SL(n) contravariant vector valuations

Jin Li[1], Dan Ma[2,3], and Wei Wang[4]

[1]Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, 1040 Vienna, Austria, lijin2955@gmail.com

[2]Department of Mathematics, Shanghai Normal University, Shanghai 200234, China, madan@shnu.edu.cn

[3]Department of Mathematics, Karlsruhe Institute of Technology (KIT), D-76128 Karlsruhe, Germany

[4]School of Mathematics and Computational Science, Hunan University of Science and Technology, Xiangtan 411201, China, wwang@hnust.edu.cn

Abstract

All SL(n) contravariant vector valuations on polytopes in $\mathbb{R}^n$ are completely classified without any additional assumptions. The facet vector is defined. It turns out to be the unique class of such valuations for $n \geq 3$. In dimension two, the classification corresponds to the known case of SL(2) covariant valuations.

1 Introduction

The study of geometric notions which are compatible with transformation groups are important tasks in geometry as proposed in Felix Klein's Erlangen program in 1872. As many functions defined on geometric objects satisfy the inclusion-exclusion principle, the property of being a valuation is natural to consider in the classification of those functions. Here, a function Z defined on $\mathcal{P}^n$, the space of all polytopes in $\mathbb{R}^n$, and taking values in an abelian semigroup is called a *valuation* if

$$Z(P) + Z(Q) = Z(P \cup Q) + Z(P \cap Q) \tag{1.1}$$

for every $P, Q, P \cup Q \in \mathcal{P}^n$. A function Z defined on some subspace of $\mathcal{P}^n$ is also called a valuation if (1.1) holds whenever $P, Q, P \cup Q, P \cap Q$ contained in this subspace. Valuations

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also have their origins in Dehn's solution of Hilbert's Third Problem in 1901. The most famous result is Hadwiger's characterization theorem which classifies all continuous and rigid motion invariant real valuations on the space of convex bodies in $\mathbb{R}^n$. This celebrated result initiated a systematic study on the classification of valuations compatible with certain linear transforms.

These studies are also a classical part of geometry with important applications in integral geometry (see [6, Chap. 7], [13], [29, Chap. 6]). They turned out to be extremely fruitful and useful especially in the affine geometry of convex bodies (see [2–4, 9, 12, 17]). Examples of valuations are intrinsic volumes [8, 20], affine surface areas [24, 25], the projection bodies [7, 16, 19, 22], the intersection bodies [23] and other Minkowski valuations [5, 30, 32].

The aim of this paper is to obtain a complete classification of $\mathrm{SL}(n)$ contravariant vector valuations on polytopes without any additional assumptions.

A function $Z : \mathcal{P}^n \to \mathbb{R}^n$ is called a vector valuation if the addition in (1.1) is the vector addition. It is called $\mathrm{SL}(n)$ *contravariant* if $Z(\phi P) = \phi^{-t} Z(P)$ for all $P \in \mathcal{P}^n$ and $\phi \in \mathrm{SL}(n)$, and is called $\mathrm{SL}(n)$ *covariant* if $Z(\phi P) = \phi Z(P)$ for all $P \in \mathcal{P}^n$ and $\phi \in \mathrm{SL}(n)$. If Z is either $\mathrm{SL}(n)$ contravariant or $\mathrm{SL}(n)$ covariant, then Z is $\mathrm{SL}(n)$ *intertwining*. In 2002, Ludwig [18] established the first classification of measurable, $\mathrm{SL}(n)$ intertwining vector valuations on $\mathcal{P}^n_{(o)}$ with some assumptions of homogeneity, where $\mathcal{P}^n_{(o)}$ is the space of polytopes in $\mathbb{R}^n$ that contain the origin in their interiors. Later, Haberl and Parapatits [10] removed the homogeneity assumption in Ludwig's result. Recently, Zeng and the second author [34] obtained a complete classification of $\mathrm{SL}(n)$ covariant vector valuations on $\mathcal{P}^n$ without any additional assumptions. There are also some interesting characterizations of matrix and tensor valuations (see [1, 11, 21, 27, 28]). Surprisingly, classifications of $\mathrm{SL}(n)$ contravariant vector valuations are still missing on $\mathcal{P}^n$ with any conditions and it should be indispensable for further classifications of $\mathrm{SL}(n)$ contravariant tensor valuations. We remark that the $\mathrm{SL}(n)$ invariant real valued valuations classified by Ludwig and Reitzner [26] are $\mathrm{SL}(n)$ contravariant tensor valuations of order 0 and the $\mathrm{SL}(n)$ contravariant vector valuations considered in this paper are $\mathrm{SL}(n)$ contravariant tensor valuations of order 1.

An intuitive example of $\mathrm{SL}(n)$ contravariant vector valuation is the sum of all facet normals. However, the Minkowski relation shows that it vanishes (see [29, §8.2.1]). More precisely, in the case of polytopes, it means the following. For $u \in S^{n-1}$, we define $a_P(u)$ by the $(n-1)$-dimensional volume of $F(P,u) = P \cap \{x \in \mathbb{R}^n : x \cdot u = h_P(u)\}$, where $h_P(u) = \max\{x \cdot u : x \in P\}$ denotes the support function of P. For $P \in \mathcal{P}^n$, we have

$$\sum_{u \in \mathcal{N}(P)} a_P(u) u = o,$$

where $\mathcal{N}(P)$ denotes the set of all outer unit normals of facets of P. However, it is also meaningful to consider the partial sum over facets that do not contain the origin, for example, the new projection body $\Pi_o P$ defined by Ludwig [22].

In general, for $\zeta \in \mathcal{C} = \{\text{solutions of Cauchy's functional equation } f : [0,\infty) \to \mathbb{R}\}$, the *facet vector* $M_\zeta^{0,1}(P)$ of $P \in \mathcal{P}^n$ is defined by

$$M_\zeta^{0,1}(P) = \sum_{u \in \mathcal{N}(P) \setminus \mathcal{N}_o(P)} \frac{\zeta(V(P,u))}{|h_P(u)|} u,$$

where $\mathcal{N}_o(P)$ denotes the set of outer unit normals of facets of P that contain the origin in their affine hulls, and $V(P,u)$ denotes the volume of the cone $[o, F(P,u)]$, the convex hull of $F(P,u)$ and the origin. We use the notation $M^{0,1}$ coinciding with $(0,1)$-tensor in [11]. Also, it is related with $(0,1)$-Minkowsi tensor in [29, §5.4.2].

Here, we say $f : [0,\infty) \to \mathbb{R}$ is a solution of Cauchy's functional equation if

$$f(x+y) = f(x) + f(y)$$

for every $x, y \in [0,\infty)$. If we assume some "regularity" conditions on f (e.g., continuous, bounded, or measurable), then f has to be a linear function. However, if no further conditions are assumed, then there are infinitely many other solutions of Cauchy's functional equation.

