cdot \mu(P)$ for every $P \in \mathcal{Q}^n$ and $\phi \in \mathrm{SL}(n)$ (see Section 2 for details on the operation $\phi \cdot \mu(P)$).

Theorem 1.4. *Let $n \geq 3$. A function $\mu : \mathcal{P}_0^n \to \mathbb{M}^n$ is an* $\mathrm{SL}(n)$ *equivariant valuation if and only if there is a constant $c \in \mathbb{R}$ such that*

$$\mu(P) = c\mathrm{M}P$$

for every $P \in \mathcal{P}_0^n$.

Similar to the classification of convex body valuations by Schuster and Wannerer [41] and Wannerer [42], we further extend this result to $\mathcal{P}^n$.

Theorem 1.5. *Let $n \geq 3$. A function $\mu : \mathcal{P}^n \to \mathbb{M}^n$ is an* $\mathrm{SL}(n)$ *equivariant valuation if and only if there are constants $c_1, c_2 \in \mathbb{R}$ such that*

$$\mu(P) = c_1 \mathrm{M}P + c_2 \mathrm{M}[0, P]$$

for every $P \in \mathcal{P}^n$.

These results can be viewed as a first step to establish such classifications of tensor valuations.

2 Notation and preliminary results

We work in n-dimensional Euclidean space $\mathbb{R}^n$. Here, we denote the orthonormal basis by $\{e_1, \dots, e_n\}$. We write a vector $x \in \mathbb{R}^n$ in coordinates by $x = (x_1, \dots, x_n)$. The inner product of $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$. The affine hull, the dimension and the relative interior of a given set in $\mathbb{R}^n$ are denoted by aff, dim and relint , respectively.

Let $A = (a_{ij}) \in \mathbb{M}^n$. We use the tensor representation, namely

$$A = \sum_{1 \leq i \leq j \leq n} a_{ij} e_i \otimes e_j,$$

and write $a_{ij} = A(e_i, e_j)$. Moreover, in the view of tensor, for every $\phi \in \mathrm{GL}(n)$ and $y_1, y_2 \in \mathbb{R}^n$, we define $(\phi \cdot A)(y_1, y_2) = (A \circ \phi^t)(y_1, y_2) = A(\phi^t y_1, \phi^t y_2)$. Indeed, this notation coincides with the action $\phi A \phi^t$ in Ludwig [26, 30, 31] in the following way

$$\phi \cdot A = \sum_{1 \le i \le j \le n} a_{ij} (\phi e_i) \otimes (\phi e_j) = \sum_{1 \le i \le j \le n} a_{ij} \phi (e_i \otimes e_j) \phi^t = \phi A \phi^t.$$

Denote by $[v_1, \ldots, v_k]$ the convex hull of $v_1, \ldots, v_k \in \mathbb{R}^n$. A convex polytope is the convex hull of finitely many points in $\mathbb{R}^n$. Two basic classes of polytopes are the k-dimensional standard simplex $T^k = [0, e_1, \ldots, e_k]$ and one of their $(k-1)$-dimensional facets $\tilde{T}^k = [e_1, \ldots, e_k]$. Indeed, every polytope can be triangulated into simplices. We define a *triangulation* of a k-dimensional polytope P into simplices as a set of k-dimensional simplices $\{T_1, \ldots, T_r\}$ which have pairwise disjoint interiors, with $P = \cup T_i$ and with the property that for arbitrary $1 \le i_1 < \cdots < i_j \le r$ the intersections $T_{i_1} \cap \cdots \cap T_{i_j}$ are again simplices.

We refer to [6, Chap. 7], [17] and [40, Chap. 6] for classical backgroud on valuations. Let $\mathcal{Q}^n$ be either $\mathcal{P}_0^n$ or $\mathcal{P}^n$ and $\mathcal{A}$ be an abelian group. First, we have the inclusion-exclusion principle (see [17] and [38, Theorem 3.1 and Lemma 3.3]).

Lemma 2.1. *Let $\mu : \mathcal{Q}^n \to \mathcal{A}$ be a valuation. Then*

$$\mu(P_1 \cup \cdots \cup P_k) = \sum_{\varnothing \ne S \subseteq \{1,2,\ldots,k\}} (-1)^{|S|-1} \mu(\bigcap_{i \in S} P_i)$$

for all $k \in \mathbb{N}$ and $P_1, P_2, \ldots, P_k \in \mathcal{Q}^n$ with $P_1 \cup \cdots \cup P_k \in \mathcal{Q}^n$.

A valuation on $\mathcal{Q}^n$ is called *simple* if it vanishes on every lower dimensional $P \in \mathcal{Q}^n$. Using triangulations of polytopes, one can determine a simple valuation on $\mathcal{P}_0^n$ by its value on n-dimensional simplices with one vertex at the origin (see [38, Lemma 3.4]). Furthermore, since these simplices are $\mathrm{SL}(n)$ images of dilated standard simplices, we only need to consider sT^n for $s > 0$. Similarly, it also suffices to consider $s\tilde{T}^k$'s for $s > 0$ and $k = 1, \ldots, n$ to determine a valuation on the space of polytopes that do not contain the origin in their affine hull.

Next, we mention a series of triangulations that will be used several times in this paper. Let $\lambda \in (0, 1)$ and denote by H the hyperplane through the origin with the normal vector $(1 - \lambda)e_1 - \lambda e_2$. Write H^+ and H^- for the two halfspaces bounded by H. This hyperplane induces the series of triangulations of T^i's as well as $\tilde{T}^i$'s for $i = 2, \ldots, n$. There are two representations corresponding to these triangulations due to the following definitions.

Definition 2.1. *For $\lambda \in (0, 1)$, define the linear transform $\phi_1 \in \mathrm{SL}(n)$ by*

$$\phi_1 e_1 = \lambda e_1 + (1 - \lambda)e_2,\ \phi_1 e_2 = e_2,\ \phi_1 e_n = e_n/\lambda,\ \phi_1 e_j = e_j,\ \textit{where } j \ne 1, 2, n,$$

and $\psi_1 \in \mathrm{SL}(n)$ by

$$\psi_1 e_1 = e_1,\ \psi_1 e_2 = \lambda e_1 + (1 - \lambda)e_2,\ \psi_1 e_n = e_n/(1 - \lambda),\ \psi_1 e_j = e_j,\ \textit{where } j \ne 1, 2, n.$$

It is clear that $T^i \cap H^+ = \psi_1 T^i$, $T^i \cap H^- = \phi_1 T^i$ and $T^i \cap H = \phi_1 T^{i-1}$. Similarly, $\tilde{T}^i \cap H^+ = \psi_1 \tilde{T}^i$, $\tilde{T}^i \cap H^- = \phi_1 \tilde{T}^i$ and $\tilde{T}^i \cap H = \phi_1 \tilde{T}^{i-1}$, for $i = 2, \ldots, n-1$.

