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The lower dimensional Busemann-Petty problem on entropy of log-concave functions

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Abstract: In this paper, we introduce the lower dimensional Busemann-Petty problem on the entropy of log-concave functions: for two even log-concave functions f and g with positive and finite integrals in $\mathbb{R}^n$, if $\int_{\mathbb{R}^n\cap H} f(x)\mathrm{d}x$ is smaller than $\int_{\mathbb{R}^n\cap H} g(x)\mathrm{d}x$ for every i-dimensional subspace H, whether the entropy of f is larger than the entropy of g? Furthermore, partial answers to this problem are given, which might provide a new path to study the long-standing lower dimensional Busemann-Petty problem on convex bodies.

Key words: lower dimensional Busemann-Petty problem; log-concave functions; i-intersection bodies; i-intersection functions; entropy

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对数凹函数熵的低维 Busemann-Petty 问题

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摘　要：引入对数凹函数熵的 Busemann-Petty 问题，即对于 2 个 $\mathbb{R}^n$ 上的偶的对数凹函数 f 和 g, 且 f 和 g 具有正的、有限的积分，假设 $\int_{\mathbb{R}^n\cap H} f(x)\mathrm{d}x \leqslant \int_{\mathbb{R}^n\cap H} g(x)\mathrm{d}x$ 对于任意 i 维子空间 H 均成立，是否能够得到 $\mathrm{Ent}(f) \geqslant \mathrm{Ent}(g)$. 得到了该问题的部分解答，为解决凸体上的低维 Busemann-Petty 问题提供了一种新的途径.

关键词：低维 Busemann-Petty 问题；对数凹函数；i- 相交体；i- 相交函数；熵函数

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1 Introduction

Let $V_i(\cdot)$ and $V(\cdot)$ denote the i-dimensional and the n-dimensional Lebesgue measure respectively, and let $G_{n,i}$ denote the Grassmann manifold of i-dimensional linear subspaces in $\mathbb{R}^n$. Let $\mathcal{K}_o^n$ denote the set of n-dimensional convex bodies containing the origin in their interiors, and let S^{n-1} denote the Euclidean sphere. Throughout this paper, we let $1 \leqslant i \leqslant n-1$.

The lower dimensional Busemann-Petty problem which also called the generalized Busemann-Petty problem asks: suppose that K and L are two origin-symmetric convex bodies in $\mathbb{R}^n$ so that

$$V_{n-i}(K \cap H) \leqslant V_{n-i}(L \cap H),$$

for every $H \in G_{n,n-i}$. Does it follow

$$V(K) \leqslant V(L)?$$

For $i = 1$, it is the celebrated Busemann-Petty problem. To solve this problem, in [1], LUTWAK introduced the notion of the intersection body of a star body, and the problem has affirmative answer for $2 \leqslant n \leqslant 4$ and has negative answer for $n \geqslant 5$ (see more references in [2-8]). For $2 \leqslant i \leqslant n-1$, in [9], BOURGAIN etc. proved that the problem has a negative answer for $4 \leqslant i \leqslant n-1$. For $i = 2$ and $i = 3$, it has still been an open problem in the last two decades, and has gained extensive attention. In some special cases, the problem has made breakthroughs. In [9], BOURGAIN etc. gave an affirmative answer for the case $i = 2$, when L is a ball and K is close to L. In [10], RUBIN gave an affirmative answer when the body with smaller sections is a body of revolution. However, the problem remains unsolved in a general situation. In order to better study this problem, ZHANG in [11] and KOLDOBSKY in [12] gave the definition of i-intersection body respectively. In [12], MILMAN pointed that two types of generalizations of the notion of intersection bodies are not equivalent.

A function $f\colon \mathbb{R}^n \to [0, +\infty)$ is log-concave if for every $x, y \in \mathbb{R}^n$ and $0 < t < 1$, then we have

$$f((1-t)x + ty) \geqslant f(x)^{1-t} f(y)^t. \tag{1}$$

The class of log-concave functions is an important concept in convex geometry. It helps establish the relationship between functions and convex bodies. In some cases, we can translate the properties of convex bodies into some special classes of log-concave functions. In recent years, questions related to the log-concave functions have been widely discussed and studied to solve various kinds of problems, especially in Brunn-Minkwoski theory. Moreover, it has been applied in many fields, not only in mathematics. See more references in [14-17].