For $\zeta \in \mathcal{C}$, the mapping $\zeta \mapsto M_\zeta^{0,1}$ is injective. Indeed, we have $M_\zeta^{0,1}(s^{\frac{1}{n}} T^n) = \zeta(s/n!)\mathbf{1}$, for $s \geq 0$ and $\zeta \in \mathcal{C}$ (see Section 2 for notation). Let $\zeta_1, \zeta_2 \in \mathcal{C}$. Then, $M_{\zeta_1}^{0,1} = M_{\zeta_2}^{0,1}$ implies $M_{\zeta_1}^{0,1}(s^{\frac{1}{n}} T^n) = M_{\zeta_2}^{0,1}(s^{\frac{1}{n}} T^n)$. Thus, $\zeta_1(s/n!) = \zeta_2(s/n!)$. Setting $t = s/n!$, we obtain $\zeta_1(t) = \zeta_2(t)$, for $t \geq 0$.

In this paper, we show that the facet vector is essentially the unique class of $\mathrm{SL}(n)$ contravariant vector valuations on $\mathcal{P}_o^n$ for $n \geq 3$.

Let $\mathcal{P}_o^n$ be the space of polytopes in $\mathbb{R}^n$ that contain the origin,

Theorem 1.1. *Let $n \geq 3$. A function $Z : \mathcal{P}_o^n \to \mathbb{R}^n$ is an $\mathrm{SL}(n)$ contravariant valuation if and only if there exists $\zeta \in \mathcal{C}$*

$$Z(P) = M_\zeta^{0,1}(P)$$

for every $P \in \mathcal{P}_o^n$.

Using a relation with $\mathrm{SL}(2)$ covariant vector valuations, we obtain the classification in the case of dimension two. We also find that the vector B_ζ defined in [34] turns out to be a rotation of the facet vector in this case (see Section 3 for details).

Theorem 1.2. *A function $Z : \mathcal{P}_o^2 \to \mathbb{R}^2$ is an $\mathrm{SL}(2)$ contravariant valuation if and only if there exist constants $c_1, c_2 \in \mathbb{R}$ and $\zeta \in \mathcal{C}$ such that*

$$Z(P) = M_\zeta^{0,1}(P) + c_1 \rho_{\frac{\pi}{2}} M^{1,0}(P) + c_2 \rho_{\frac{\pi}{2}} A(P)$$

for every $P \in \mathcal{P}_o^2$, where $\rho_{\frac{\pi}{2}}$ is the counter-clockwise rotation in $\mathbb{R}^2$ of the angle $\frac{\pi}{2}$.

Here, for $P \in \mathcal{P}^n$, $M^{1,0}(P)$ is the *moment vector* of P, which is defined by $M^{1,0}(P) = \int_P x dx$. The notation also coincides with $(1,0)$-tensor in [11] and is related with $(1,0)$-Minkowsi tensor in [29, §5.4.2]. The valuation $A : \mathcal{P}_o^2 \to \mathbb{R}^2$ is defined by

$$A(P) = \begin{cases} v + w, & \text{if } \dim P = 2 \text{ and } P \text{ has two edges } [o,v], [o,w], \\ 2(v+w), & \text{if } P = [v,w] \text{ contains the origin,} \\ o, & \text{otherwise,} \end{cases}$$

where we view $[v,w]$ that contains the origin as two edges $[o,v]$ and $[o,w]$.

If we identify A as a valuation taking values in the dual space of $\mathbb{R}^n$, then we have

$$A(P)\cdot x = h_P(x) - h_{-P}(x) - h_P^o(x) + h_{-P}^o(x),\ x\in\mathbb{R}^n$$

for all $P\in\mathcal{P}_o^2$. Here $h_P^o := -\sum_{u\in\mathcal{F}_o(P)}(-1)^{\dim F}h_{-F}$ and $\mathcal{F}_o(P)$ is the set of (all dimensional) faces (including P itself) of P which contain the origin. Let $\mathcal{F}(P)$ be the set of faces of P. Shephard [33] established the following Euler-type relation:

$$h_P = -\sum_{F\in\mathcal{F}(P)}(-1)^{\dim F}h_{-F},$$

Thus, we also have

$$A(P)\cdot x = \sum_{u\in\mathcal{F}(P)\setminus\mathcal{F}_o(P)}(-1)^{\dim F}h_F(x) - \sum_{u\in\mathcal{F}(P)\setminus\mathcal{F}_o(P)}(-1)^{\dim F}h_{-F}(x),\ x\in\mathbb{R}^n$$

for all $P\in\mathcal{P}_o^2$. We refer to [15] for further study of h_P^o and Euler-type relations in a more general setting.

Similar to the classification of Minkowski valuations by Schuster and Wannerer [31], we further extend these results to $\mathcal{P}^n$.

Theorem 1.3. *Let $n\geq 3$. A function $Z:\mathcal{P}^n\to\mathbb{R}^n$ is an $\mathrm{SL}(n)$ contravariant valuation if and only if there exist $\zeta_1,\zeta_2\in\mathcal{C}$ such that*

$$Z(P) = M_{\zeta_1}^{0,1}(P) + M_{\zeta_2}^{0,1}([o,P]) \tag{1.2}$$

for every $P\in\mathcal{P}^n$, where $[o,P]$ is the convex hull of P and the origin.

Again, the case of dimension two is different.

Theorem 1.4. *A function $Z:\mathcal{P}^2\to\mathbb{R}^2$ is an $\mathrm{SL}(2)$ contravariant valuation if and only if there exist constants $c_1,c_2,\tilde{c}_1,\tilde{c}_2\in\mathbb{R}$ and $\zeta_1,\zeta_2\in\mathcal{C}$ such that*

$$\begin{aligned}Z(P) = M_{\zeta_1}^{0,1}(P) + M_{\zeta_2}^{0,1}([o,P]) + c_1\rho_{\frac{\pi}{2}}M^{1,0}(P) + \tilde{c}_1\rho_{\frac{\pi}{2}}M^{1,0}([o,P])\\ + c_2\rho_{\frac{\pi}{2}}A([o,P]) + \tilde{c}_2\rho_{\frac{\pi}{2}}A([o,v_1,\dots,v_r])\end{aligned}$$

for every polytope $P\in\mathcal{P}^2$ with vertices $v_1,\dots,v_r$ visible from the origin and labeled counter-clockwisely, where a vertex v of P is called visible *from the origin if $P\cap\operatorname{relint}[o,v]=\varnothing$.*

It should be remarked that vector valuations are special Minkowski valuations [7], since vectors can be viewed as convex bodies and the vector addition coincides with the Minkowski addition. Also, vectors can be viewed as linear functions on $\mathbb{R}^n$. Hence vector valuations are also embedded in the space of continuous-function valued valuations [14]. However, classifications of valuations in [7, 14] both need some assumptions of regularity. But as we have seen, it is not a problem for vector valuations.

