Complex L_p mixed Petty projection inequalities

MA Dan, SHI Sunping, XU Wenshuai

(Mathematics and Science College, Shanghai Normal University, Shanghai 200234, China)

Abstract: In this paper, we introduce the concept of complex L_p mixed projection bodies by giving its support function. Then, we establish the complex L_p mixed Petty projection inequalities. Finally, the monotonicity for complex L_p mixed projection bodies is obtained.

Key words: complex L_p projection bodies; mixed projection bodies; Petty projection inequality; monotonicity

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复 L_p 混合 Petty 投影不等式

马 丹, 施孙平, 徐文帅

(上海师范大学 数理学院, 上海 200234)

摘 要: 通过支撑函数引入了复 L_p 混合投影体的概念. 在此基础上, 建立了复 L_p 混合 Petty 投影不等式, 得到了复 L_p 混合投影体的单调性.

关键词: 复 L_p 投影体; 混合投影体; Petty 投影不等式; 单调性

0 Introduction

In the late 19th century, projection bodies have been extensively studied. The important properties of projection bodies have significant applications not only in convex geometry, but also in other aspects such as geometric tomography, stochastic geometry, optimization and functional analysis[1]. Let $\mathcal{K}^n$ denote the set of convex bodies (non-empty compact convex subsets in $\mathbb{R}^n$). In [2,3], it states that, for $K \in \mathcal{K}^n$, Minkowski introduced the projection body ΠK as the convex body, which has support function

$$h_{\Pi K}(u) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v| \mathrm{d}S(K, v),$$

where, $S(K, \cdot)$ is the surface area measure of $K \in \mathcal{K}^n$. There are also some important inequalities about projection bodies such as the Petty projection inequality[3]. The Petty projection inequality shows that ellipsoids precisely have polar projection bodies of maximal volume in all convex bodies of given volume.

Recently, complex convex bodies have gradually attracted increasing attention[4,5]. Some classical convex geometric concepts in real vector spaces were generalized to complex cases such as complex projection bodies[6], complex difference bodies[4] and complex intersection bodies[7]. A recent important result by WANG et al[8] has introduced complex L_p projection bodies, which uses the properties of the asymmetric L_p zonoid (see [9] for related interesting work).

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Biography: MA Dan (1985-), female, associate professor, research area: convex geometric analysis. E-mail: madan@shnu.edu.cn.

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Let $\mathcal{K}(\mathbb{C}^n)$ denote the set of convex bodies in $\mathbb{C}^n$, and $\mathcal{K}_0(\mathbb{C}^n)$ represent the set of convex bodies in $\mathbb{C}^n$ that contain the origin in their interiors. B denotes the unit ball in $\mathbb{C}^n$, and its surface is denoted by $\mathbb{S}^n$. If there exists a finite even Borel measure $\mu_{p,C}$ on the unit sphere $\mathbb{S}^1$ such that

$$h_C(u)^p = \int_{\mathbb{S}^1} (\Re[cu \cdot_H v])_+^p \mathrm{d}\mu_{p,C}(v), u \in \mathbb{S}^1,$$

then a convex body $C \in \mathcal{K}(\mathbb{C})$ is called an asymmetric L_p zonoid. Let $p \geq 1$, $K \in \mathcal{K}_0(\mathbb{C}^n)$, $C \in \mathcal{K}(\mathbb{C})$ is an asymmetric L_p zonoid and $Cu = \{cu : c \in C\}$, then the asymmetric complex L_p projection body $\Pi_{p,C}^+ K$ as the convex body, which has the support function

$$h_{\Pi_{p,C}^+ K}(u)^p = 2nV_p(K, Cu) = \int_{\mathbb{S}^n} \int_{\mathbb{S}^1} (\Re[cu \cdot_H v])_+^p \mathrm{d}\mu_{p,C}(c)\mathrm{d}S_p(K, v), \tag{1}$$

for every $u \in \mathbb{S}^n$, where V_p is the L_p-mixed volume, $\cdot_H$ denotes the Hermitian inner product on $\mathbb{C}^n$, $S_p(K, \cdot)$ denotes the L_p surface area measure of K on $\mathbb{S}^n$, and $\mu_{p,C}$ is a finite even Borel measure on the unit sphere $\mathbb{S}^1$ (see [8] and section 2 for definitions).