Second, we consider the triangulation of sT^n for $s > 0$.

Definition 2.2. *For $\lambda \in (0,1)$ and $s > 0$, define the linear transform $\phi_2 \in \mathrm{GL}(n)$ by*

$$\phi_2 e_1 = \lambda e_1 + (1-\lambda) e_2, \quad \phi_2 e_2 = e_2, \quad \phi_2 e_j = e_j, \quad \textit{where } j = 3, \ldots, n,$$

and $\psi_2 \in \mathrm{GL}(n)$ by

$$\psi_2 e_1 = e_1, \quad \psi_2 e_2 = \lambda e_1 + (1-\lambda) e_2, \quad \psi_2 e_j = e_j, \quad \textit{where } j = 3, \ldots, n.$$

It is clear that $sT^n \cap H^+ = \psi_2 sT^n$, $sT^n \cap H^- = \phi_2 sT^n$ and $sT^n \cap H = \phi_2 sT^{n-1}$. Similarly, $s\tilde{T}^n \cap H^+ = \psi_2 s\tilde{T}^n$, $s\tilde{T}^n \cap H^- = \phi_2 s\tilde{T}^n$ and $s\tilde{T}^n \cap H = \phi_2 s\tilde{T}^{n-1}$.

Finally, we have several reduction steps for $\mathrm{SL}(n)$ equivariant functions towards the classification.

Lemma 2.2. *Let $\mu : \mathcal{P}_0^n \to \mathbb{M}^n$ be an $\mathrm{SL}(n)$ equivariant function. Then, for $k = 1, \ldots, n$, the coefficients $\mu(T^k)(e_i, e_i)$'s are equal for all $i = 1, \ldots, k$.*

Proof. The case $k = 1$ is trivial. For $k = 2$, we consider $\sigma_1 \in \mathrm{SL}(n)$ such that

$$\sigma_1 e_1 = e_2, \quad \sigma_1 e_2 = e_1, \quad \sigma_1 e_n = -e_n, \quad \sigma_1 e_j = e_j, \quad \text{where } j \neq 1, 2, n.$$

Since σ_1 fixes T^2, the $\mathrm{SL}(n)$ equivariance of μ gives

$$\mu(T^2)(e_1, e_1) = \mu(\sigma_1 T^2)(e_1, e_1) = \mu(T^2)(\sigma_1^t e_1, \sigma_1^t e_1) = \mu(T^2)(e_2, e_2).$$

Now we assume $k \geq 3$. For $l = 0, \ldots, k-3$, consider the permutation θ_1 such that

$$\theta_1 e_{l+1} = e_{l+3},\ \theta_1 e_{l+2} = e_{l+1},\ \theta_1 e_{l+3} = e_{l+2},\ \theta_1 e_j = e_j,\ \text{where } j \neq l+1, l+2, l+3.$$

Since θ_1 fixes T^k, the $\mathrm{SL}(n)$ equivariance of μ gives

$$\mu(T^k)(e_{l+1}, e_{l+1}) = \mu(T^k)(e_{l+2}, e_{l+2}) = \mu(T^k)(e_{l+3}, e_{l+3}).$$

Repeating the above for different l's, we obtain the desired result. □

Lemma 2.3. *Let $\mu : \mathcal{P}_0^n \to \mathbb{M}^n$ be an $\mathrm{SL}(n)$ equivariant function. Then, for $k = 2, \ldots, n$, the coefficients $\mu(T^k)(e_i, e_j)$'s are equal for all $1 \leq i < j \leq k$.*

Proof. The case $k = 2$ is trivial. For $k \geq 3$, we apply all the possible permutations θ_1 defined in Lemma 2.2 within $e_1, \ldots, e_k$. Since these permutations fix T^k, the $\mathrm{SL}(n)$ equivariance of μ implies the lemma. □

Lemma 2.4. *Let $\mu : \mathcal{P}_0^n \to \mathbb{M}^n$ be an $\mathrm{SL}(n)$ equivariant function. Then, for $k = 1, \ldots, n-1$, the coefficients $\mu(T^k)(e_i, e_j) = 0$ for all $i = 1, \ldots, k$ and $j = k+1, \ldots, n$.*

Proof. For $k = 1, \ldots, n-2$ and $k+1 \leq l_1 < l_2 \leq n$, we consider $\sigma_2 \in \mathrm{SL}(n)$ such that σ_2 reflects e_{l_1} and e_{l_2}, i.e.

$$\sigma_2 e_{l_1} = -e_{l_1}, \qquad \sigma_2 e_{l_2} = -e_{l_2}, \qquad \sigma_2 e_p = e_p, \qquad \text{where } p \neq l_1, l_2.$$

Since σ_2 fixes T^k, the $\mathrm{SL}(n)$ equivariance of μ gives

$$\begin{aligned} \mu(T^k)(e_i, e_{l_1}) &= \mu(\sigma_2 T^k)(e_i, e_{l_1}) = \mu(T^k)(\sigma_2^t e_i, \sigma_2^t e_{l_1}) \\ &= \mu(T^k)(e_i, -e_{l_1}) = -\mu(T^k)(e_i, e_{l_1}), \end{aligned}$$

which implies $\mu(T^k)(e_i, e_{l_1}) = 0$, where $i = 1, \ldots, k$. Repeating the above for different l_1's and l_2's, we obtain the desired result in this case.

To see the case $k = n-1$, we apply σ_1 defined in Lemma 2.2. Similarly, since σ_1 fixes T^{n-1}, the $\mathrm{SL}(n)$ equivariance of μ implies the lemma. □

Lemma 2.5. *Let $\mu : \mathcal{P}_0^n \to \mathbb{M}^n$ be an $\mathrm{SL}(n)$ equivariant function. Then, for $k = 0, \ldots, n-1$, the coefficients $\mu(T^k)(e_i, e_i) = 0$ for all $i = k+1, \ldots, n$.*

Proof. For $k = n-1$, we consider $\rho_1 \in \mathrm{SL}(n)$ such that

$$\rho_1 e_n = e_{n-1} + e_n, \qquad \rho_1 e_j = e_j, \qquad \text{where } j = 1, \ldots, n-1.$$

Since ρ_1 fixes T^{n-1}, the $\mathrm{SL}(n)$ equivariance of μ gives

$$\begin{aligned} \mu(T^{n-1})(e_{n-1}, e_n) &= \mu(\rho_1 T^{n-1})(e_{n-1}, e_n) = \mu(T^{n-1})(\rho_1^t e_{n-1}, \rho_1^t e_n) \\ &= \mu(T^{n-1})(e_{n-1} + e_n, e_n) = \mu(T^{n-1})(e_{n-1}, e_n) + \mu(T^{n-1})(e_n, e_n), \end{aligned}$$

which implies $\mu(T^{n-1})(e_n, e_n) = 0$.