2 Notation and preliminaries

2.1 Basic settings

We work in n-dimensional Euclidean space $\mathbb{R}^n$ with the standard basis $\{e_1, \dots, e_n\}$. We write a vector $x \in \mathbb{R}^n$ in coordinates by $x = (x_1, \dots, x_n)^t$. The inner product of $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$. Denote the vector with all coordinates 1 by $\mathbf{1}$, the $n \times n$ identity matrix by $I_n = (e_1, \dots, e_n)$ and the determinant of a matrix A by $\det A$. The affine hull, the boundary, the dimension, the interior and the relative interior of a given set in $\mathbb{R}^n$ are denoted by aff, bd, dim, int and relint, respectively.

Denote by $[v_1, \dots, v_k]$ the convex hull of $v_1, \dots, v_k \in \mathbb{R}^n$. A polytope is the convex hull of finitely many points in $\mathbb{R}^n$. Two basic classes of polytopes are the k-dimensional standard simplex $T^k = [o, e_1, \dots, e_k]$ and one of their $(k-1)$-dimensional facets $\tilde{T}^k = [e_1, \dots, e_k]$. In general, an i-dimensional simplex is the convex hull of $i+1$ affinely independent points. For $i = 1, \dots, n$, let $\mathcal{T}^i$ denote the set of i-dimensional simplices with one vertex at the origin, and $\tilde{\mathcal{T}}^i$ denote the set of $(i-1)$-dimensional simplices $T \subset \mathbb{R}^n$ with $o \notin \operatorname{aff} T$. Indeed, every polytope can be triangulated into simplices. We define a *triangulation* of a k-dimensional polytope P into simplices as a set of k-dimensional simplices $\{T_1, \dots, T_r\}$ which have pairwise disjoint interiors, with $P = \cup T_i$ and with the property that for arbitrary $1 \le i_1 < \cdots < i_j \le r$ the intersections $T_{i_1} \cap \cdots \cap T_{i_j}$ are again simplices.

2.2 Backgrounds on valuations

We refer to [6, Chap. 7], [13] and [29, Chap. 6] for classical backgroud on valuations. Let $\mathcal{Q}^n$ be either $\mathcal{P}^n_o$ or $\mathcal{P}^n$. First, we have the inclusion-exclusion principle (see [13]).

Lemma 2.1. *Let $Z : \mathcal{Q}^n \to \mathbb{R}^n$ be a valuation. Then*

$$Z(P_1 \cup \cdots \cup P_k) = \sum_{\varnothing \neq S \subseteq \{1,2,\dots,k\}} (-1)^{|S|-1} Z\Big(\bigcap_{i \in S} P_i\Big)$$

for all $k \in \mathbb{N}$ and $P_1, P_2, \dots, P_k \in \mathcal{Q}^n$ with $P_1 \cup \cdots \cup P_k \in \mathcal{Q}^n$.

We can use triangulations and the inclusion-exclusion principle to get the following result (see e.g., [14, Lemma 4.5 and Lemma 4.6]).

Lemma 2.2. *Let Z and Z' be $\mathrm{SL}(n)$ contravariant vector valuations on $\mathcal{P}^n_o$. If $Z(sT^d) = Z'(sT^d)$ for every $s > 0$ and $0 \le d \le n$, then $ZP = Z'P$ for every $P \in \mathcal{P}^n_o$.*

Lemma 2.3. *Let Z and Z' be $\mathrm{SL}(n)$ contravariant vector valuations on $\mathcal{P}^n$. If $Z(sT^d) = Z'(sT^d)$ and $Z(s\tilde{T}^d) = Z'(s\tilde{T}^d)$ for every $s > 0$ and $0 \le d \le n$, then $ZP = Z'P$ for every $P \in \mathcal{P}^n$.*

2.3 Basic results

A valuation on $\mathcal{Q}^n$ is called *simple* if it vanishes on $P \in \mathcal{Q}^n$ with $\dim P < n$.

Next, we mention a series of triangulations that will be used several times in this paper. Let $\lambda \in (0,1)$ and denote by H the hyperplane through the origin with the normal vector $(1-\lambda)e_1 - \lambda e_2$. Write

$$H^+ = \{x \in \mathbb{R}^n : x \cdot ((1-\lambda)e_1 - \lambda e_2) \geq 0\} \text{ and } H^- = \{x \in \mathbb{R}^n : x \cdot ((1-\lambda)e_1 - \lambda e_2) \leq 0\}.$$

Clearly, H^+ and H^- are the two halfspaces bounded by H. This hyperplane induces the series of triangulations of T^i as well as $\tilde{T}^i$ for $i = 2, \dots, n$. There are two representations corresponding to these triangulations due to the following definitions.

Definition 1. *For $\lambda \in (0,1)$, define the linear transform $\phi_1 \in \mathrm{SL}(n)$ by*

$$\phi_1 e_1 = \lambda e_1 + (1-\lambda)e_2,\ \phi_1 e_2 = e_2,\ \phi_1 e_n = e_n/\lambda,\ \phi_1 e_j = e_j, \quad \text{where } j \neq 1, 2, n,$$

and $\psi_1 \in \mathrm{SL}(n)$ by

$$\psi_1 e_1 = e_1,\ \psi_1 e_2 = \lambda e_1 + (1-\lambda)e_2,\ \psi_1 e_n = e_n/(1-\lambda),\ \psi_1 e_j = e_j, \quad \text{where } j \neq 1, 2, n.$$

Let $\hat{T}^{k-1} = [o, e_1, e_3, \dots, e_k]$ for $2 \leq k \leq n$.

Proposition 2.4. *Let $Z : \mathcal{P}_o^n \to \mathbb{R}^n$ be an $\mathrm{SL}(n)$ contravariant valuation. Then,*

$$\left(\phi_1^{-t} + \psi_1^{-t} - I_n\right) Z(T^i) = \phi_1^{-t} Z(\hat{T}^{i-1}), \tag{2.1}$$

for $2 \leq i < n$.