In this paper, we study the lower dimensional Busemann-Petty problem on the entropy of log-concave functions. For an integrable function $f\colon \mathbb{R}^n \to \mathbb{R}$, the total mass functional $J(f)$ is defined by

$$J(f) = \int_{\mathbb{R}^n} f(x)\mathrm{d}x.$$

If f is log-concave with positive integral, the entropy of f is defined by

$$\mathrm{Ent}(f) = \int_{\mathbb{R}^n} f(x) \log f(x)\mathrm{d}x - J(f) \log J(f).$$

In [18], FANG etc. introduced the Busemann-Petty problem on the entropy of log-concave functions. They established the relationship between convex bodies and log-concave functions, and the problem has been solved completely.

Theorem 1[18] The Busemann-Petty problem on the entropy of log-concave functions has affirmative answers when $2 \leqslant n \leqslant 4$ and has negative answers when $n \geqslant 5$.

Motivated by FANG etc. [18], we consider the following question, which is called the lower dimensional Busemann-Petty problem on the entropy of log-concave functions.

Question Suppose that i is fixed. Let f and g be two even log-concave functions with finite positive integrals in $\mathbb{R}^n$, so that

$$\int_{\mathbb{R}^n \cap H} f(x)\mathrm{d}x \leqslant \int_{\mathbb{R}^n \cap H} g(x)\mathrm{d}x,$$

for every $H \in G_{n,n-i}$. Does it follow

$$\mathrm{Ent}(f) \geqslant \mathrm{Ent}(g)?$$

To solve our problem, we give the definition of i-intersection function $I_{n,i}f$. It helps to establish the relationship between the log-concave functions and the lower dimensional Busemann-Petty problem. This is based on the work of KOLDOBSKY etc. For a convex body $K \in \mathcal{K}_o^n$, they defined its i-intersection body $I_{n,i}K$ in [12]. Additionally, in [18], FANG etc. provided the definition of intersection functions to solve the functional Busemann-Petty problem. We derive the following partial answer.

Theorem 2 The lower dimensional Busemann-Petty Problem on the entropy of log-concave functions includes the lower dimensional Busemann-Petty Problem.

2 Preliminaries

For a convex body $K \in \mathcal{K}_o^n$, the support function of K is defined by

$$h(K,x) = \sup\{x \cdot y : y \in K\}, \quad \text{for } x \in \mathbb{R}^n,$$

where $x \cdot y$ is the usual inner product of x and y in $\mathbb{R}^n$, and K° denotes the polar body of K,

$$K^\circ = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1 \text{ for all } y \in K\}.$$

The radial function of a convex body K is defined by

$$\rho_K(x) = \rho(K,x) = \max\{\lambda \geqslant 0 : \lambda x \in K\}, \text{ for } x \in \mathbb{R}^n \backslash \{o\}. \tag{2}$$

If a compact star-shaped set (with respect to the origin) with a positive continuous radial function about the variable x is called a star body. Let S_o^n denote the set of n-dimensional all star bodies containing origin in their interiors. The radial function is positively homogeneous of degree -1, that is,

$$\rho(K,tx) = t^{-1}\rho(K,x), \quad \text{for } t > 0.$$

For a star body $K \in S_o^n$, the Minkwoski functional of K is defined by

$$\|x\|_K = \min\{\lambda \geqslant 0 : x \in \lambda K\}.$$

Then, for $x \in \mathbb{R}^n \backslash \{o\}$, we have $\rho(K,x) = \|x\|_K^{-1}$.

Letting $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, φ is convex if for every $x, y \in \mathbb{R}^n$ and $t \in (0,1)$,

$$\varphi((1-t)x + ty) \leqslant (1-t)\varphi(x) + t\varphi(y).$$

From the definition of log-concave function (1), every log-concave function f: $\mathbb{R}^n \to [0,+\infty)$ has the form

$$f = \mathrm{e}^{-\varphi}.$$

For an integrable function $f : \mathbb{R}^n \to [0,+\infty)$ with $f(o) > 0$ and any $p > 0$, in [19], BALL introduced the set of $K_p(f)$,

$$K_p(f) = \left\{ x \in \mathbb{R}^n : \int_0^{+\infty} f(rx) r^{p-1} \mathrm{d}r \geqslant \frac{f(o)}{p} \right\}.$$