For $k = n-2$, we apply ρ_1 again and similarly obtain $\mu(T^{n-2})(e_n, e_n) = 0$. Now, consider $\sigma_3 \in \mathrm{SL}(n)$ such that

$$\sigma_3 e_{n-1} = e_n, \qquad \sigma_3 e_n = -e_{n-1}, \qquad \sigma_3 e_j = e_j, \qquad \text{where } j = 1, \ldots, n-2.$$

Since σ_3 fixes T^{n-2}, the $\mathrm{SL}(n)$ equivariance of μ gives

$$\begin{aligned} \mu(T^{n-2})(e_{n-1}, e_{n-1}) &= \mu(\sigma_3 T^{n-2})(e_{n-1}, e_{n-1}) = \mu(T^{n-2})(\sigma_3^t e_{n-1}, \sigma_3^t e_{n-1}) \\ &= \mu(T^{n-2})(-e_n, -e_n) = \mu(T^{n-2})(e_n, e_n) = 0. \end{aligned}$$

For $k = 0, \ldots, n-3$, we also apply ρ_1 and similarly obtain $\mu(T^k)(e_n, e_n) = 0$. Next, we use θ_1 defined in Lemma 2.2 for different $l = k, \ldots, n-3$. Similarly, since these permutations fix T^k, the $\mathrm{SL}(n)$ equivariance of μ implies the lemma. □

Lemma 2.6. *Let $\mu : \mathcal{P}_0^n \to \mathbb{M}^n$ be an $\mathrm{SL}(n)$ equivariant function. Then, for $k = 0, \dots, n-2$, the coefficients $\mu(T^k)(e_i, e_j) = 0$ for all $k+1 \le i < j \le n$.*

Proof. For $k = n-2$, we apply ρ_1 defined in Lemma 2.5 and similarly since ρ_1 fixes T^{n-2}, the $\mathrm{SL}(n)$ equivariance of μ gives

$$\begin{aligned}\mu(T^{n-2})(e_{n-1}, e_{n-1}) &= \mu(\rho_1 T^{n-2})(e_{n-1}, e_{n-1}) = \mu(T^{n-2})(\rho_1^t e_{n-1}, \rho_1^t e_{n-1}) \\ &= \mu(T^{n-2})(e_{n-1} + e_n, e_{n-1} + e_n) \\ &= \mu(T^{n-2})(e_{n-1}, e_{n-1}) + 2\mu(T^{n-2})(e_{n-1}, e_n) + \mu(T^{n-2})(e_n, e_n).\end{aligned}$$

Now, Lemma 2.5 implies $\mu(T^{n-2})(e_{n-1}, e_n) = 0$.

For $k = 0, \dots, n-3$, we apply ρ_1 as above again and similarly obtain $\mu(T^k)(e_{n-1}, e_n) = 0$. Next, consider all the possible permutations θ_1 defined in Lemma 2.2 within $e_{k+1}, \dots, e_n$. Similarly, since these permutations fix T^k, the $\mathrm{SL}(n)$ equivariance of μ implies the lemma. $\square$

We remark that there are some similar results for T^1 in [34].

3 Characterizations on $\mathcal{P}_0^n$

First, we consider sT^2 for $s > 0$ and obtain the following representation of such valuations.

Lemma 3.1. *Let $n \ge 3$, $s > 0$ and $\mu : \mathcal{P}_0^n \to \mathbb{M}^n$ be an $\mathrm{SL}(n)$ equivariant valuation. Then, there is a constant $c \in \mathbb{R}$ such that $\mu(sT^2) = cs^2(e_1 \otimes e_1 + e_2 \otimes e_2)$.*

Proof. Due to Lemmas 2.2-2.6, it suffices to show the following. First, we use the triangulation in Definition 2.1. Since μ is a valuation, we have

$$\mu(T^2) + \mu(\phi_1 T^1) = \mu(\phi_1 T^2) + \mu(\psi_1 T^2).$$

Then, the equivariance of μ gives

$$\mu(T^2) + \mu(T^1) \circ \phi_1^t = \mu(T^2) \circ \phi_1^t + \mu(T^2) \circ \psi_1^t.$$

On one hand,

$$\mu(T^2)(e_1, e_1) + \mu(T^1)(\lambda e_1, \lambda e_1) = \mu(T^2)(\lambda e_1, \lambda e_1) + \mu(T^2)(e_1 + \lambda e_2, e_1 + \lambda e_2),$$

which implies

$$\mu(T^2)(e_1, e_1) + \lambda^2 \mu(T^1)(e_1, e_1) = \lambda^2 \mu(T^2)(e_1, e_1) + \mu(T^2)(e_1, e_1) + 2\lambda\mu(T^2)(e_1, e_2) + \lambda^2 \mu(T^2)(e_2, e_2).$$

By Lemma 2.2, we get

$$\lambda\mu(T^1)(e_1,e_1) = 2\lambda\mu(T^2)(e_1,e_1) + 2\mu(T^2)(e_1,e_2). \tag{3.1}$$

On the other hand,

$$\mu(T^2)(e_1,e_2)+\mu(T^1)(\lambda e_1,(1-\lambda)e_1+e_2) = \mu(T^2)(\lambda e_1,(1-\lambda)e_1+e_2)+\mu(T^2)(e_1+\lambda e_2,(1-\lambda)e_2).$$

It follows from Lemma 2.4 that

$$\begin{aligned}&\mu(T^2)(e_1,e_2) + \lambda(1-\lambda)\mu(T^1)(e_1,e_1)\\ =&\lambda(1-\lambda)\mu(T^2)(e_1,e_1) + \lambda\mu(T^2)(e_1,e_2) + (1-\lambda)\mu(T^2)(e_1,e_2) + \lambda(1-\lambda)\mu(T^2)(e_2,e_2).\end{aligned}$$

By Lemma 2.2, we get

$$\mu(T^1)(e_1,e_1) = 2\mu(T^2)(e_1,e_1). \tag{3.2}$$

Thus, equations (3.1) and (3.2) imply $\mu(T^2)(e_1,e_2) = 0$.

Next, let $s > 0$ and consider $\rho_2 \in \mathrm{SL}(n)$ such that

$$\rho_2 e_1 = se_1, \quad \rho_2 e_2 = se_2, \quad \rho_2 e_n = e_n/s^2, \quad \rho_2 e_j = e_j, \quad \text{where } j \neq 1,2,n.$$

Since $\rho_2 T^2 = sT^2$, the $\mathrm{SL}(n)$ equivariance of μ gives

$$\begin{aligned}\mu(sT^2)(e_1,e_1) &= \mu(\rho_2 T^2)(e_1,e_1) = \mu(T^2)(\rho_2^t e_1,\rho_2^t e_1)\\ &= \mu(T^2)(se_1,se_1) = s^2\mu(T^2)(e_1,e_1).\end{aligned}$$

Finally, we finish the proof by applying Lemma 2.2 again. □

Next, we treat the case $n = 3$ and obtain the following characterization.