Proof. It is clear that $T^i \cap H^+ = \psi_1 T^i$, $T^i \cap H^- = \phi_1 T^i$ and $T^i \cap H = \phi_1 \hat{T}^{i-1}$. By the inclusion-exclusion principle, we have

$$Z(T^i) + Z(T^i \cap H) = Z(T^i \cap H^+) + Z(T^i \cap H^-).$$

Thus,

$$Z(T^i) + Z(\phi_1 \hat{T}^{i-1}) = Z(\phi_1 T^i) + Z(\psi_1 T^i).$$

Since Z is $\mathrm{SL}(n)$ contravariant, we derive

$$\left(\phi_1^{-t} + \psi_1^{-t} - I_n\right) Z(T^i) = \phi_1^{-t} Z(\hat{T}^{i-1}).$$

□

Definition 2. *For $\lambda \in (0,1)$, define the linear transform $\phi_2 \in \mathrm{GL}(n)$ by*

$$\phi_2 e_1 = \lambda e_1 + (1-\lambda)e_2, \quad \phi_2 e_2 = e_2, \quad \phi_2 e_j = e_j, \quad \text{where } j = 3, \dots, n,$$

and $\psi_2 \in \mathrm{GL}(n)$ by

$$\psi_2 e_1 = e_1, \quad \psi_2 e_2 = \lambda e_1 + (1-\lambda)e_2, \quad \psi_2 e_j = e_j, \quad \text{where } j = 3, \dots, n.$$

Proposition 2.5. *Let $Z : \mathcal{P}_o^n \to \mathbb{R}^n$ be an* $\mathrm{SL}(n)$ *contravariant valuation. Then,*

$$\begin{aligned} &Z(s^{\frac{1}{n}}T^n) + \lambda^{\frac{1}{n}}\phi_2^{-t}Z((\lambda s)^{\frac{1}{n}}\hat{T}^{n-1}) \\ =&\lambda^{\frac{1}{n}}\phi_2^{-t}Z((\lambda s)^{\frac{1}{n}}T^n) + (1-\lambda)^{\frac{1}{n}}\psi_2^{-t}Z(((1-\lambda)s)^{\frac{1}{n}}T^n), \end{aligned} \tag{2.2}$$

for $s > 0$.

Proof. It is clear that $sT^n \cap H^+ = \psi_2 sT^n$, $sT^n \cap H^- = \phi_2 sT^n$ and $sT^n \cap H = \phi_2 s\hat{T}^{n-1}$. By the inclusion-exclusion principle, we have

$$Z(sT^n) + Z(sT^n \cap H) = Z(sT^n \cap H^+) + Z(sT^n \cap H^-).$$

Thus,

$$Z(sT^n) + Z(\phi_2 s\hat{T}^{n-1}) = Z(\phi_2 sT^n) + Z(\psi_2 sT^n).$$

Since $\phi_2/\lambda^{\frac{1}{n}}$ and $\psi_2/(1-\lambda)^{\frac{1}{n}}$ belong to $\mathrm{SL}(n)$, we obtain

$$Z(sT^n) + \lambda^{\frac{1}{n}}\phi_2^{-t}Z(\lambda^{\frac{1}{n}}s\hat{T}^{n-1}) = \lambda^{\frac{1}{n}}\phi_2^{-t}Z(\lambda^{\frac{1}{n}}sT^n) + (1-\lambda)^{\frac{1}{n}}\psi_2^{-t}Z((1-\lambda)^{\frac{1}{n}}sT^n).$$

Replacing s by $s^{\frac{1}{n}}$ in the equation above yields

$$\begin{aligned} &Z(s^{\frac{1}{n}}T^n) + \lambda^{\frac{1}{n}}\phi_2^{-t}Z((\lambda s)^{\frac{1}{n}}\hat{T}^{n-1}) \\ =&\lambda^{\frac{1}{n}}\phi_2^{-t}Z((\lambda s)^{\frac{1}{n}}T^n) + (1-\lambda)^{\frac{1}{n}}\psi_2^{-t}Z(((1-\lambda)s)^{\frac{1}{n}}T^n). \end{aligned}$$

□

3 The facet vector

First, we show that the facet vector is a simple valuation on $\mathcal{P}^n$.

Lemma 3.1. *Let $\zeta \in \mathcal{C}$. Then, the facet vector $M_\zeta^{0,1} : \mathcal{P}^n \to \mathbb{R}^n$ is a simple valuation.*

Proof. In order to prove that $M_\zeta^{0,1}$ is a valuation, we need to show that

$$M_\zeta^{0,1}(P \cup Q) + M_\zeta^{0,1}(P \cap Q) = M_\zeta^{0,1}(P) + M_\zeta^{0,1}(Q) \tag{3.1}$$

for all $P, Q \in \mathcal{P}^n$ with $P \cup Q \in \mathcal{P}^n$. We distinguish three sets of unit vectors:

$$\begin{aligned} I_1 &:= \{u \in S^{n-1} : h_P(u) < h_Q(u)\}, \\ I_2 &:= \{u \in S^{n-1} : h_P(u) = h_Q(u)\}, \\ I_3 &:= \{u \in S^{n-1} : h_P(u) > h_Q(u)\}. \end{aligned}$$

Note that the sets I_1, I_3 are both open and that $h_{P\cup Q} = \max\{h_P, h_Q\}$ and $h_{P\cap Q} = \min\{h_P, h_Q\}$ if $P \cup Q$ is convex. Recall that $a_P(u)$ is the $(n-1)$-dimensional volume of $F(P,u)$. Then,

$$V(P,u) = \frac{1}{n} a_P(u) h_P(u).$$

For $u \in I_1$, we have

$$F(P\cup Q, u) = F(Q,u), \quad F(P\cap Q, u) = F(P,u).$$

Hence,

$$a_{P\cup Q}(u) = a_Q(u),\ h_{P\cup Q}(u) = h_Q(u),\ a_{P\cap Q}(u) = a_P(u),\ h_{P\cap Q}(u) = h_P(u).$$

Thus,

$$V(P\cup Q,u) = V(Q,u) \text{ and } V(P\cap Q) = V(P,u), \quad \text{for } u \in I_1.$$