For $p \geq 1$, K, $L \in \mathcal{K}_0(\mathbb{C}^n)$ and $\alpha, \beta \geq 0$, the L_p Minkowski combination $\alpha \cdot K +_p \beta \cdot L$ is defined by $h^p_{\alpha \cdot K +_p \beta \cdot L} = \alpha h^p_K + \beta {h_L}^p$, where the relationship between the L_p Minkowski and the usual scalar multiplication is $\alpha \cdot K = \alpha^{\frac{1}{p}} K$[3]. The complex L_p projection bodies, $\Pi^\lambda_{p,C} K$, are defined by

$$\Pi^\lambda_{p,C} K = \lambda \cdot \Pi^+_{p,C} K +_p (1 - \lambda) \cdot \Pi^-_{p,C} K, \tag{2}$$

for every $\lambda \in [0, 1]$, where $\Pi^-_{p,C} K = \Pi^+_{p,C}(-K)$. Then, by making full use of Haberl's method in [10], WANG et al[8] established the Petty projection inequality about the general complex L_p projection. The results can be stated as follows:

Theorem 1 [8] Let $p > 1$ and $K \in \mathcal{K}_0(\mathbb{C}^n)$. If $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric L_p zonoid which satisfies $\dim C \geq 1$, then for every $\lambda \in [0, 1]$, we have

$$V(K)^{\frac{2n}{p}-1} V(\Pi^{\lambda,*}_{p,C} K) \leq V(B)^{\frac{2n}{p}-1} V(\Pi^{\lambda,*}_{p,C} B),$$

where, $\Pi^{\lambda,*}_{p,C} K$ is the polar body of $\Pi^\lambda_{p,C} K$. With equality, if and only if K is an origin-symmetric ellipsoid when $\dim C = 1$, and with equality, if and only if K is an origin-symmetric Hermitian ellipsoid, when $\dim C = 2$.

Mixed projection bodies are related to ordinary projection bodies in the same way that mixed volumes are related to ordinary volume. In [3], it states that mixed projection bodies $\Pi(K_1, K_2, \ldots, K_{n-1})$ first appeared in the work of Süss. For $K_1, K_2, \ldots, K_{n-1} \in \mathcal{K}^n$, $\Pi(K_1, K_2, \ldots, K_{n-1})$ are defined as the convex bodies with the support function

$$h_{\Pi(K_1,K_2,\ldots,K_{n-1})}(u) = \frac{1}{2} \int_{S^{n-1}} |u \cdot v| \mathrm{d}S(K_1, K_2, \ldots, K_{n-1}, v),$$

for every $u \in S^{n-1}$. For $K_1, K_2, \ldots, K_{n-i-1} = K$ and $K_{n-i}, K_{n-i+1}, \ldots, K_{n-1} = B$, the mixed projection body $\Pi(K_1, K_2, \ldots, K_{n-1})$ is usually written as $\Pi_i K$.

In [11], WAN et al gave the proof of the Petty projection inequality and the monotonicity for the general L_p mixed projection bodies in $\mathbb{R}^n$. In this paper, we extend these results to $\mathbb{C}^n$.