Lemma 3.2. *Let $\mu : \mathcal{P}_0^3 \to \mathbb{M}^3$ be an $\mathrm{SL}(3)$ equivariant valuation. Then, there is a constant $c \in \mathbb{R}$ such that*

$$\mu(P) = c\mathrm{M}P$$

for every $P \in \mathcal{P}_0^3$.

Proof. Let $s > 0$. Due to Lemmas 2.2-2.6, it suffices to show that μ is simple, $\mu(sT^3)(e_1,e_1) = 2\mu(sT^3)(e_1,e_2)$ and $\mu(sT^3) = s^5\mu(T^3)$ as the moment matrix satisfies those properties. We use the triangulation in Definition 2.2. Since μ is a valuation, we have

$$\mu(sT^3) + \mu(\phi_2 sT^2) = \mu(\phi_2 sT^3) + \mu(\psi_2 sT^3).$$

Note that both $\phi_2/\sqrt[3]{\lambda}$ and $\psi_2/\sqrt[3]{1-\lambda}$ belong to SL(3). The SL(3) equivariance of μ gives

$$\mu(sT^3) + \mu(\sqrt[3]{\lambda}sT^2) \circ (\lambda^{-\frac{1}{3}}\phi_2^t) = \mu(\sqrt[3]{\lambda}sT^3) \circ (\lambda^{-\frac{1}{3}}\phi_2^t) + \mu(\sqrt[3]{1-\lambda}sT^3) \circ ((1-\lambda)^{-\frac{1}{3}}\psi_2^t).$$

Replacing s by $\sqrt[3]{s}$, we get

$$\mu(\sqrt[3]{s}T^3) + \mu(\sqrt[3]{\lambda s}T^2) \circ (\lambda^{-\frac{1}{3}}\phi_2^t) = \mu(\sqrt[3]{\lambda s}T^3) \circ (\lambda^{-\frac{1}{3}}\phi_2^t) + \mu(\sqrt[3]{(1-\lambda)s}T^3) \circ ((1-\lambda)^{-\frac{1}{3}}\psi_2^t).$$

On one hand, Lemma 3.1 implies

$$\mu(\sqrt[3]{s}T^3)(e_1,e_3) = \lambda^{-\frac{2}{3}}\mu(\sqrt[3]{\lambda s}T^3)(\lambda e_1,e_3) + (1-\lambda)^{-\frac{2}{3}}\mu(\sqrt[3]{(1-\lambda)s}T^3)(e_1+\lambda e_2,e_3).$$

It follows from Lemma 2.3 that

$$\mu(\sqrt[3]{s}T^3)(e_1,e_2) = \lambda^{\frac{1}{3}}\mu(\sqrt[3]{\lambda s}T^3)(e_1,e_2) + (1+\lambda)(1-\lambda)^{-\frac{2}{3}}\mu(\sqrt[3]{(1-\lambda)s}T^3)(e_1,e_2). \tag{3.3}$$

On the other hand, Lemma 3.1 also implies

$$\mu(\sqrt[3]{s}T^3)(e_2,e_3) = \lambda^{-\frac{2}{3}}\mu(\sqrt[3]{\lambda s}T^3)((1-\lambda)e_1+e_2,e_3) + (1-\lambda)^{-\frac{2}{3}}\mu(\sqrt[3]{(1-\lambda)s}T^3)((1-\lambda)e_2,e_3).$$

It follows from Lemma 2.3 that

$$\mu(\sqrt[3]{s}T^3)(e_1,e_2) = (2-\lambda)\lambda^{-\frac{2}{3}}\mu(\sqrt[3]{\lambda s}T^3)(e_1,e_2) + (1-\lambda)^{\frac{1}{3}}\mu(\sqrt[3]{(1-\lambda)s}T^3)(e_1,e_2). \tag{3.4}$$

Thus, equations (3.3) and (3.4) imply

$$\lambda^{-\frac{5}{3}}\mu(\sqrt[3]{\lambda s}T^3)(e_1,e_2) = (1-\lambda)^{-\frac{5}{3}}\mu(\sqrt[3]{(1-\lambda)s}T^3)(e_1,e_2).$$

Let $a,b>0$. Setting $s=a+b$ and $\lambda=\frac{a}{a+b}$ we obtain

$$a^{-\frac{5}{3}}\mu(\sqrt[3]{a}T^3)(e_1,e_2) = b^{-\frac{5}{3}}\mu(\sqrt[3]{b}T^3)(e_1,e_2).$$

Hence,

$$\mu(sT^3)(e_1,e_2) = s^5\mu(T^3)(e_1,e_2). \tag{3.5}$$

Moreover, Lemma 3.1 similarly implies the following equations. First,

$$\begin{aligned}&\mu(\sqrt[3]{s}T^3)(e_1,e_1) + \mu(\sqrt[3]{s}T^2)(\lambda e_1,\lambda e_1)\\ =&\lambda^{-\frac{2}{3}}\mu(\sqrt[3]{\lambda s}T^3)(\lambda e_1,\lambda e_1) + (1-\lambda)^{-\frac{2}{3}}\mu(\sqrt[3]{(1-\lambda)s}T^3)(e_1+\lambda e_2,e_1+\lambda e_2).\end{aligned}$$

It follows from equation (3.5) and Lemma 2.2 that

$$\begin{aligned}
&\mu(\sqrt[3]{s}T^3)(e_1,e_1)+\lambda^2\mu(\sqrt[3]{s}T^2)(e_1,e_1)\\
=&\lambda^{\frac{4}{3}}\mu(\sqrt[3]{\lambda s}T^3)(e_1,e_1)+(1+\lambda^2)(1-\lambda)^{-\frac{2}{3}}\mu(\sqrt[3]{(1-\lambda)s}T^3)(e_1,e_1)\\
&+2\lambda(1-\lambda)\mu(\sqrt[3]{s}T^3)(e_1,e_2).
\end{aligned} \tag{3.6}$$

Second,

$$\begin{aligned}
&\mu(\sqrt[3]{s}T^3)(e_2,e_2)+\mu(\sqrt[3]{s}T^2)((1-\lambda)e_1+e_2,(1-\lambda)e_1+e_2)\\
=&\lambda^{-\frac{2}{3}}\mu(\sqrt[3]{\lambda s}T^3)((1-\lambda)e_1+e_2,(1-\lambda)e_1+e_2)+(1-\lambda)^{-\frac{2}{3}}\mu(\sqrt[3]{(1-\lambda)s}T^3)((1-\lambda)e_2,(1-\lambda)e_2).
\end{aligned}$$