Analogous for I_3. Note that

$$\begin{aligned}
(\mathcal{N}(P\cup Q) \setminus \mathcal{N}_o(P\cup Q)) \cap I_1 &= (\mathcal{N}(Q) \setminus \mathcal{N}_o(Q)) \cap I_1,\\
(\mathcal{N}(P\cap Q) \setminus \mathcal{N}_o(P\cap Q)) \cap I_1 &= (\mathcal{N}(P) \setminus \mathcal{N}_o(P)) \cap I_1,\\
(\mathcal{N}(P\cup Q) \setminus \mathcal{N}_o(P\cup Q)) \cap I_3 &= (\mathcal{N}(P) \setminus \mathcal{N}_o(P)) \cap I_3,\\
(\mathcal{N}(P\cap Q) \setminus \mathcal{N}_o(P\cap Q)) \cap I_3 &= (\mathcal{N}(Q) \setminus \mathcal{N}_o(Q)) \cap I_3.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sum_{u\in(\mathcal{N}(P\cup Q)\setminus\mathcal{N}_o(P\cup Q))\cap I_1} \frac{\zeta(V(P\cup Q,u))}{h_{P\cup Q}(u)}u + \sum_{u\in(\mathcal{N}(P\cap Q)\setminus\mathcal{N}_o(P\cap Q))\cap I_1} \frac{\zeta(V(P\cap Q,u))}{h_{P\cap Q}(u)}u\\
&+ \sum_{u\in(\mathcal{N}(P\cup Q)\setminus\mathcal{N}_o(P\cup Q))\cap I_3} \frac{\zeta(V(P\cup Q,u))}{h_{P\cup Q}(u)}u + \sum_{u\in(\mathcal{N}(P\cap Q)\setminus\mathcal{N}_o(P\cap Q))\cap I_3} \frac{\zeta(V(P\cap Q,u))}{h_{P\cap Q}(u)}u\\
&= \sum_{u\in(\mathcal{N}(Q)\setminus\mathcal{N}_o(Q))\cap I_1} \frac{\zeta(V(Q,u))}{h_Q(u)}u + \sum_{u\in(\mathcal{N}(P)\setminus\mathcal{N}_o(P))\cap I_1} \frac{\zeta(V(P,u))}{h_P(u)}u\\
&+ \sum_{u\in(\mathcal{N}(P)\setminus\mathcal{N}_o(P))\cap I_3} \frac{\zeta(V(P,u))}{h_P(u)}u + \sum_{u\in(\mathcal{N}(Q)\setminus\mathcal{N}_o(Q))\cap I_3} \frac{\zeta(V(Q,u))}{h_Q(u)}u.
\end{aligned}$$

It follows that (3.1) is equivalent to

$$\begin{aligned}
&\sum_{u\in(\mathcal{N}(P\cup Q)\setminus\mathcal{N}_o(P\cup Q))\cap I_2} \frac{\zeta(V(P\cup Q,u))}{h_{P\cup Q}(u)}u + \sum_{u\in(\mathcal{N}(P\cap Q)\setminus\mathcal{N}_o(P\cap Q))\cap I_2} \frac{\zeta(V(P\cap Q,u))}{h_{P\cap Q}(u)}u\\
&= \sum_{u\in(\mathcal{N}(P)\setminus\mathcal{N}_o(P))\cap I_2} \frac{\zeta(V(P,u))}{h_P(u)}u + \sum_{u\in(\mathcal{N}(Q)\setminus\mathcal{N}_o(Q))\cap I_2} \frac{\zeta(V(Q,u))}{h_Q(u)}u.
\end{aligned} \tag{3.2}$$

Fix $u \in S^{n-1}$. Since for $P \in \mathcal{P}^n$, $P \mapsto a_P(u)$ is a valuation, we have

$$a_{P\cup Q}(u) + a_{P\cap Q}(u) = a_P(u) + a_Q(u)$$

for all $P, Q \in \mathcal{P}^n$ with $P \cup Q \in \mathcal{P}^n$. Note that

$$h_{P\cup Q}(u) = h_{P\cap Q}(u) = h_P(u) = h_Q(u)$$

for $u \in I_2$. Then,

$$V(P \cup Q, u) + V(P \cap Q, u) = V(P, u) + V(Q, u)$$

for $u \in I_2$. Since ζ is a solution of Cauchy's functional equation, we obtain

$$\frac{\zeta(V(P\cup Q,u))}{h_{P\cup Q}(u)} + \frac{\zeta(V(P\cap Q,u))}{h_{P\cap Q}(u)} = \frac{\zeta(V(P,u))}{h_P(u)} + \frac{\zeta(V(Q,u))}{h_Q(u)} \tag{3.3}$$

for $u \in I_2$, where $P, Q \in \mathcal{P}^n$ with $P \cup Q \in \mathcal{P}^n$. Also, note that

$$\begin{aligned} &(\mathcal{N}(P\cup Q) \setminus \mathcal{N}_o(P\cup Q)) \cap I_2 = (\mathcal{N}(P\cap Q) \setminus \mathcal{N}_o(P\cap Q)) \cap I_2 \\ =&(\mathcal{N}(P) \setminus \mathcal{N}_o(P)) \cap I_2 = (\mathcal{N}(Q) \setminus \mathcal{N}_o(Q)) \cap I_2. \end{aligned} \tag{3.4}$$

Combined with (3.3) and (3.4), we obtain (3.2), and therefore the desired valuation property.

Next, we will show that the facet vector operator vanishes in the following two cases.

If $\dim P \le n-2$, it is clear that $M_\zeta^{0,1}(P) = 0$ as $\mathcal{N}(P) = \varnothing$.

If $\dim P = n-1$, then $h_P(u) = -h_P(-u)$, where $u, -u$ are the outer unit normals of P. By the definition of the facet vector, we obtain $M_\zeta^{0,1}(P) = 0$. □

Next, we prove the $\mathrm{SL}(n)$ contravariance of the facet vector.

Lemma 3.2. *Let $\zeta \in \mathcal{C}$. Then, the facet vector operator $M_\zeta^{0,1} : \mathcal{P}^n \to \mathbb{R}^n$ is $\mathrm{SL}(n)$ contravariant.*

Proof. Let $\phi \in \mathrm{SL}(n)$. Note that

$$u \in \mathcal{N}(P) \setminus \mathcal{N}_o(P) \quad \Leftrightarrow \quad \tilde{u} \in \mathcal{N}(\phi P) \setminus \mathcal{N}_o(\phi P) \tag{3.5}$$

with

$$\tilde{u} := \left\|\phi^{-t}u\right\|^{-1} \phi^{-t}u$$

and that

$$h_{\phi P}(\tilde{u}) = h_P(\phi^t \tilde{u}) = \left\|\phi^{-t}u\right\|^{-1} h_P(u),\ a_{\phi P}(\tilde{u}) = \left\|\phi^{-t}u\right\| a_P(u).$$

We have

$$V(\phi P, \tilde{u}) = V(P, u). \tag{3.6}$$

Applying (3.5), (3.6) and the definition of the facet vector, we obtain

$$\begin{aligned} M_\zeta^{0,1}(\phi P) &= \sum_{\tilde u\in\mathcal{N}(\phi P)\setminus\mathcal{N}_o(\phi P)} \frac{\zeta(V(\phi P,\tilde u))}{h_{\phi P}(\tilde u)}\tilde u \\ &= \sum_{u\in\mathcal{N}(P)\setminus\mathcal{N}_o(P)} \frac{\zeta(V(P,u))}{\|\phi^{-t}u\|^{-1}h_P(u)}(\|\phi^{-t}u\|^{-1}\phi^{-t}u) \\ &= \sum_{u\in\mathcal{N}(P)\setminus\mathcal{N}_o(P)} \frac{\zeta(V(P,u))}{h_P(u)}\phi^{-t}u \\ &= \phi^{-t}M_\zeta^{0,1}(P). \end{aligned}$$

