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Convergence in time-periodic quasilinear parabolic equations in one space dimension ☆

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Abstract

We consider time-periodic quasilinear parabolic equations in the domain $\{(t, x) \in \mathbb{R}^2 \mid 0 < x < r(t), t > 0\}$, where the right boundary $r(t)$ of the spatial interval is a positive function which might be periodic, or asymptotically periodic, or a function tending to infinity, or infinity. We show that, in the first case (that is, $r(t)$ is a periodic function), any bounded solution of the equations converges as $t \to \infty$ to a periodic one; in the other three cases, any positive bounded solution converges as $t \to \infty$ to a nonnegative periodic one. Using such a result, we study the long time dynamics of the initial-boundary value problem on the half line, as well as the Stefan free boundary problem, of a general heterogeneous reaction–diffusion equation. Also, we use the convergence result to study the long time dynamics of the initial-boundary value problem for a time-periodic (mean) curvature flow equation.

MSC: 35K55; 35K20; 35B40; 35R35

Keywords: Nonlinear parabolic equation; Time-periodic solution; Initial-boundary value problem; Asymptotic behavior; Free boundary problem

1. Introduction

In this paper we first consider the convergence of bounded solutions of the following quasilinear parabolic equation in one space dimension

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$$u_t = d(t, x, u, u_x)u_{xx} + f(t, x, u, u_x), \quad x \in (0, r(t)),\ t > 0, \tag{1.1}$$

where $d, f \in C^1$ with $d > 0$ are periodic in t with a common period T, the right boundary $r(t)$ of the spatial interval is a positive function (see details below). We adopt the homogeneous Dirichlet boundary conditions

$$u(t, 0) = 0, \quad u(t, r(t)) = 0, \quad t > 0. \tag{1.2}$$

When $r(t) = \infty$, the second boundary condition is neglected automatically. So our problem to be studied is (1.1)–(1.2).

We will present some convergence results for bounded solutions. Such kind of results are of special importance in the study of qualitative properties of the solutions. Let us recall some known results in this field for equations in one space dimension.

1. *Autonomous equations in a fixed bounded interval*. When d, f are independent of t, the equation is an autonomous one. If, in addition, $r(t)$ is a positive constant, then the problem is a rather simpler one which has been well studied. For example, Zelenyak [34] (see also Matano [23,25]) proved that any bounded solution of such a simple problem converges as $t \to \infty$ to a stationary one.

2. *Autonomous reaction–diffusion equations in unbounded intervals*. Consider

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R},\ t > 0, \tag{1.3}$$

with f locally Lipschitz and $f(0) = 0$. Du and Matano [16] proved that any bounded nonnegative solution starting from a compactly supported initial data converges to a stationary one. Chen et al. [8] proved a similar result for the problem on the half line $[0, \infty)$ (though they considered only the equations with bistable type of nonlinearity, their argument applies to general equations like (1.3)). When the initial data is not a compactly supported one, say, it is in $C_0(\mathbb{R})$ (the space of continuous functions with limits at $\pm\infty$ equaling to 0), Matano and Polacik [27] obtained similar convergence result very recently.

3. *Time periodic equations in fixed bounded interval*. Chen and Matano [10] considered the equation $u_t = u_{xx} + f(t, u)$ with f being periodic in t in a fixed interval $(0, \bar{r})$, and proved that any bounded solution of this equation with Dirichlet, Neumann or periodic boundary conditions converges to a periodic one. Later Brunovsky et al. [4] extended such a result to the general equation (1.1).

4. *Reaction–diffusion equations in variable intervals*. Variable intervals appear in some applied fields, to say, in the free boundary problems. Du and Lin [14] and Du and Lou [15] studied the equation (1.3) in the interval $(0, r(t))$ with r satisfying the Stefan free boundary conditions:

$$u(t, r(t)) = 0, \quad r'(t) = -u_x(t, r(t)),\ \ t > 0. \tag{1.4}$$

They used such a problem to model the spreading of a new or invasive species, with u denoting the population density and the free boundary $r(t)$ representing the (monotone increasing) expanding front of the species. Du and Lou [15] proved that, for general $f(u)$ satisfying $f(0) = 0$, any bounded positive solution of this problem converges to a stationary one. In the last few years, using the free boundary condition as in (1.4), some authors studied various reaction–diffusion equations (in advective or spatially heterogeneous environments) (cf. [13,18,19,31]), or time periodic reaction–diffusion equations (cf. [12,29,32]). Moreover, Bao et al. [3] and Cai et al. [5,6] also considered the equation (1.3) with other types of free boundary conditions like

$$u(t,r(t))=0, \quad r'(t)=-u_x(t,r(t))+\alpha_0, \ \ t>0,$$

where α_0 may be a sign-changing function to reflect the change of the climate near the spreading front. Thus $r'(t)$ may change sign. Any of these free boundary problems involves a variable boundary. Many of the above papers also gave the convergence for bounded positive solutions, but only for reaction–diffusion equations.