From the definition of radial function (2), we have

$$\rho_{K_p(f)}(x) = \left(\frac{1}{f(o)} \int_0^{+\infty} p r^{p-1} f(rx) \mathrm{d}r \right)^{\frac{1}{p}}, \quad \text{for } x \in \mathbb{R}^n \setminus \{o\}.$$

In [20], we get the following properties. Let $f : \mathbb{R}^n \to [0,+\infty)$ be an integrable function with $f(o) > 0$. For every $p > 0$, we have that

(i) $o \in K_p(f)$;

(ii) $K_p(f)$ is a star-shaped set;

(iii) $K_p(f)$ is symmetric if f is even;

(iv) $V(K_n(f))$=$\frac{1}{f(o)} \int_{\mathbb{R}^n} f(x)\mathrm{d}x$.

From [12], KOLDOBSKY gave the definition of i-intersection bodies, that is, for two origin-symmetric star bodies K and L in $\mathbb{R}^n$, K is an i-intersection body of L if for every $H \in G_{n,n-i}$ such that

$$V_i(K \cap H^{\perp}) = V_{n-i}(L \cap H).$$

By the polar formula for the volume, the above equality can be written as the form

$$\int_{S^{n-1}\cap H^{\perp}} \|u\|_K^{-i} \mathrm{d}u = \frac{i}{n-i} \int_{S^{n-1}\cap H} \|u\|_L^{-n+i} \mathrm{d}u.$$

So a star body K is an i-intersection body, if there exists a star body L such that K=$I_{n,i}L$.

3 Main results and proofs

In this section, we study the lower dimensional Busemann-Petty problem on the entropy of log-concave functions. First, we give the definition of i-intersection function.

Definition 1 Let $f : \mathbb{R}^n \to [0,+\infty)$ be a positive integrable function. The i-interscetion function $I_{n,i}f$: $\mathbb{R}^n \to [0,+\infty)$ is defined as

$$I_{n,i}f(x) = \exp\left\{ -\frac{1}{f(o)} \|x\|^i_{I_{n,i}K_{n-i}(f)} \right\}.$$

$f : \mathbb{R}^n \to [0,+\infty)$ with $f(o) > 0$ is an i-intersection function, if there exists a positive integral function g such that $f(x)$=$I_{n,i}g(x)$.

Now, we study the equivalence between i-intersection bodies and i-intersection functions.

Lemma 1 Let $f:\mathbb{R}^n\to[0,+\infty)$ be a positive continuous integrable function with $f(o)>0$. Then, f is an i-intersection function, if and only if $K_{n-i}(f)$ is an i-intersection body.

Proof If f is an i-intersection function, according to definition 1, there exists a positive continuous integrable function g with $g(o)>0$ such that

$$f(x)=I_{n,i}g(x)=\exp\left\{-\frac{1}{g(o)}\|x\|^i_{I_{n,i}K_{n-i}(g)}\right\}.$$

Meanwhile, by the definition of $K_{n-i}(g)$, we have

$$\rho_{K_{n-i}(f)}(x)=f(o)^{-\frac{1}{n-i}}\Gamma\left(\frac{n}{i}\right)^{\frac{1}{n-i}}g(o)^{\frac{1}{i}}\rho_{I_{n,i}K_{n-i}(g)}(x).$$

It is clear that $K_{n-i}(f)$ is an i-intersection body.

Reversely, if $K_{n-i}(f)$ is an i-intersection body, there exists an origin-symmetric star body L such that

$$\rho_{K_{n-i}(f)}(x)=\rho_{I_{n,i}L}(x).$$

Set $g(x)$=$\mathrm{e}^{-\|x\|_L}$, $g(o)=1$. A direct calculation yields $K_{n-i}(g)$=$\Gamma(n-i+1)^{-\frac{1}{n-i}}L$. Then, L=$\Gamma(n-i+1)^{\frac{1}{n-i}}K_{n-i}(g)$, and from the definition of i-intersection body, we have

$$K_{n-i}(f)=I_{n,i}L=I_{n,i}(\Gamma(n-i+1)^{\frac{1}{n-i}}K_{n-i}(g))=\Gamma(n-i+1)^{\frac{1}{i}}I_{n,i}K_{n-i}(g).$$