1 Preliminaries

For a complex number $c \in \mathbb{C}$, let $\overline{c}$ denote its complex conjugate and $|c|$ for its norm. Write $\cdot_H$ for the Hermitian inner product on $\mathbb{C}^n$, i.e. $x \cdot_H y = x^* y$ for all $x, y \in \mathbb{C}^n$, where x^* denote the conjugate transpose of x. Let B stand for the complex unit ball $\{c \in \mathbb{C}^n : c \cdot_H c \leq 1\}$, and $\mathbb{S}^n$ its sphere. Use ι to denote the canonical isomorphism between $\mathbb{C}^n$ (viewed as a real vector space) and $\mathbb{R}^{2n}$, i.e.,

$$\iota(c) = (\Re[c_1], \Re[c_2], \ldots, \Re[c_n], \Im[c_1], \Im[c_2], \ldots, \Im[c_n]), \quad c \in \mathbb{C}^n,$$

where, $\Re$, $\Im$ are the real and imaginary part respectively. Note that $\Re[x \cdot_H y] = \iota x \cdot \iota y$ for all $x, y \in \mathbb{C}^n$, where the inner product on the right hand side is the standard Euclidean inner product on $\mathbb{R}^{2n}$. The volume of the unit ball in $\mathbb{C}^n$ is denoted by ω_{2n}.

Let $K \in \mathcal{K}(\mathbb{C}^n)$. K is called an origin-symmetric ellipsoid, if there exists some positive definite symmetric matrix $\phi \in \mathrm{GL}(2n, \mathbb{R})$ such that $K = \{x \in \mathbb{C}^n : \iota x \cdot \phi \iota x \leq 1\}$. K is called an origin-symmetric Hermitian ellipsoid, if $K = \{x \in \mathbb{C}^n : x \cdot_H \xi x \leq 1\}$, for a positive definite Hermitian matrix $\xi \in \mathrm{GL}(n, \mathbb{C})$.

If K is a nonempty set in $\mathbb{C}^n$, the polar set of K, K^* is defined by

$$K^* = \{x \in \mathbb{C}^n : \Re[x \cdot_H y] \leq 1, y \in K\}.$$

If $K \in \mathcal{K}_0(\mathbb{C}^n)$, then K^* is called polar body and $K^* \in \mathcal{K}_0(\mathbb{C}^n)$. The radial function $\rho_K = \rho(K, \cdot) : \mathbb{C}^n \setminus \{0\} \to [0, \infty)$, of a compact star-shaped (about the orgin) $K \subset \mathbb{C}^n$, is defined by $\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}$. Moreover, on $\mathbb{C}^n \setminus \{0\}$, we have $\rho_{K^*} = h_K^{-1}$. If ρ_K is positive and continuous, then K is called a star body (about the origin). We write $S(\mathbb{C}^n)$ for the set of star bodies in $\mathbb{C}^n$.

The following results follow immediately from the real counterparts since all the quantities (volume, L_p mixed volume, L_p surface area measure, support function) are compatible with the canonical isomorphism ι. In [12], the dual L_p mixed volume $\widetilde{V}_{-p}(K, L)$ is defined by

$$\widetilde{V}_{-p}(K, L) = \frac{1}{2n} \int_{\mathbb{S}^n} \rho_K^{2n+p} \rho_L^{-p} \mathrm{d}\sigma, \tag{3}$$

where, σ stands for the push forward with respect to ι^{-1} of $\mathcal{H}^{2n-1}$ on the $(2n-1)$-dimensional Euclidean unit sphere.

In [8], authors introduced the notion of the complex L_p moment body. Let $p \geq 1$, $K \in \mathcal{K}_0(\mathbb{C}^n)$ and $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric L_p zonoid. The asymmetric complex L_p moment body $\mathrm{M}_{p,C}^+ K$ is the convex body with the support function

$$h_{\mathrm{M}_{p,C}^+ K}(u)^p = 2 \int_K h_{Cu}(x)^p \mathrm{d}x = \frac{2}{2n+p} \int_{\mathbb{S}^n} \int_{\mathbb{S}^1} (\Re[cu \cdot_H v])_+^p \rho_K(v)^{2n+p} \mathrm{d}\mu_{p,C}(c) \mathrm{d}\sigma(v),$$

for all $u \in \mathbb{S}^n$. The complex L_p moment bodies $\mathrm{M}_{p,C}^\lambda K$ are defined by

$$\mathrm{M}_{p,C}^\lambda K = \lambda \cdot \mathrm{M}_{p,C}^+ K +_p (1 - \lambda) \cdot \mathrm{M}_{p,C}^- K, \tag{4}$$

for every $\lambda \in [0, 1]$, where, $\mathrm{M}_{p,C}^- K = \mathrm{M}_{p,C}^+(-K)$.