It follows from equation (3.5), Lemma 2.2 and Lemma 3.1 that

$$\begin{aligned}
&\mu(\sqrt[3]{s}T^3)(e_1,e_1)+(\lambda^2-2\lambda+2)\mu(\sqrt[3]{s}T^2)(e_1,e_1)\\
=&(\lambda^2-2\lambda+2)\lambda^{-\frac{2}{3}}\mu(\sqrt[3]{\lambda s}T^3)(e_1,e_1)+(1-\lambda)^{\frac{4}{3}}\mu(\sqrt[3]{(1-\lambda)s}T^3)(e_1,e_1)\\
&+2\lambda(1-\lambda)\mu(\sqrt[3]{s}T^3)(e_1,e_2).
\end{aligned} \tag{3.7}$$

Next,

$$\begin{aligned}
&\mu(\sqrt[3]{s}T^3)(e_1,e_2)+\mu(\sqrt[3]{s}T^2)(\lambda e_1,(1-\lambda)e_1+e_2)\\
=&\lambda^{-\frac{2}{3}}\mu(\sqrt[3]{\lambda s}T^3)(\lambda e_1,(1-\lambda)e_1+e_2)+(1-\lambda)^{-\frac{2}{3}}\mu(\sqrt[3]{(1-\lambda)s}T^3)(e_1+\lambda e_2,(1-\lambda)e_2).
\end{aligned}$$

It follows from equation (3.5), Lemma 2.2 and Lemma 3.1 that

$$\begin{aligned}
&\lambda(1-\lambda)\mu(\sqrt[3]{s}T^2)(e_1,e_1)\\
=&(1-\lambda)\lambda^{\frac{1}{3}}\mu(\sqrt[3]{\lambda s}T^3)(e_1,e_1)+\lambda(1-\lambda)^{\frac{1}{3}}\mu(\sqrt[3]{(1-\lambda)s}T^3)(e_1,e_1)\\
&+2\lambda(\lambda-1)\mu(\sqrt[3]{s}T^3)(e_1,e_2).
\end{aligned} \tag{3.8}$$

Thus, equations (3.6) and (3.7) imply

$$\mu(\sqrt[3]{s}T^2)(e_1,e_1)=\lambda^{-\frac{2}{3}}\mu(\sqrt[3]{\lambda s}T^3)(e_1,e_1)-\lambda(1-\lambda)^{-\frac{5}{3}}\mu(\sqrt[3]{(1-\lambda)s}T^3)(e_1,e_1). \tag{3.9}$$

Furthermore, equations (3.8) and (3.9) imply

$$(1-\lambda)^{-\frac{5}{3}}\mu(\sqrt[3]{(1-\lambda)s}T^3)(e_1,e_1)=2\mu(\sqrt[3]{s}T^3)(e_1,e_2).$$

Hence,

$$\mu(sT^3)(e_1,e_1)=2s^5\mu(T^3)(e_1,e_2). \tag{3.10}$$

Putting (3.10) back to (3.9) we get $\mu(sT^2)(e_1,e_1)=0$. By Lemma 2.2 and Lemma 3.1, we obtain $\mu(sT^2)=0$. Finally, (3.2) and Lemmas 2.4-2.6 imply $\mu(T^1)=0$. $\square$

Now, we assume $n \geq 4$ and obtain the following Lemma.

Lemma 3.3. *Let $n \geq 4$ and $\mu : \mathcal{P}_0^n \to \mathbb{M}^n$ be an* $\mathrm{SL}(n)$ *equivariant valuation. Then, μ is simple.*

Proof. Due to Lemmas 2.2-2.6, it suffices to show that $\mu(T^k)(e_1, e_1) = \mu(T^k)(e_1, e_2) = 0$ for $k = 1, \dots, n-1$. First, consider T^3. We use the triangulation in Definition 2.1. Since μ is a valuation, we have

$$\mu(T^3) + \mu(\phi_1 T^2) = \mu(\phi_1 T^3) + \mu(\psi_1 T^3).$$

Then, the equivariance of μ gives

$$\mu(T^3) + \mu(T^2) \circ \phi_1^t = \mu(T^3) \circ \phi_1^t + \mu(T^3) \circ \psi_1^t.$$

It follows from Lemma 3.1 that

$$\mu(T^3)(e_1, e_3) = \mu(T^3)(\lambda e_1, e_3) + \mu(T^3)(e_1 + \lambda e_2, e_3).$$

Thus, Lemma 2.3 imply

$$\mu(T^3)(e_1, e_2) = 0. \tag{3.11}$$

Moreover, we similarly have, on one hand,

$$\mu(T^3)(e_1, e_1) + \mu(T^2)(\lambda e_1, \lambda e_1) = \mu(T^3)(\lambda e_1, \lambda e_1) + \mu(T^3)(e_1 + \lambda e_2, e_1 + \lambda e_2).$$

Then, equation (3.11) and Lemma 2.2 imply

$$\mu(T^2)(e_1, e_1) = 2\mu(T^3)(e_1, e_1). \tag{3.12}$$

On the other hand,

$$\begin{aligned} &\mu(T^3)(e_2, e_2) + \mu(T^2)((1-\lambda)e_1 + e_2, (1-\lambda)e_1 + e_2) \\ =&\mu(T^3)((1-\lambda)e_1 + e_2, (1-\lambda)e_1 + e_2) + \mu(T^3)((1-\lambda)e_2, (1-\lambda)e_2). \end{aligned}$$

It follows from equation (3.11), Lemma 2.2 and Lemma 3.1 that

$$\mu(T^3)(e_1, e_1) = 0.$$

Therefore, using equations (3.2) and (3.12), we obtain

$$\mu(T^1) = \mu(T^2) = \mu(T^3) = 0.$$

Now, assume $n \geq 5$ and

$$\mu(T^{k-1}) = 0, \tag{3.13}$$

for $k=4,\ldots,n-1$. We consider T^k and use the triangulation in Definition 2.1. Since μ is a valuation, we have

$$\mu(T^k)+\mu(\phi_1 T^{k-1})=\mu(\phi_1 T^k)+\mu(\psi_1 T^k).$$

Then, equation (3.13) and the equivariance of μ give

$$\mu(T^k)=\mu(T^k)\circ\phi_1^t+\mu(T^k)\circ\psi_1^t.$$

On one hand,

$$\mu(T^k)(e_1,e_3)=\mu(T^k)(\lambda e_1,e_3)+\mu(T^k)(e_1+\lambda e_2,e_3).$$

It follows from Lemma 2.3 that

$$\mu(T^k)(e_1,e_2)=0. \tag{3.14}$$

On the other hand,

$$\mu(T^k)(e_1,e_1)=\mu(T^k)(\lambda e_1,\lambda e_1)+\mu(T^k)(e_1+\lambda e_2,e_1+\lambda e_2).$$

It follows from equation (3.14) and Lemma 2.2 that

$$\mu(T^k)(e_1,e_1)=0.$$

Hence, we obtain $\mu(T^k)=0$. Finally, we use the induction on k and finish the proof. $\square$

Finally, we finish the proof of the classification.