Clearly, our problem (1.1)–(1.2) includes most of the above problems as special cases.

Before stating our main result, we present some hypotheses and notations. Denote $Q:=\mathbb{R}\times[0,\infty)\times\mathbb{R}^2$, for the functions $d(t,x,u,p)$ and $f(t,x,u,p)$ we assume

$$\begin{cases} d,f\in C^1(Q) \text{ if } d_p\equiv 0, \text{ or, } d,f\in C^2(Q) \text{ if } d_p\not\equiv 0; \ \text{ both are } T\text{-periodic in } t; \\ \text{there exists a positive continuous function } K(u) \text{ such that} \\ |d,\ d^{-1},\ f,\ d_t,\ d_x,\ d_u,\ d_p,\ f_t,\ f_x,\ f_u,\ f_p|\le K(u) \text{ for } (t,x,u,p)\in Q. \end{cases} \tag{H}$$

For the right boundary $r(t)$ of the spatial interval, we take it from the set $\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3\cup\mathcal{B}_4$, where

$$\begin{aligned} \mathcal{B}_1 &:= \{r\in C^{1+\alpha/2}_{loc}(\mathbb{R}) \mid r(t)\equiv r(t+T)\}, \quad \text{for some } \alpha\in(0,1); \\ \mathcal{B}_2 &:= \{r\in C^{1+\alpha/2}_{loc}(\mathbb{R}) \mid r(t)\not\equiv r(t+T) \text{ for } t\ge 0, \\ &\qquad |r(t)-\bar r(t)|\to 0 \text{ as } t\to\infty \text{ for some } \bar r(t)\in\mathcal{B}_1\}; \\ \mathcal{B}_3 &:= \{r\in C(\mathbb{R}) \mid r(t+T)>r(t)>0 \text{ for } t\ge 0, \text{ and } r(t)\to\infty \text{ as } t\to\infty\}; \\ \mathcal{B}_4 &:= \{r(t)\equiv\infty\}. \end{aligned}$$

We always use $\bar r(t)$ to denote the limit of $r(t)$. Then $\bar r(t)=r(t)$ in case $r\in\mathcal{B}_1$, $\bar r(t)=\infty$ in case $r\in\mathcal{B}_3\cup\mathcal{B}_4$. We choose initial data $u(0,x)=u_0(x)\in C([0,\infty))$ such that it has a compact support $\mathrm{spt}(u_0)\subset[0,r(0)]$, and say that u is a classical solution in $(0,\tau)$ for $\tau\in(0,\infty]$ if, with $\Omega(\tau):=\{(t,x)\mid x\in[0,r(t)], t\in(0,\tau)\}$,

$$u(t,x)\in C^{1+\alpha/2,2+\alpha}_{loc}\big(\Omega(\tau)\big)\cap C\big(\overline{\Omega(\tau)}\big) \quad \text{for the constant } \alpha \text{ in } \mathcal{B}_1, \tag{1.5}$$

and if u satisfies (1.1)–(1.2). It is called a bounded time global solution if $\tau=\infty$ and if $|u(t,x)|\le C$ for all $0<x<r(t)$, $t>0$ and some $C>0$. By [22, Chapter VI, Theorem 5.2] we know that, under certain structural conditions (including some conditions in (*H*)), the problem (1.1)–(1.2) does have time global classical solutions (not necessarily to be bounded). We will study the convergence of bounded time global solutions of (1.1)–(1.2) to time-periodic ones, that is, to solutions of the following problem

$$\begin{cases} P_t=d(t,x,P,P_x)P_{xx}+f(t,x,P,P_x), & 0<x<\bar r(t),\ t\in\mathbb{R}, \\ P(t,0)=0,\ P(t,\bar r(t))=0, & t\in\mathbb{R}, \\ P(t,x)=P(t+T,x), & 0<x<\bar r(t),\ t\in\mathbb{R}. \end{cases} \tag{1.6}$$

(As above, the second boundary condition is neglected automatically in case $\bar r(t)=\infty$.)