For $K \in \mathcal{K}(\mathbb{C}^n)$ and $i = 0, 1, \ldots, 2n-1$, the quermassintegrals, $W_i(K)$, of K are defined by

$$W_i(K) = \frac{1}{2n} \int_{\mathbb{S}^n} h(K, u) \mathrm{d}S_i(K, u).$$

LUTWAK[13] introduced the notion of L_p mixed quermassintegrals. Let $p \geq 1$, $i = 0, 1, \ldots, 2n-1$. For $K, L \in \mathcal{K}_0(\mathbb{C}^n)$, the integral represention of the L_p mixed quermassintegrals $W_{p,i}(K, L)$ of K and L is

$$W_{p,i}(K, L) = \frac{1}{2n} \int_{\mathbb{S}^n} h^p(L, u) \mathrm{d}S_{p,i}(K, u). \tag{5}$$

The integral represention of the L_p mixed volume $V_p(K, L)$ is defined by [13]

$$V_p(K, L) = \frac{1}{2n} \int_{\mathbb{S}^n} h^p(L, u) \mathrm{d}S_p(K, u). \tag{6}$$

2 The complex L_p mixed Petty projection inequalities

To start with, the following definitions, theorems and lemmas are needed.

Definition 1 Let $p \geq 1$, $i = 0, 1, \ldots, 2n-1$ and $K \in \mathcal{K}_0(\mathbb{C}^n)$. If $C \in \mathcal{K}(\mathbb{C})$ is an asymmetric L_p zonoid, then the asymmetric complex L_p mixed projection body $\Pi^+_{p,C,i}K$ is the convex body with the support function

$$h_{\Pi^+_{p,C,i}K}(u)^p = 2nW_{p,i}(K, Cu) = \int_{\mathbb{S}^n}\int_{\mathbb{S}^1} (\Re[cu \cdot_H v])^p_+ \mathrm{d}\mu_{p,C}(c)\mathrm{d}S_{p,i}(K, v), \tag{7}$$

for every $u \in \mathbb{S}^n$, where, the positive Borel measure $S_{p,i}(K, \cdot)$ on $\mathbb{S}^n$ is absolutely continuous with respect to $S_i(K, \cdot)$, and has Radon-Nikodym derivative

$$\frac{\mathrm{d}S_{p,i}(K, \cdot)}{\mathrm{d}S_i(K, \cdot)} = h(K, \cdot)^{1-p}. \tag{8}$$

The complex L_p mixed projection bodies $\Pi^\lambda_{p,C,i}K$ are defined by

$$\Pi^\lambda_{p,C,i}K = \lambda \cdot \Pi^+_{p,C,i}K +_p (1-\lambda) \cdot \Pi^-_{p,C,i}K, \tag{9}$$

for every $\lambda \in [0, 1]$, where $\Pi^-_{p,C,i}K = \Pi^+_{p,C,i}(-K)$.

Theorem 2 [8] Let $p > 1$ and $K \in \mathcal{K}_0(\mathbb{C}^n)$. If $C \in \mathcal{K}(\mathbb{C})$ be an asymmetric L_p zonoid which satisfies $\dim C \geq 1$, then for every $\lambda \in [0, 1]$, we have

$$V(K)^{-\frac{2n}{p}-1}V(\mathrm{M}^\lambda_{p,C}K) \geq V(B)^{-\frac{2n}{p}-1}V(\mathrm{M}^\lambda_{p,C}B)$$

with equality, if and only if K is an origin-symmetric ellipsoid when $\dim C = 1$, and with equality, if and only if K is an origin-symmetric Hermitian ellipsoid, when $\dim C = 2$.