Thus, we have finished the proof of the $\mathrm{SL}(n)$ contravariance of the facet vector. □

Finally, the facet vector is related to an $\mathrm{SL}(2)$ covariant valuation in dimension two up to a rotation. Let $\zeta\in\mathcal{C}$. Define $B_\zeta:\mathcal{P}_o^2\to\mathbb{R}^2$ by

$$B_\zeta(P)=\sum_{i=2}^r\frac{\zeta\left(\det(v_{i-1},v_i)\right)}{\det(v_{i-1},v_i)}(v_{i-1}-v_i)$$

if $\dim P=2$ and $P=[o,v_1,\dots,v_r]$ with $o\in\mathrm{bd}\,P$ and the vertices $\{o,v_1,\dots,v_r\}$ are labeled counter-clockwisely;

$$B_\zeta(P)=\frac{\zeta\left(\det(v_r,v_1)\right)}{\det(v_r,v_1)}(v_r-v_1)+\sum_{i=2}^r\frac{\zeta\left(\det(v_{i-1},v_i)\right)}{\det(v_{i-1},v_i)}(v_{i-1}-v_i)$$

if $o\in\mathrm{int}\,P$ and $P=[v_1,\dots,v_r]$ with the vertices $\{v_1,\dots,v_r\}$ are labeled counter-clockwisely;

$$B_\zeta(P)=o$$

if $P=\{o\}$ or P is a line segment. We remark that if v_1,v_2 are vertices of $P\in\mathcal{P}_o^2$ in counter-clockwise order, then $\det(v_1,v_2)>0$. Indeed, there exist $r_1,r_2>0$ and $\theta_1,\theta_2\in[0,2\pi)$ with $0<\theta_2-\theta_1<\pi$, such that $v_1=r_1(\cos\theta_1,\sin\theta_1)^t$ and $v_2=r_2(\cos\theta_2,\sin\theta_2)^t$. Thus, $\det(v_1,v_2)=r_1r_2\sin(\theta_2-\theta_1)>0$.

Lemma 3.3. *Let $\zeta\in\mathcal{C}$. Then*

$$M_\zeta^{0,1}(P)=\frac{1}{2}\rho_{\frac{\pi}{2}}B_\zeta(P),$$

for all $P\in\mathcal{P}_o^2$.

Proof. For $\dim P=2$ and $P=[o,v_1,\dots,v_r]$ with $o\in\mathrm{bd}\,P$ and the vertices $\{0,v_1,\dots,v_r\}$ are labeled counter-clockwisely, we have

$$B_\zeta(P)=\sum_{i=2}^r\frac{\zeta\left(\det(v_{i-1},v_i)\right)}{\det(v_{i-1},v_i)}(v_{i-1}-v_i)=\sum_{i=2}^r\frac{\zeta\left(2V([o,v_{i-1},v_i])\right)}{2V([o,v_{i-1},v_i])}(v_{i-1}-v_i).$$

Write $u_i = \frac{\rho_{\frac{\pi}{2}}(v_{i-1}-v_i)}{\|v_{i-1}-v_i\|}$. Then, u_i is the outer unit normal of $[v_{i-1}, v_i]$ and $[o, v_{i-1}, v_i]$ is the cone $[o, F(P, u_i)]$. Therefore,

$$\begin{aligned}
\rho_{\frac{\pi}{2}} B_\zeta(P) &= \sum_{i=2}^{r} \frac{\zeta\left(2V(P,u_i)\right)}{2V(P,u_i)} \|v_{i-1} - v_i\| \, u_i \\
&= 2\sum_{i=2}^{r} \frac{\zeta\left(V(P,u_i)\right)}{\|v_{i-1} - v_i\| \, h_P(u_i)} \|v_{i-1} - v_i\| \, u_i \\
&= 2 \sum_{u\in\mathcal{N}(P)\setminus\mathcal{N}_o(P)} \frac{\zeta(V(P,u))}{h_P(u)} u \\
&= 2M_\zeta^{0,1}(P).
\end{aligned}$$

Similar arguments also prove other cases. □

4 Proof of the main results on $\mathcal{P}_o^n$

4.1 The two-dimensional case

First, we show a relation between SL(2) covariant functions and SL(2) contravariant functions. Let $\mathcal{Q}^2$ be either $\mathcal{P}_o^2$ or $\mathcal{P}^2$.

Lemma 4.1. *Let $Z : \mathcal{Q}^2 \to \mathbb{R}^2$. Then, Z is* SL(2) *covariant if and only if $\rho_{\frac{\pi}{2}} Z$ is* SL(2) *contravariant.*

Proof. A direct calculation shows that $\rho_{\frac{\pi}{2}}\phi = \phi^{-t}\rho_{\frac{\pi}{2}}$ for all $\phi \in \mathrm{SL}(2)$, which implies the Lemma. □

We will use the following result.

Theorem 4.2 (Zeng & Ma [34])**.** *A function $Z : \mathcal{P}_o^2 \to \mathbb{R}^2$ is an* SL(2) *covariant valuation if and only if there exist constants $c_1, c_2 \in \mathbb{R}$ and $\zeta \in \mathcal{C}$ such that*

$$Z(P) = c_1 M^{1,0}(P) + c_2 A(P) + B_\zeta(P)$$

for every $P \in \mathcal{P}_o^2$.

Now, Theorem 1.2 follows immediately from Lemma 4.1, Theorem 4.2 and Lemma 3.3.

4.2 The higher-dimensional case

First, we state the following simple proposition.

Proposition 4.3. *Let $n \geq 3$ and $Z : \mathcal{P}_o^n \to \mathbb{R}^n$ be an* SL(n) *contravariant function. Then, there exists a function $f : [0, \infty) \to \mathbb{R}$ such that $Z(sT^n) = f(s)\mathbf{1}$, for $s \geq 0$.*

Proof. Let $s \geq 0$. We first consider $n = 3$. Write $Z(sT^3) = (x_1, x_2, x_3)^t$ and

$$\sigma_0 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \mathrm{SL}(3).$$

The SL(3) contravariance of Z implies

$$Z(sT^3) = Z(\sigma_0 sT^3) = \sigma_0^{-t} Z(sT^3),$$

i.e.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_1 \\ x_2 \end{pmatrix}.$$

Thus, $x_1 = x_2 = x_3$.

Next, we consider $n \geq 4$. Write $Z(sT^n) = (x_1, \dots, x_n)^t$ and

$$\sigma = \begin{pmatrix} I_r & & \\ & \sigma_0 & \\ & & I_{n-r-3} \end{pmatrix} \in \mathrm{SL}(n),$$

where $r = 0, 1, \dots, n-3$ and σ_0 moves along the main diagonal of σ. Using the SL(n) contravariance of Z, we have $Z(sT^n) = Z(\sigma sT^n) = \sigma^{-t} Z(sT^n)$. This yields $x_1 = \cdots = x_n$. Therefore, there exists a function $f : [0, \infty) \to \mathbb{R}$ such that $Z(sT^n) = f(s)\mathbf{1}$, since the coordinates depend on s. $\square$

Next, we obtain a sufficient condition for a valuation becoming a simple valuation.