Our main result is as follows.

Theorem 1.1. *Assume (H). Let* $u(t,x)$ *be a time global classical solution of* (1.1)–(1.2) *with* $u(0,x) = u_0(x) \in C([0, r(0)])$*, which is bounded:* $|u(t,x)| \le C$ *for* $0 < x < r(t)$, $t > 0$ *and some* $C > 0$.

(i). *Assume* $r \in \mathcal{B}_1$*. Then* $u(t,\cdot)$ *converges as* $t \to \infty$ *to a solution of* (1.6)*;*
(ii). *Assume* $r \in \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$*. If* $u_0 \ge, \not\equiv 0$ *has compact support in* $[0, r(0)]$ *and if* $f(t,x,0,0) \ge 0$*, then* u *converges as* $t \to \infty$ *to a nonnegative solution of* (1.6).

A direct consequence of this theorem is that when $d, f, \bar{r}$ are all independent of t, any bounded time global solution of (1.1)–(1.2) converges to a stationary one. Our main method to prove the above theorem is the so-called zero-number argument (cf. [2,9,24]), which is a powerful tool in studying equations in one space dimension (cf. [4,16,27] etc.). Our approach applies with minor modification to the problems with the following general boundary conditions at $x = 0$:

$$\gamma(t)u_x(t,0) - (1-\gamma(t))u(t,0) = g(t), \qquad \text{for } T\text{-periodic } g \text{ and } \gamma \text{ with } \gamma(t) \in [0,1],$$

as well as the radially symmetric problems in high space dimension. Moreover, the initial data can be chosen as a compactly supported continuous function or a function in the space $C_0(\mathbb{R})$ as in [27]. They are both of interest, mathematically as well as physically. Since we mainly focus on the generality of the equations in this paper, we use compactly supported initial data to avoid the extra complexity induced by $u_0 \in C_0(\mathbb{R})$.

In Section 2 we first present the known results on the zero-number properties and then use them to prove the main theorem. In Section 3 we use the main theorem to study the asymptotic behavior for the solutions of a general heterogeneous reaction–diffusion equation. More precisely, we will consider the initial-boundary value problem on the half line

$$\begin{cases} u_t = u_{xx} + g(t,x,u)[m(t,x) - u], & x > 0, t > 0, \\ u(t,0) = 0, & t > 0, \end{cases}$$

and the free boundary problem

$$\begin{cases} u_t = u_{xx} + g(t,x,u)[m(t,x) - u], & x \in (0, r(t)), t > 0, \\ u(t, r(t)) = 0, \ r'(t) = -u_x(t, r(t)), & t > 0, \end{cases}$$

with nonnegative initial data, where both g and $m > 0$ are T-periodic in t. We show that, under certain conditions, each solution of these problems is a time global classical one, and it converges as $t \to \infty$ to a nonnegative time-periodic solution. Moreover, when the initial data is large enough, spreading happens in the sense that the limit time-periodic solution takes values in the range of $m(t,x)$ near $x = \infty$. In Section 4, we study a time-dependent curvature flow equation with driving force:

$$u_t = a(t,u)\frac{u_{xx}}{1+u_x^2} + b(t,u)\sqrt{1+u_x^2}, \quad x \in (-L(t), L(t)), \ t > 0,$$

where all the coefficients are T-periodic in t. We will show that any solution starting from a suitable initial data converges to a time-periodic one.

2. Proof of the main theorem

2.1. Zero-number properties

Now we present some results on the zero-number properties which will be used below. For any continuous function $w(\cdot)$ defined in $I \subset \mathbb{R}$, denote the number of its zeros as

$$\mathcal{Z}_I[w(\cdot)] := \sup\left\{ k \in \mathbb{Z} \;\middle|\; \begin{array}{l} \text{there exist } x_1, \cdots, x_k \in I \text{ with } x_1 < x_2 < \cdots < x_k \\ \text{such that } w(x_j)w(x_{j+1}) < 0 \text{ for } j = 1, 2, \cdots, k-1 \end{array} \right\}.$$

Moreover, we say that $x_0 \in I$ is a degenerate zero of $w(x)$ if $w(x_0) = w'(x_0) = 0$. The following result is a modification of the analogue in [2].