Theorem 3 [13] Let $K, L \in \mathcal{K}_0(\mathbb{C}^n)$, $p > 1$ and $0 \leq i < n$. Then, $W_{p,i}(K, L)^{n-i} \geq W_i(K)^{n-p-i}W_i(L)^p$ with equality, if and only if K and L are dilations.

Theorem 4 [14] Let $K, L \in \mathcal{K}^n_0$, $p > 1$, $0 \leq i < n$, $n - i \neq p$. If $W_{p,i}(K, Q) = W_{p,i}(L, Q)$, for any $Q \in \mathcal{K}^n_o$, then $K = L$.

Lemma 1 [15] Let $0 < i < n$. If $K \in \mathcal{K}^n$, then $W_i(K) \geq \omega_n^{\frac{i}{n}}V(K)^{\frac{n-i}{n}}$ with equality, if and only if K is a ball.

Lemma 2 Let $K \in \mathcal{K}_0(\mathbb{C}^n)$, $L \in S(\mathbb{C}^n)$. For $0 \leq i < 2n$, $p \geq 1$, $\lambda \in [0, 1]$, then

$$W_{p,i}(K, \mathrm{M}^\lambda_{p,C}L) = \frac{2}{2n+p}\widetilde{V}_{-p}(L, \Pi^{\lambda,*}_{p,C,i}K).$$

Proof From (5), (3), (4), (7) and Fubini theorem, we have

$$W_{p,i}(K, \mathrm{M}^+_{p,C}L) = \frac{1}{2n} \cdot \frac{2}{2n+p}\int_{\mathbb{S}^n} h_{\Pi^+_{p,C,i}K}(v)^p \rho_L(v)^{2n+p}\mathrm{d}\sigma(v) = \frac{2}{2n+p}\widetilde{V}_{-p}(L, \Pi^{+,*}_{p,C,i}K).$$

Therefore, from (9) and (4), we conclude the desired result.

Lemma 3 Let $p \geq 1$. For an asymmetric L_p zonoid $C \in \mathcal{K}(\mathbb{C})$, $\Pi^\lambda_{p,C,i}$ maps a symmetric ball about the origin to a symmetric ball about the origin.

Proof From (8), we have

$$S_{p,i}(rB,v)=r^{1-p}S_i(rB,v)=r^{1-p+i}S(rB,v)=r^{2n-p+i}\sigma(v),$$

for every $r>0, v\in\mathbb{S}^n$. Plug this into (7) to get

$$h_{\Pi^+_{p,C,i}(rB)}(u)^p=r^{2n-p+i}\int_{\mathbb{S}^n}\int_{\mathbb{S}^1}(\Re[cu\cdot_H v])_+^p\mathrm{d}\mu_{p,C}(c)\mathrm{d}\sigma(u), \tag{10}$$

for every $u\in\mathbb{S}^n$. Now, we fix some $u_0\in\mathbb{S}^n$ and recall $Cu=\{cu:c\in C\}$. For every $u\in\mathbb{S}^n$, there is a $\phi_u\in\mathrm{SU}(n)$ satisfies $\phi_u u_0=u$, then $Cu=\phi_u Cu_0$. Substituting this into (10) gives

$$h_{\Pi^+_{p,C,i}(rB)}(u)^p=r^{2n-p+i}\int_{\mathbb{S}^n}\int_{\mathbb{S}^1}(\Re[cu_0\cdot_H \phi_u^* v])_+^p\mathrm{d}\mu_{p,C}(c)\mathrm{d}\sigma(v).$$

Since σ is $\mathrm{SU}(n)$-invariant and $\dim C>0$, the right hand is independent from u and greater than zero. Therefore, $\Pi^+_{p,C,i}(rB)$ is a symmetric ball about the origin, and $\Pi^+_{p,C,i}$ maps a symmetric ball about the origin to a symmetric ball about the origin.