Lemma 4.4. *Let $n \geq 2$ and $Z : \mathcal{P}_o^n \to \mathbb{R}^n$ be an* SL(n) *contravariant valuation. Then, Z is simple if $Z(T^k) = 0$ for $k = 0, 1, \dots, n-1$.*

Proof. First, using triangulations of polytopes, it suffices to prove Z vanishes on $\mathcal{T}^k$ for $k = 0, 1, \dots, n-1$. Since every $T \in \mathcal{T}^k$ is an SL(n) image of sT^k for $s \neq 0$, we only need to consider sT^k. Now, write

$$\rho = \begin{pmatrix} sI_k & & \\ & I_{n-k-1} & \\ & & s^{-k} \end{pmatrix} \in \mathrm{SL}(n).$$

The SL(n) contravariance of Z gives $Z(sT^k) = Z(\rho T^k) = \rho^{-t} Z(T^k)$. By the assumption that $Z(T^k) = 0$ for $k = 0, 1, \dots, n-1$, we obtain that Z vanishes on all sT^k for $s \neq 0$ and $k = 0, 1, \dots, n-1$. Therefore, Z is simple. $\square$

Now, we investigate SL(n) contravariant valuations on $\mathcal{T}^k$.

Lemma 4.5. *Let $n \geq 3$ and $Z : \mathcal{P}_o^n \to \mathbb{R}^n$ be an* $\mathrm{SL}(n)$ *contravariant valuation. Then, Z is simple.*

Proof. Due to Lemma 4.4, it suffices to prove Z vanishes on T^k for $k = 0, 1, \ldots, n-1$. We prove the statement by induction on the dimension k.

For $k = 0$, write $Z(\{o\}) = (v_1, \ldots, v_n)^t$,

$$\sigma_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \sigma_2 = \begin{pmatrix} I_r & & \\ & \sigma_1 & \\ & & I_{n-r-2} \end{pmatrix} \in \mathrm{SL}(n),$$

where $r = 0, 1, \ldots, n-2$. Using the $\mathrm{SL}(n)$ contravariance of Z, we have $Z(\{o\}) = Z(\sigma_2 \{o\}) = \sigma_2^{-t} Z(\{o\})$. Hence, $v_1 = \cdots = v_n = 0$.

For $k = 1$, write $Z(T^1) = (w_1, \ldots, w_n)^t$ and

$$\sigma_3 = \begin{pmatrix} I_r & & \\ & \sigma_1 & \\ & & I_{n-r-2} \end{pmatrix} \in \mathrm{SL}(n),$$

where $r = 1, \ldots, n-2$. Using the $\mathrm{SL}(n)$ contravariance of Z, we have $Z(T^1) = Z(\sigma_3 T^1) = \sigma_3^{-t} Z(T^1)$. Thus, $w_2 = \cdots = w_n = 0$ and $Z(T^1) = w_1 e_1$.

For $k = 2$, write $Z(T^2) = (x_1, \ldots, x_n)^t$. If $n = 3$, we consider

$$\sigma_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathrm{SL}(3).$$

The $\mathrm{SL}(3)$ contravariance of Z implies $Z(T^2) = Z(\sigma_4 T^2) = \sigma_4^{-t} Z(T^2)$. Thus, $x_1 = x_2$ and $x_3 = 0$. If $n \geq 4$, we consider

$$\sigma_5 = \begin{pmatrix} \sigma_4 & 0 \\ 0 & I_{n-3} \end{pmatrix} \in \mathrm{SL}(n) \text{ and } \sigma_6 = \begin{pmatrix} I_r & & \\ & \sigma_1 & \\ & & I_{n-r-2} \end{pmatrix} \in \mathrm{SL}(n),$$

where $r = 2, \ldots, n-2$. By the $\mathrm{SL}(n)$ contravariance of Z, we have $Z(T^2) = Z(\sigma_5 T^2) = \sigma_5^{-t} Z(T^2)$ and $Z(T^2) = Z(\sigma_6 T^2) = \sigma_6^{-t} Z(T^2)$. Thus, $x_1 = x_2$, $x_3 = \cdots = x_n = 0$ and $Z(T^2) = x_1(e_1 + e_2)$. Now, we use the triangulation in Definition 1. Equation (2.1) is equivalent to

$$\begin{pmatrix} \frac{1}{\lambda} & -\frac{1-\lambda}{\lambda} & 0 & \cdots & 0 \\ -\frac{\lambda}{1-\lambda} & \frac{1}{1-\lambda} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda} & -\frac{1-\lambda}{\lambda} & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix} \begin{pmatrix} w_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This yields $x_1 = w_1 = 0$. Therefore, Z vanishes on T^1 and T^2.

Next, assume $Z(T^{k-1}) = 0$ for $3 \le k \le n-1$. Write $Z(T^k) = (y_1, \dots, y_n)^t$ and

$$\sigma_7 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & I_{n-3} & \\ & & & -1 \end{pmatrix} \in \mathrm{SL}(n).$$

By the $\mathrm{SL}(n)$ contravariance of Z, we have $Z(T^k) = Z(\sigma_7 T^k) = \sigma_7^{-t} Z(T^k)$. Thus $y_1 = y_2$.

Finally, we use the triangulation in Definition 1. Equation (2.1) is equivalent to

$$\begin{pmatrix} \frac{1}{\lambda} & -\frac{1-\lambda}{\lambda} & 0 & \cdots & 0 \\ -\frac{\lambda}{1-\lambda} & \frac{1}{1-\lambda} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Together with $y_1 = y_2$, this yields $y_1 = \cdots = y_n = 0$. Therefore, $Z(T^k) = 0$, which completes the proof. $\square$

Finally, we obtain the following classification.

Proof of Theorem 1.1. Let $\zeta \in \mathcal{C}$. Due to Lemmas 3.1 and 3.2, $M_\zeta^{0,1}$ is an $\mathrm{SL}(n)$ contravariant valuation on $\mathcal{P}_o^n$. It remains to show the reverse statement.