Lemma 2.1. *Let $\xi(t)$ be a positive, continuous function in (t_1, t_2), and $\eta : (t_1, t_2) \times (0, \xi(t)) \to \mathbb{R}$ be a bounded classical solution of*

$$\eta_t = a(t,x)\eta_{xx} + b(t,x)\eta_x + c(t,x)\eta \tag{2.1}$$

with boundary conditions

$$\begin{cases} \eta(t,0) \equiv 0, \quad \text{or} \quad \eta(t,0) \neq 0 \text{ for all } t \in (t_1,t_2), \\ \eta(t,\xi(t)) \equiv 0 \text{ and } \xi \in C^1, \quad \text{or} \quad \eta(t,\xi(t)) \neq 0 \text{ for all } t \in (t_1,t_2). \end{cases}$$

Suppose also that

$$a \in C, \quad a, 1/a, a_t, a_x, b, c \in L^\infty. \tag{2.2}$$

Then for each $t \in (t_1, t_2)$, $\mathcal{Z}_{[0,\xi(t)]}[\eta(t,\cdot)] < \infty$. Moreover, $\mathcal{Z}_{[0,\xi(t)]}[\eta(t,\cdot)]$ is nonincreasing in t for $t \in (t_1, t_2)$, and if for some $t_0 \in (t_1, t_2)$ the function $\eta(t_0, \cdot)$ has a degenerate zero $x_0 \in [0, \xi(t_0)]$, then $\mathcal{Z}_{[0,\xi(s_1)]}[\eta(s_1,\cdot)] > \mathcal{Z}_{[0,\xi(s_2)]}[\eta(s_2,\cdot)]$ for all s_1, s_2 satisfying $t_1 < s_1 < t_0 < s_2 < t_2$.

Proof. If $\eta(t,\xi(t)) \neq 0$ for all $t \in (t_1, t_2)$, then at each time moment $\tilde{t} \in (t_1, t_2)$, by the continuity of η we can find $\tilde{x} < \xi(\tilde{t})$ and $\tilde{\delta} > 0$ small such that $\tilde{x} \le \xi(t)$ for $t \in \tilde{J} := [\tilde{t} - \tilde{\delta}, \tilde{t} + \tilde{\delta}]$, and $\eta(t,x) \neq 0$ for all $x \in [\tilde{x}, \xi(t)]$ and $t \in \tilde{J}$. Regarding $\eta(t,x)$ as a solution in the domain $\{(t,x) \mid x \in [0,\tilde{x}], t \in \tilde{J}\}$ (which has straight boundaries $x = 0$ and $x = \tilde{x}$), we see by the results in [2,9] that the zero-number properties as in the lemma is true in the time interval $\tilde{J}$, and so it is true in $t \in (t_1, t_2)$ since $\tilde{t} \in (t_1, t_2)$ is arbitrary. (In [2], the conditions on a, b, c are $a, 1/a, a_t, a_x, a_{xx}, b, b_t, b_x, c \in L^\infty$. These, however, were weakened in Chen [9] to be as that in (2.2), by considering $W^{1,2}_{p,loc}$ functions η satisfying $|\eta_t - \eta_{xx}| \le M_1|\eta_x| + M_0|\eta|$ for some $M_0,\ M_1 > 0$).

When the boundary condition at $x = \xi(t)$ is $\eta(t,\xi(t)) \equiv 0$, we change the variable x to be $y := x/\xi(t)$. Denote $w(t,y) := \eta(t, y\xi(t)) = \eta(t,x)$, then w satisfies

$$\begin{cases} w_t = a\xi^{-2}w_{yy} + (b + y\xi'(t))\xi^{-1}w_y + c(t,x)w, & y \in (0,1),\ t \in (t_1,t_2), \\ w(t,0) \equiv 0, \quad \text{or} \quad w(t,0) \neq 0 \text{ for all } t \in (t_1,t_2), & w(t,1) \equiv 0. \end{cases}$$

When (2.2) holds, the coefficients in this new problem again satisfy (2.2) by the extra condition $\xi \in C^1$. Now we can use the results in [2,9] as above to conclude that the zero-number properties in the lemma hold for w, and so they hold for η. □