Finally, by the definition of $\Pi^\lambda_{p,C,i}$ and the proof above, we conclude this lemma.

Lemma 4 [8] Let $p\geq 1$. For an asymmetric L_p zonoid $C\in\mathcal{K}(\mathbb{C})$, $\mathrm{M}^\lambda_{p,C}$ has the same conclusion of lemma 3.

2.1 The Petty projection inequality for complex L_p mixed projection body

In this part, we establish the Petty projection inequality for complex L_p mixed projection body.

Theorem 5 Let $1<p<2n-i$, $0<i<2n-1$. If $K\in\mathcal{K}_0(\mathbb{C}^n)$ is smooth and $C\in\mathcal{K}(\mathbb{C})$ be an asymmetric L_p zonoid which satisfies $\dim C\geq 1$, then

$$\omega_{2n}^{\frac{i}{2n}}W_i(K)^{\frac{2n-i-p}{p}}V(\Pi^{\lambda,*}_{p,C,i}K)^{\frac{2n-i}{2n}}\leq\left(\frac{2}{2n+p}\right)^{\frac{2n-i}{p}}r_C^{-2n+i}\omega_{2n}^{\frac{2n-i}{p}}, \tag{11}$$

for every $\lambda\in[0,1]$,where $r_C>0$ such that $M^\lambda_{p,C}B=r_C B$. Equality holds if and only if K is an origin-symmetric ball.

Proof From lemma 2, taking $L=\Pi^{\lambda,*}_{p,C,i}K$, we get

$$\frac{2}{2n+p}V(\Pi^{\lambda,*}_{p,C,i}K)=W_{p,i}(K,\mathrm{M}^\lambda_{p,C}\Pi^{\lambda,*}_{p,C,i}K).$$

By using theorem 3 and lemma 1, we have

$$\frac{2}{2n+p}V(\Pi^{\lambda,*}_{p,C,i}K)\geq W_i(K)^{\frac{2n-p-i}{2n-i}}(\omega_{2n}^{\frac{i}{2n}}V(\mathrm{M}^\lambda_{p,C}\Pi^{\lambda,*}_{p,C,i}K)^{\frac{2n-i}{2n}})^{\frac{p}{2n-i}}.$$

Next by theorem 2, lemma 3 and lemma 4, when taking $\mathrm{M}^\lambda_{p,C}B=r_C B$, we obtain

$$\frac{2}{2n+p}\omega_{2n}\geq\omega_{2n}^{\frac{pi}{2n\cdot(2n-i)}}W_i(K)^{\frac{2n-p-i}{2n-i}}V(\Pi^{\lambda,*}_{p,C,i}K)^{\frac{p}{2n}}\cdot r_C^p.$$

Raising both sides of the inequality to the power of $(2n-i)/p$, we obtain

$$\omega_{2n}^{\frac{i}{2n}}W_i(K)^{\frac{2n-i-p}{p}}V(\Pi^{\lambda,*}_{p,C,i}K)^{\frac{2n-i}{2n}}\leq\left(\frac{2}{2n+p}\right)^{\frac{2n-i}{p}}r_C^{-2n+i}\omega_{2n}^{\frac{2n-i}{p}}.$$

Finally, according to the conditions of the three equalities in theorem 3, lemma 1 and theorem 2, combining with lemma 3, equality holds in (11), if and only if K is a symmetric ball about the origin.

2.2 The monotonicity of the complex L_p mixed projection bodies

In this part, we give the monotonicity of the complex L_p mixed projection body. In the following, $\mathrm{A}_{p,C}^{\lambda,n}$ denotes the set of all complex L_p projection bodies.