Thus,

$$\rho_{K_{n-i}(f)}(x)=\Gamma(n-i+1)^{-\frac{1}{i}}\rho_{I_{n,i}K_{n-i}(g)}(x).$$

By the definition of $K_{n-i}(f)$ and its radial function, we have

$$\begin{aligned}\rho^{n-i}_{K_{n-i}(I_{n,i}g)}(x)&=(n-i)\int_0^{+\infty}r^{n-i-1}I_{n,i}g(rx)\mathrm{d}r\\&=\Gamma\left(\frac{n}{i}\right)\rho^{n-i}_{I_{n,i}K_{n-i}(g)}(x).\end{aligned}$$

Therefore,

$$\rho_{K_{n-i}(f)}(x)=c\rho_{K_{n-i}(I_{n,i}g)}(x),\quad c=\Gamma\left(\frac{n}{i}\right)^{-\frac{1}{n-i}}\Gamma(n-i+1)^{-\frac{1}{i}}.$$

By [18], for fixed $p>0$ and $t>0$, we have $K_p(f)=tK_p(g)$, if and only if $f(x)=g(t^{-1}x)$. Therefore, f is an i-intersection function.

Next, we give the following lemma to better understand the relationship between the integral of log-concave functions and the volume of star bodies.

Lemma 2 Let K be an origin-symmetric star body in $\mathbb{R}^n$, $f(x)=\mathrm{e}^{-\|x\|_K}$, for $x\in\mathbb{R}^n$, $H\in G_{n,n-i}$. Then,

$$\begin{aligned}&\int_{\mathbb{R}^n\cap H}f(x)\mathrm{d}x=\Gamma(n-i+1)V_{n-i}(K\cap H),\\&\int_{\mathbb{R}^n}f(x)\mathrm{d}x=\Gamma(n+1)V(K),\\&\int_{\mathbb{R}^n}f(x)\log f(x)\mathrm{d}x=-n\Gamma(n+1)V(K).\end{aligned}$$

Proof By polar coordinates and the volume formula, we have

$$\begin{aligned}\int_{\mathbb{R}^n\cap H} f(x)\mathrm{d}x &= \int_{\mathbb{R}^n\cap H} \mathrm{e}^{-\rho^{-1}(K,x)}\mathrm{d}x\\ &= \int_0^{+\infty}\int_{S^{n-1}\cap H} r^{n-i-1}\mathrm{e}^{-\rho^{-1}(K,ru)}\mathrm{d}u\mathrm{d}r\\ &= (n-i)V_{n-i}(K\cap H)\int_0^{+\infty} t^{n-i-1}\mathrm{e}^{-t}\mathrm{d}t\\ &= \Gamma(n-i+1)V_{n-i}(K\cap H).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\mathrm{d}x &= \int_{\mathbb{R}^n} \mathrm{e}^{-\rho^{-1}(K,x)}\mathrm{d}x\\ &= \int_0^{+\infty}\int_{S^{n-1}} r^{n-1}\mathrm{e}^{-\rho^{-1}(K,ru)}\mathrm{d}u\mathrm{d}r\\ &= nV(K)\int_0^{+\infty} t^{n-1}\mathrm{e}^{-t}\mathrm{d}t\\ &= \Gamma(n+1)V(K),\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\log f(x)\mathrm{d}x &= -\int_{\mathbb{R}^n} \mathrm{e}^{-\rho^{-1}(K,x)}\rho^{-1}(K,x)\mathrm{d}x\\ &= -\int_0^{+\infty}\int_{S^{n-1}} r^{n-1}\mathrm{e}^{-\rho^{-1}(K,ru)}\rho^{-1}(K,ru)\mathrm{d}u\mathrm{d}r\\ &= -n\Gamma(n+1)V(K).\end{aligned}$$

We also need the following geometric result.

Lemma 3[11] Let K be an i-intersection body in $\mathbb{R}^n$ and let L be an origin-symmetric star body in $\mathbb{R}^n$ satisfying

$$V_{n-i}(K\cap H)\leqslant V_{n-i}(L\cap H),$$

for every $H\in G_{n,n-i}$. Then,

$$V(K)\leqslant V(L).$$

Now, we give the main results and their proofs.