Proof of Theorem 1.4. On one hand, it is clear that the moment matrix operator is an $\mathrm{SL}(n)$ equivariant valuation. On the other hand, since the case $n=3$ is already included in Lemma 3.2, it remains to consider $n\geq 4$. Let $s>0$. Due to Lemmas 2.2-2.6, it suffices to show that $\mu(sT^n)=s^{n+2}\mu(T^n)$ and $\mu(T^n)(e_1,e_1)=2\mu(T^n)(e_1,e_2)$ as the moment matrix satisfies those properties. We use the triangulation in Definition 2.2. Since μ is a valuation, we have

$$\mu(sT^n)+\mu(\phi_2 sT^{n-1})=\mu(\phi_2 sT^n)+\mu(\psi_2 sT^n).$$

Then, Lemma 3.3 implies

$$\mu(sT^n)=\mu(\phi_2 sT^n)+\mu(\psi_2 sT^n).$$

Note that both $\phi_2/\sqrt[n]{\lambda}$ and $\psi_2/\sqrt[n]{1-\lambda}$ belong to $\mathrm{SL}(n)$. The $\mathrm{SL}(n)$ equivariance of μ gives

$$\mu(sT^n)=\mu(\sqrt[n]{\lambda}sT^n)\circ(\lambda^{-\frac{1}{n}}\phi_2^t)+\mu(\sqrt[n]{1-\lambda}sT^n)\circ((1-\lambda)^{-\frac{1}{n}}\psi_2^t).$$

Replace s by $\sqrt[n]{s}$ and we have

$$\mu(\sqrt[n]{s}T^n)=\mu(\sqrt[n]{\lambda s}T^n)\circ(\lambda^{-\frac{1}{n}}\phi_2^t)+\mu(\sqrt[n]{(1-\lambda)s}T^n)\circ((1-\lambda)^{-\frac{1}{n}}\psi_2^t).$$

On one hand,

$$\mu(\sqrt[n]{s}T^n)(e_1,e_3)=\lambda^{-\frac{2}{n}}\mu(\sqrt[n]{\lambda s}T^n)(\lambda e_1,e_3)+(1-\lambda)^{-\frac{2}{n}}\mu(\sqrt[n]{(1-\lambda)s}T^n)(e_1+\lambda e_2,e_3).$$

It follows from Lemma 2.3 that

$$\mu(\sqrt[n]{s}T^n)(e_1,e_2)=\lambda^{1-\frac{2}{n}}\mu(\sqrt[n]{\lambda s}T^n)(e_1,e_2)+(1+\lambda)(1-\lambda)^{-\frac{2}{n}}\mu(\sqrt[n]{(1-\lambda)s}T^n)(e_1,e_2). \tag{3.15}$$

On the other hand,

$$\mu(\sqrt[n]{s}T^n)(e_2,e_3)=\lambda^{-\frac{2}{n}}\mu(\sqrt[n]{\lambda s}T^n)((1-\lambda)e_1+e_2,e_3)+(1-\lambda)^{-\frac{2}{n}}\mu(\sqrt[n]{(1-\lambda)s}T^n)((1-\lambda)e_2,e_3).$$

It follows from Lemma 2.3 that

$$\mu(\sqrt[n]{s}T^n)(e_1,e_2)=(2-\lambda)\lambda^{-\frac{2}{n}}\mu(\sqrt[n]{\lambda s}T^n)(e_1,e_2)+(1-\lambda)^{1-\frac{2}{n}}\mu(\sqrt[n]{(1-\lambda)s}T^n)(e_1,e_2). \tag{3.16}$$

Thus, equations (3.15) and (3.16) imply

$$\lambda^{-1-\frac{2}{n}}\mu(\sqrt[n]{\lambda s}T^n)(e_1,e_2)=(1-\lambda)^{-1-\frac{2}{n}}\mu(\sqrt[n]{(1-\lambda)s}T^n)(e_1,e_2).$$

Hence,

$$\mu(sT^n)(e_1,e_2)=s^{n+2}\mu(T^n)(e_1,e_2). \tag{3.17}$$

Moreover, we similarly have the following equations. First,

$$\mu(\sqrt[n]{s}T^n)(e_1,e_1)=\lambda^{-\frac{2}{n}}\mu(\sqrt[n]{\lambda s}T^n)(\lambda e_1,\lambda e_1)+(1-\lambda)^{-\frac{2}{n}}\mu(\sqrt[n]{(1-\lambda)s}T^n)(e_1+\lambda e_2,e_1+\lambda e_2).$$

It follows from equation (3.17) and Lemma 2.2 that

$$\begin{aligned}\mu(\sqrt[n]{s}T^n)(e_1,e_1)=&\lambda^{2-\frac{2}{n}}\mu(\sqrt[n]{\lambda s}T^n)(e_1,e_1)+(1+\lambda^2)(1-\lambda)^{-\frac{2}{n}}\mu(\sqrt[n]{(1-\lambda)s}T^n)(e_1,e_1)\\&+2\lambda(1-\lambda)\mu(\sqrt[n]{s}T^n)(e_1,e_2).\end{aligned} \tag{3.18}$$

Second,

$$\begin{aligned}\mu(\sqrt[n]{s}T^n)(e_2,e_2)=&\lambda^{-\frac{2}{n}}\mu(\sqrt[n]{\lambda s}T^n)((1-\lambda)e_1+e_2,(1-\lambda)e_1+e_2)\\&+(1-\lambda)^{-\frac{2}{n}}\mu(\sqrt[n]{(1-\lambda)s}T^n)((1-\lambda)e_2,(1-\lambda)e_2).\end{aligned}$$

It follows from equation (3.17) and Lemma 2.2 that

$$\begin{aligned}\mu(\sqrt[n]{s}T^n)(e_1,e_1)=&(\lambda^2-2\lambda+2)\lambda^{-\frac{2}{n}}\mu(\sqrt[n]{\lambda s}T^n)(e_1,e_1)+(1-\lambda)^{2-\frac{2}{n}}\mu(\sqrt[n]{(1-\lambda)s}T^n)(e_1,e_1)\\&+2\lambda(1-\lambda)\mu(\sqrt[n]{s}T^n)(e_1,e_2).\end{aligned} \tag{3.19}$$

Thus, equations (3.18) and (3.19) imply

$$\lambda^{-1-\frac{2}{n}}\mu(\sqrt[n]{\lambda s}T^n)(e_1,e_1)=(1-\lambda)^{-1-\frac{2}{n}}\mu(\sqrt[n]{(1-\lambda)s}T^n)(e_1,e_1).$$

Hence,

$$\mu(sT^n)(e_1,e_1)=s^{n+2}\mu(T^n)(e_1,e_1). \tag{3.20}$$

Finally, put (3.17) and (3.20) into (3.18) and we obtain $\mu(T^n)(e_1,e_1)=2\mu(T^n)(e_1,e_2)$. This completes the proof. □

4 Characterizations on $\mathcal{P}^n$

Since all the steps also work on $\tilde{T}^k$'s for $k=1,\dots,n$, including reductions in Lemmas 2.2-2.6 and triangulations in Definitions 2.1 and 2.2, we similarly obtain the following Lemma.