Lemma 5 Let $p \geq 1$, $0 < i < 2n$. If $K, L \in \mathcal{K}_0(\mathbb{C}^n)$, then for every $\lambda \in [0,1]$, we have

$$W_{p,i}(L, \Pi_{p,C}^{\lambda} K) = V_p(K, \Pi_{p,C,i}^{\lambda} L). \tag{12}$$

Proof From (1), (7), (5), (6) and Fubini theorem, we get

$$W_{p,i}(L, \Pi_{p,C}^{+} K) = \frac{1}{2n}\int_{\mathbb{S}^n} h_{\Pi_{p,C,i}^{+} L}(v)^p \mathrm{d}S_p(K, v) = V_p(K, \Pi_{p,C,i}^{+,*} L).$$

Applying (2) and (9), we conclude the desired result.

Theorem 6 Let $K, L \in \mathcal{K}_0(\mathbb{C}^n)$, $p > 1$, $0 < i < 2n$. For every $\lambda \in [0,1]$, if $\Pi_{p,C,i}^{\lambda} K \subseteq \Pi_{p,C,i}^{\lambda} L$, then

$$W_{p,i}(K, Q) \leq W_{p,i}(L, Q), \tag{13}$$

for any $Q \in \mathrm{A}_{p,C}^{\lambda,n}$. Equality holds if and only if $K = L$.

Proof Since $Q \in \mathrm{A}_{p,C}^{\lambda,n}$, there is $M \in \mathcal{K}_0(\mathbb{C}^n)$ such that $Q = \Pi_{p,C}^{\lambda} M$. From (12) and the integral representation of the L_p mixed volume, using the condition $\Pi_{p,C,i}^{\lambda} K \subseteq \Pi_{p,C,i}^{\lambda} L$, we get

$$\frac{W_{p,i}(L,Q)}{W_{p,i}(K,Q)} = \frac{W_{p,i}(L, \Pi_{p,C}^{\lambda} M)}{W_{p,i}(K, \Pi_{p,C}^{\lambda} M)} = \frac{V_p(M, \Pi_{p,C,i}^{\lambda} L)}{V_p(M, \Pi_{p,C,i}^{\lambda} K)} = \frac{\int_{\mathbb{S}^n} h^p(\Pi_{p,C,i}^{\lambda} L, u)\mathrm{d}S_p(M,u)}{\int_{\mathbb{S}^n} h^p(\Pi_{p,C,i}^{\lambda} K, u)\mathrm{d}S_p(M,u)} \geq 1.$$

Thus, we get (13). According to theorem 4, equality holds in (13), if and only if $K = L$.

Theorem 7 Let $p > 1$, $0 < i < 2n$, $K \in \mathcal{K}_0(\mathbb{C}^n)$, $L \in \mathrm{A}_{p,C}^{\lambda,n}$, and $2n - i \neq p$. For every $\lambda \in [0,1]$, if $\Pi_{p,C,i}^{\lambda} K \subseteq \Pi_{p,C,i}^{\lambda} L$, then

$$W_i(K) \geq W_i(L), \text{ for } 0 < 2n - i < p \tag{14}$$

and

$$W_i(K) \leq W_i(L), \text{ for } 2n - i > p. \tag{15}$$

Equalities hold in (14) and (15) if and only if $K = L$.

Proof For $L \in \mathrm{A}_{p,C}^{\lambda,n}$, replacing Q with L in theorem 6, from theorem 3, then

$$W_i(L) \geq W_{p,i}(K, L) \geq W_i(K)^{\frac{2n-p-i}{2n-i}} W_i(L)^{\frac{p}{2n-i}}.$$

Therefore, under the restriction of $0 < 2n - i < p$, $W_i(K) > W_i(L)$ holds. On the other hand, when $2n - i > p$, $W_i(K) \leq W_i(L)$ holds. Together with the conditions of the two equlities in theorem 6 and theorem 3, we get the equalities hold in (14) and (15), if and only if $K = L$.

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