We use the triangulation in Definition 2. By (2.2) and Lemma 4.5, we have for $s > 0$

$$Z(s^{\frac{1}{n}} T^n) = \lambda^{\frac{1}{n}} \phi_2^{-t} Z((\lambda s)^{\frac{1}{n}} T^n) + (1-\lambda)^{\frac{1}{n}} \psi_2^{-t} Z(((1-\lambda)s)^{\frac{1}{n}} T^n).$$

By Proposition 4.3, there exists a function $f : [0, \infty) \to \mathbb{R}$ such that $Z(sT^n) = f(s)\mathbf{1}$ and

$$f(s^{\frac{1}{n}})\mathbf{1} = \lambda^{\frac{1}{n}} \phi_2^{-t} f((\lambda s)^{\frac{1}{n}})\mathbf{1} + (1-\lambda)^{\frac{1}{n}} \psi_2^{-t} f(((1-\lambda)s)^{\frac{1}{n}})\mathbf{1}.$$

In other words,

$$f(s^{\frac{1}{n}}) = \lambda^{\frac{1}{n}} f((\lambda s)^{\frac{1}{n}}) + (1-\lambda)^{\frac{1}{n}} f(((1-\lambda)s)^{\frac{1}{n}}).$$

Set $s = a + b$, $\lambda = a/(a+b)$ for $a, b > 0$, and $g(x) = x^{\frac{1}{n}} f(x^{\frac{1}{n}})$ for $x > 0$ to get

$$g(a+b) = g(a) + g(b).$$

Hence, g is a solution of Cauchy's functional equation and

$$Z(s^{\frac{1}{n}} T^n) = \frac{g(s)}{s^{\frac{1}{n}}} \mathbf{1}.$$

Setting $\zeta(s) = g(n!s)$, we obtain $Z(s^{\frac{1}{n}} T^n) = M_\zeta^{0,1}(s^{\frac{1}{n}} T^n)$. The proof is now completed by Lemma 2.2. $\square$

5 Proof of the main results on $\mathcal{P}^n$

5.1 The two-dimensional case

First, we treat the case for $\mathcal{P}^2$. We need the following result.

Theorem 5.1 (Zeng & Ma [34]). *A function $Z : \mathcal{P}^2 \to \mathbb{R}^2$ is an $\mathrm{SL}(2)$ covariant valuation if and only if there exist constants $c_1, c_2, \tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$ and $\zeta_1, \zeta_2 \in \mathcal{C}$ such that*

$$Z(P) = B_{\zeta_1}([o,P]) + \sum_{i=2}^{r} B_{\zeta_2}([o, v_{i-1}, v_i]) \\ + c_1 M^{1,0}(P) + \tilde{c}_1 M^{1,0}([o,P]) + c_2 A([o,P]) + \tilde{c}_2 A([o, v_1, \dots, v_r])$$

for every polytope $P \in \mathcal{P}^2$ with vertices $v_1, \dots, v_r$ visible from the origin and labeled counter-clockwisely.

Now, similar to the proof of Theorem 1.2, we obtain the characterization in dimension two.

Proof of Theorem 1.4. By Lemma 4.1, Theorem 5.1 and Lemma 3.3, there exist constants $c_1, c_2, \tilde{c}_1, \tilde{c}_2 \in \mathbb{R}$ and $\bar{\zeta}_1, \bar{\zeta}_2 \in \mathcal{C}$ such that

$$Z(P) = M^{0,1}_{2\bar{\zeta}_1}([o,P]) + \sum_{i=2}^{r} M^{0,1}_{2\bar{\zeta}_2}([o, v_{i-1}, v_i]) \\ + c_1 \rho_{\frac{\pi}{2}} M^{1,0}(P) + \tilde{c}_1 \rho_{\frac{\pi}{2}} M^{1,0}([o,P]) + c_2 \rho_{\frac{\pi}{2}} A([o,P]) + \tilde{c}_2 \rho_{\frac{\pi}{2}} A([o, v_1, \dots, v_r]).$$

Then, Lemma 3.1 yields

$$\sum_{i=2}^{r} M^{0,1}_{2\bar{\zeta}_2}([o, v_{i-1}, v_i]) = M^{0,1}_{2\bar{\zeta}_2}([o,P]) - M^{0,1}_{2\bar{\zeta}_2}(P).$$

Finally, we set $\zeta_1 = -2\bar{\zeta}_2$ and $\zeta_2 = 2(\bar{\zeta}_1 + \bar{\zeta}_2)$ to conclude the proof. $\square$

5.2 The higher-dimensional case

In the final step, we extend Theorem 1.1 to $\mathcal{P}^n$.

Proof of Theorem 1.3. Let $\zeta_1, \zeta_2 \in \mathcal{C}$. First, due to Lemmas 3.1 and 3.2, $M^{0,1}_{\zeta_1}$ is an $\mathrm{SL}(n)$ contravariant valuation on $\mathcal{P}^n$. Next, for $P, Q \in \mathcal{P}^n$ with $P \cup Q \in \mathcal{P}^n$, we have $[o, P \cup Q] = [o,P] \cup [o,Q]$ and $[o, P \cap Q] = [o,P] \cap [o,Q]$. Notice that $[o, \phi P] = \phi\,[o,P]$ for all $\phi \in \mathrm{SL}(n)$ and $P \in \mathcal{P}^n$. Again by Lemmas 3.1 and 3.2, we obtain that the function $P \mapsto M^{0,1}_{\zeta_2}([o,P])$ for $P \in \mathcal{P}^n$ is also an $\mathrm{SL}(n)$ contravariant valuation on $\mathcal{P}^n$.

It remains to show the reverse statement. Indeed, we only need to show that Z has the corresponding representation on sT^k and $s\tilde{T}^k$ for $s > 0$ and $0 \leq k \leq n$. By Theorem 1.1, there exists $\eta_1 \in \mathcal{C}$ such that

$$Z(sT^k) = M^{0,1}_{\eta_1}(sT^k).$$

Let $\mathcal{T}^n_o$ be the set of simplices in $\mathbb{R}^n$ with one vertex at the origin. For any $T \in \mathcal{T}^n_o \setminus \{o\}$, we write $\tilde{T}$ as its facet opposite to the origin. We define the new map $\tilde{Z} : \mathcal{T}^n_o \to \mathbb{R}$ by $\tilde{Z}(T) = Z(\tilde{T})$ for every $T \in \mathcal{T}^n_o \setminus \{o\}$ and $\tilde{Z}\{o\} = o$. It is not hard to check that $\tilde{Z}$ is an $\mathrm{SL}(n)$ contravariant valuation on $\mathcal{T}^n_o$. From the proof of Theorem 1.1, one can see that Theorem 1.1 also holds on $\mathcal{T}^n_o$. Hence there exists $\eta_2 \in \mathcal{C}$ such that

$$Z(s\tilde{T}^k) = \tilde{Z}(sT^k) = M^{0,1}_{\eta_2}(sT^k).$$

Now, we set $\zeta_1 = \eta_1 - \eta_2$ and $\zeta_2 = \eta_2$ such that (1.2) holds for both sT^k and $s\tilde{T}^k$ for $0 \leq k \leq n$, which completes the proof by Lemma 2.3. □

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