Lemma 4.1. *Let $n\ge 3$ and $\mu:\mathcal{P}^n\to\mathbb{M}^n$ be an $\mathrm{SL}(n)$ equivariant valuation. Then, $\mu(P)=0$ for every $P\in\mathcal{P}^n$ with $\dim P\le n-2$ and $0\notin\operatorname{aff}P$.*

Next, we determine such valuations on every $P\in\mathcal{P}^n$ with $\dim P\le n-1$ and $0\in\operatorname{aff}P$.

Lemma 4.2. *Let $n\ge 3$ and $\mu:\mathcal{P}^n\to\mathbb{M}^n$ be an $\mathrm{SL}(n)$ equivariant valuation. Then, $\mu(P)=0$ for every $P\in\mathcal{P}^n$ with $\dim P\le n-1$ and $0\in\operatorname{aff}P$.*

Proof. Let $P\in\mathcal{P}^n$ with $\dim P\le n-1$ and $0\in\operatorname{aff}P$. The case $0\in P$ is already included in Theorem 1.4. It suffices to consider such polytopes that do not contain the origin. Let $F_1,\dots,F_r$ be the facets of P *visible* from the origin, i.e. $P\cap\operatorname{relint}[0,F_i]=\varnothing$. Since μ is a valuation, the inclusion-exclusion principle yields

$$\begin{aligned}0=&\mu([0,P])\\=&\mu(P)+\sum_{i=1}^{r}\mu(\underbrace{[0,F_i]}_{\in\mathcal{P}_0^n})-\sum_{j=2}^{r}(-1)^j\sum_{1\le i_1<\cdots<i_j\le r}\mu(\underbrace{[0,F_{i_1}]\cap\cdots\cap[0,F_{i_j}]}_{\in\mathcal{P}_0^n})\\&-\sum_{i=1}^{r}\mu(\underbrace{[0,F_i]\cap P}_{=F_i})+\sum_{j=2}^{r}(-1)^j\sum_{1\le i_1<\cdots<i_j\le r}\mu(\underbrace{[0,F_{i_1}]\cap\cdots\cap[0,F_{i_j}]\cap P}_{\dim\le n-3})\\=&\mu(P),\end{aligned}$$

where the steps follow from Theorem 1.4 and Lemma 4.1. □

Now, we obtain the following characterization for $(n-1)$-dimensional polytopes that do not contain the origin in their affine hull.

Lemma 4.3. *Let $n \geq 3$ and $\mu : \mathcal{P}^n \to \mathbb{M}^n$ be an* $\mathrm{SL}(n)$ *equivariant valuation. Then, there is a constant $c \in \mathbb{R}$ such that*

$$\mu(P) = c\mathrm{M}[0, P]$$

for every $P \in \mathcal{P}^n$ with $\dim P = n-1$ *and* $0 \notin \operatorname{aff} P$.

Proof. First, we consider $s\tilde{T}^n$ for $s > 0$ and use the triangulation in Definition 2.2. Since μ is a valuation, we have

$$\mu(s\tilde{T}^n) + \mu(\phi_2 s\tilde{T}^{n-1}) = \mu(\phi_2 s\tilde{T}^n) + \mu(\psi_2 s\tilde{T}^n).$$

Similar to Theorem 1.4, we obtain that there exists a constant $c \in \mathbb{R}$ such that

$$\mu(s\tilde{T}^n) = c\mathrm{M}[0, s\tilde{T}^n]. \tag{4.1}$$

Next, let P be an $(n-1)$-dimensional polytope with $0 \notin \operatorname{aff} P$. Triangulate P into simplices $T_1, \dots, T_r$. Using the inclusion-exclusion principle, (4.1) and Lemmas 4.1-4.2, we obtain

$$\mu(P) = \sum_{i=1}^{r} \mu(T_i) = c\mathrm{M}[0, P].$$

□

Finally, we finish the proof of the classification on $\mathcal{P}^n$.

Proof of Theorem 1.5. Let $P \in \mathcal{P}^n$. On one hand, it is clear that $\mathrm{M}P$ and $\mathrm{M}[0, P]$ are $\mathrm{SL}(n)$ equivariant valuations. On the other hand, due to Theorem 1.4, Lemma 3.3 and Lemmas 4.2-4.3, it remains to consider full dimensional polytopes. Let $F_1, \dots, F_r$ be the facets of P visible from the origin. Since μ is a valuation, the inclusion-exclusion principle yields that there are constants $\tilde{c}_1, c_2 \in \mathbb{R}$ such that

$$\begin{aligned}
\tilde{c}_1\mathrm{M}[0, P] =& \mu([0, P]) \\
=& \mu(P) + \sum_{i=1}^{r} \mu([0, F_i]) - \sum_{j=2}^{r} (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq r} \mu(\underbrace{[0, F_{i_1}] \cap \dots \cap [0, F_{i_j}]}_{\in \mathcal{P}_0^n}) \\
&- \sum_{i=1}^{r} \mu(\underbrace{[0, F_i] \cap P}_{=F_i}) + \sum_{j=2}^{r} (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq r} \mu(\underbrace{[0, F_{i_1}] \cap \dots \cap [0, F_{i_j}] \cap P}_{\dim \leq n-2}) \\
=& \mu(P) + \sum_{i=1}^{r} \mu([0, F_i]) - \sum_{i=1}^{r} \mu(F_i) \\
=& \mu(P) + \tilde{c}_1 \sum_{i=1}^{r} \mathrm{M}[0, F_i] - c_2 \sum_{i=1}^{r} \mathrm{M}[0, F_i],
\end{aligned}$$

where the steps follow from Theorem 1.4, Lemma 4.1 and Lemma 4.3. Finally, we finish the proof by setting $c_1 = \tilde{c}_1 - c_2$. $\square$

Acknowledgement

The author wishes to thank the referees for valuable suggestions and careful reading of the original manuscript. The work was supported in part by Shanghai Sailing Program 17YF1413800 and by the National Natural Science Foundation of China (Project 11701373).