Theorem 3 The lower dimensional Busemann-Petty problem on the entropy of log-concave functions has affirmative answers for $f(x)=\mathrm{e}^{-\|x\|_K}$ and $g(x)=\mathrm{e}^{-\|x\|_L}$, where $x\in\mathbb{R}^n$, K is an i-intersection body in $\mathbb{R}^n$ and L is an origin-symmetric star body in $\mathbb{R}^n$.

Proof Let K be an i-intersection body in $\mathbb{R}^n$ and let L be an origin-symmetric star body in $\mathbb{R}^n$, and let $f(x)=\mathrm{e}^{-\|x\|_K}$ and $g(x)=\mathrm{e}^{-\|x\|_L}$ for $x\in\mathbb{R}^n$. Since the function $t\log t$ is increasing for $t>\mathrm{e}^{-1}$ and $\mathrm{Ent}(cf)=c\mathrm{Ent}(f)$ for a constant c, without loss of generality, we assume $J(f)>\mathrm{e}^{-1}$ and $J(g)>\mathrm{e}^{-1}$. Now, for an arbitrary $H\in G_{n,n-i}$, suppose $\int_{\mathbb{R}^n\cap H} f(x)\mathrm{d}x\leqslant\int_{\mathbb{R}^n\cap H} g(x)\mathrm{d}x$. By lemma , this is equivalent to $V_{n-i}(K\cap H)\leqslant V_{n-i}(L\cap H)$.

By lemma , we have $V(K) \leqslant V(L)$. Thus,

$$\begin{aligned}\mathrm{Ent}(f) &= \int_{\mathbb{R}^n} f(x)\log f(x)\mathrm{d}x - J(f)\log J(f)\\ &= -n\Gamma(n+1)V(K) - \Gamma(n+1)V(K)\log(\Gamma(n+1)V(K))\\ &\geqslant -n\Gamma(n+1)V(L) - \Gamma(n+1)V(L)\log(\Gamma(n+1)V(L))\\ &= \mathrm{Ent}(g),\end{aligned}$$

which completes the proof.

Finally, we further investigate the relationship between the original lower dimensional Busemann-Petty problem and its version on the entropy of log-concave functions, which might provide a new path to study this long-standing open problem.

Proof of theorem 2 Let $K, L \in \mathcal{K}_o^n$. Similar to [18], without loss of generality, we assume $V(K) > 1$ and $V(L) > 1$. Let H be an arbitrary $(n-i)$-dimensional subspace in $\mathbb{R}^n$, and let $f(x) = \mathrm{e}^{-\|x\|_K}$ and $g(x) = \mathrm{e}^{-\|x\|_L}$, for $x \in \mathbb{R}^n$. By lemma 2 , we have

$$\int_{\mathbb{R}^n\cap H} f(x)\mathrm{d}x = \Gamma(n-i+1)V_{n-i}(K\cap H).$$

Similarly,

$$J(f) = \Gamma(n+1)V(K),$$
$$\int_{\mathbb{R}^n} f(x)\log f(x)\mathrm{d}x = -n\Gamma(n+1)V(K).$$

Hence,

$$\begin{aligned}\mathrm{Ent}(f) &= \int_{\mathbb{R}^n} f(x)\log f(x)\mathrm{d}x - J(f)\log J(f)\\ &= -\Gamma(n+1)V(K)(n+\log\Gamma(n+1)\log V(K)).\end{aligned}$$

Analogies hold for L.

Now, we assume $V_{n-i}(K\cap H) \leqslant V_{n-i}(L\cap H)$ for every $H \in G_{n,n-i}$, by lemma 2 , which is equivalent to $\int_{\mathbb{R}^n\cap H} f(x)\mathrm{d}x \leqslant \int_{\mathbb{R}^n\cap H} g(x)\mathrm{d}x$. If the lower dimensional Busemann-Petty problem on the entropy of log-concave functions has affirmative answer, we have $\mathrm{Ent}(f) \geqslant \mathrm{Ent}(g)$, which implies $V(K) \leqslant V(L)$. Therefore, the lower dimensional Busemann-Petty problem also has affirmative answer.

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