References

[1] S. Alesker. Continuous rotation invariant valuations on convex sets. *Ann. Math. (2)*, 149(3):977–1005, 1999.

[2] S. Alesker. Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture. *Geom. Funct. Anal.*, 11(2):244–272, 2001.

[3] S. Alesker, A. Bernig, and F. E. Schuster. Harmonic analysis of translation invariant valuations. *Geom. Funct. Anal.*, 21(4):751–773, 2011.

[4] S. Artstein-Avidan, A. Giannopoulos, and V. D. Milman. *Asymptotic Geometric Analysis, Part I*, volume 202 of *Mathematical Surveys and Monographs*. Amer. Math. Soc., Providence, RI, 2015.

[5] A. Bernig and D. Hug. Kinematic formulas for tensor valuations. *J. Reine Angew. Math.*, 736(141-191), 2018.

[6] P. M. Gruber. *Convex and Discrete Geometry*. Springer-Verlag, Berlin Heidelberg, 2007.

[7] C. Haberl. Blaschke valuations. *Amer. J. Math.*, 133(3):717–751, 2011.

[8] C. Haberl. Minkowski valuations intertwining the special linear group. *J. Eur. Math. Soc.*, 14(5):1565–1597, 2012.

[9] C. Haberl and M. Ludwig. A characterization of L_p intersection bodies. *Int. Math. Res. Not.*, 10548:1–29, 2006.

[10] C. Haberl and L. Parapatits. The centro-affine Hadwiger theorem. *J. Amer. Math. Soc.*, 27(3):685–705, 2014.

[11] C. Haberl and L. Parapatits. Valuations and surface area measures. *J. Reine Angew. Math.*, 687:225–245, 2014.

[12] C. Haberl and L. Parapatits. Moments and valuations. *Amer. J. Math.*, 138(6):1575–1603, 2016.

[13] C. Haberl and L. Parapatits. Centro-affine tensor valuations. *Adv. Math.*, 316:806–865, 2017.

[14] D. Hug and R. Schneider. Local tensor valuations. *Geom. Funct. Anal.*, 24(5):1516–1564, 2014.

[15] D. Hug, R. Schneider, and R. Schuster. Integral geometry of tensor valuations. *Adv. Appl. Math.*, 41(4):482–509, 2008.

[16] D. A. Klain. Star valuations and dual mixed volumes. *Adv. Math.*, 121(1):80–101, 1996.

[17] D. A. Klain and G. C. Rota. *Introduction to Geometric Probability.* Cambridge University Press, Cambridge, 1997.

[18] B. Klartag and E. Milman. Centroid bodies and the logarithmic Laplace transform — A unified approach. *J. Funct. Anal.*, 262(1):10–34, 2012.

[19] J. Li and G. Leng. L_p Minkowski valuations on polytopes. *Adv. Math.*, 299:139–173, 2016.

[20] J. Li and D. Ma. Laplace transforms and valuations. *J. Funct. Anal.*, 272(2):738–758, 2017.

[21] J. Li, S. Yuan, and G. Leng. L_p-Blaschke valuations. *Trans. Amer. Math. Soc.*, 367(5):3161–3187, 2015.

[22] J. Lindenstrauss and V. D. Milman. The local theory of normed spaces and its applications to convexity. In P. M. Gruber and J. M. Wills, editors, *Handbook of Convex Geometry*, pages 1149–1220, Amesterdam, 1993. North-Holland.

[23] M. Ludwig. Moment vectors of polytopes. *Rend. Circ. Mat. Pale. (2) Suppl.*, 70:123–138, 2002.

[24] M. Ludwig. Projection bodies and valuations. *Adv. Math.*, 172(2):158–168, 2002.

[25] M. Ludwig. Valuations on polytopes containing the origin in their interiors. *Adv. Math.*, 170(2):239–256, 2002.

[26] M. Ludwig. Ellipsoids and matrix-valued valuations. *Duke Math. J.*, 119(1):159–188, 2003.

[27] M. Ludwig. Minkowski valuations. *Trans. Amer. Math. Soc.*, 357(10):4191–4213, 2005.

[28] M. Ludwig. Intersection bodies and valuations. *Amer. J. Math.*, 128(6):1409–1428, 2006.

[29] M. Ludwig. Minkowski areas and valuations. *J. Differential Geom.*, 86(1):133–161, 2010.

[30] M. Ludwig. Fisher information and matrix-valued valuations. *Adv. Math.*, 226(3):2700–2711, 2011.

[31] M. Ludwig. Covariance matrices and valuations. *Adv. Appl. Math.*, 51(3):359–366, 2013.

[32] M. Ludwig and M. Reitzner. A classification of $\mathrm{SL}(n)$ invariant valuations. *Ann. Math. (2)*, 172(2):1219–1267, 2010.

[33] M. Ludwig and M. Reitzner. $\mathrm{SL}(n)$ invariant valuations on polytopes. *Discrete Comput. Geom.*, 57(3):571–581, 2017.

[34] M. Ludwig and L. Silverstein. Tensor valuations on lattice polytopes. *Adv. Math.*, 319:76–110, 2017.

[35] E. Lutwak, D. Yang, and G. Zhang. A new ellipsoid associated with convex bodes. *Duke. Math. J.*, 104(3):375–390, 2000.

[36] E. Lutwak, D. Yang, and G. Zhang. L_p John ellipsoids. *Proc. London Math. Soc. (3)*, 90(2):497–520, 2005.

[37] V. D. Milman and A. Pajor. Isotropic position and inertia ellipsoids and zonoinds of the unit ball of a normed n-dimensional space. In *Geometric Aspects of Functional Analysis*, volume 1376 of *Lecture Notes in Math.*, pages 64–104, Berlin, 1989. Springer.

[38] L. Parapatits. $\mathrm{SL}(n)$-contravariant L_p-Minkowski valuations. *Trans. Amer. Math. Soc.*, 366(3):1195–1211, 2014.

[39] L. Parapatits. $\mathrm{SL}(n)$-covariant L_p-Minkowski valuations. *J. London Math. Soc. (2)*, 89(2):397–414, 2014.

[40] R. Schneider. *Convex Bodies: the Brunn-Minkowski Theory.* Cambridge University Press, Cambridge, 2nd expanded edition, 2014.

[41] F. E. Schuster and T. Wannerer. $\mathrm{GL}(n)$ contravariant Minkowski valuations. *Trans. Amer. Math. Soc.*, 364(2):815–826, 2012.

[42] T. Wannerer. GL(n) equivariant Minkowski valuations. *Indiana Univ. Math. J.*, 60(5):1655–1672, 2011.

[43] C. Zeng and D. Ma. SL(n) covariant vector valuations on polytopes. *Trans. Amer. Math. Soc.*, 370(12):8999–9023, 2018.