2.2. Proof of Theorem 1.1

Under the hypotheses (H) and $|u| \leq C$, one can see that the Schauder estimate holds for u. More precisely, in case $r \in \mathcal{B}_1 \cup \mathcal{B}_2$, by straightening the boundary $x = r(t)$ and using the standard parabolic theory we can obtain the following uniform estimate:

$$\|u(\cdot + t_0, \cdot)\|_{C^{1+\alpha/2, 2+\alpha}(\Omega(t_0;T))} \leq C,$$

for all $t_0 > T$ and some $C > 0$ independent of t_0, where $\Omega(t_0;T) := \{(t,x) \mid 0 \leq x \leq r(t+t_0), t \in [-T,T]\}$; In case $r \in \mathcal{B}_3 \cup \mathcal{B}_4$ we have the uniform interior estimate:

$$\|u(\cdot + t_0, \cdot + x_0)\|_{C^{1+\alpha/2, 2+\alpha}(\Omega(t_0,x_0;T,X))} \leq C(T,X),$$

for all $t_0 > T$, $x_0 \geq 0$, $X > 0$ and some $C(T,X) > 0$ independent of t_0, x_0, where $\Omega(t_0,x_0;T,X) := \{(t,x) \mid x_0 \leq x \leq x_0 + X, t \in [-T,T]\}$.

For each $s \in (-T,T)$ we will compare $u(t-s,x)$ with its time shift $u(t-s+T,x)$. For this purpose we first determine their common spatial domain. Set

$$\hat{r}(t-s) := \min\{r(t-s), r(t-s+T)\}, \quad t \geq s. \tag{2.3}$$

In case $r \in \mathcal{B}_1$ it is clear that $\hat{r}(t-s) \equiv r(t-s)$; In case $r \in \mathcal{B}_2$, $r(t-s) < r(t-s+T)$ for all $t \geq s$, or $r(t-s) > r(t-s+T)$ for all $t \geq s$. Since the proofs for both cases are similar, for definiteness we assume the former holds, then $\hat{r}(t-s) \equiv r(t-s)$; In case $r \in \mathcal{B}_3$, we also have $\hat{r}(t-s) \equiv r(t-s)$; In case $r \in \mathcal{B}_4$, of course, $\hat{r}(t-s) \equiv \infty$.

For each $s \in (-T,T)$, define

$$\eta^s(t,x) := u(t-s+T,x) - u(t-s,x), \quad x \in [0, \hat{r}(t-s)], t \geq s.$$

Then η^s is a classical solution of

$$\begin{cases} \eta^s_t = a^s(t,x)\eta^s_{xx} + b^s(t,x)\eta^s_x + c^s(t,x)\eta^s, & x \in (0, \hat{r}(t-s)), t > s, \\ \eta^s(t,0) = 0, & t > s, \end{cases} \tag{2.4}$$

where, with $u_1 := u(t-s+T,x)$, $u_2 := u(t-s,x)$,

$$
\begin{aligned}
a^s(t,x) &= d(t-s,x,u_1,u_{1x}),\\
b^s(t,x) &= [f(t-s,x,u_1,u_{1x}) - f(t-s,x,u_1,u_{2x})]/(u_{1x}-u_{2x})\\
&\quad + [d(t-s,x,u_1,u_{1x}) - d(t-s,x,u_1,u_{2x})]u_{2xx}/(u_{1x}-u_{2x}),\\
c^s(t,x) &= [f(t-s,x,u_1,u_{2x}) - f(t-s,x,u_2,u_{2x})]/(u_1-u_2)\\
&\quad + [d(t-s,x,u_1,u_{2x}) - d(t-s,x,u_2,u_{2x})]u_{2xx}/(u_1-u_2),
\end{aligned}
$$

for $u_1 \neq u_2, u_{1x} \neq u_{2x}$, and extended continuously to the case where $u_1 = u_2$ or $u_{1x} = u_{2x}$. Note that a^s, b^s, c^s satisfies the conditions in (2.2) by the hypothesis (H) and the above uniform Schauder estimates. (Note that, when $d_p \not\equiv 0$, to ensure the term $d_p u_{1xt}$ involved in a^s_t is bounded, we use the hypothesis $d, f \in C^2$, which guarantees the boundedness of u_{1xt}.)

Claim 1. *$\eta^s(t,\cdot)$ has no degenerate zeros in $[0, \hat{r}(t-s)]$ for large t.*

In case $r \in \mathcal{B}_1$, by Lemma 2.1, $\mathcal{Z}_{[0,r(t-s)]}[\eta^s(t,\cdot)] < \infty$ for $t-s>0$ small. Using Lemma 2.1 again we see that, when t is sufficiently large, $\eta^s(t,\cdot)$ has no degenerate zeros.

In case $r \in \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$, using the conditions (H), $f(t,x,0,0) \geq 0$ and $u_0 \geq, \not\equiv 0$ in (ii), we have

$$
u(t,x) > 0 \text{ for } x \in (0, r(t)),\ \ t > 0
$$

by the maximum principle. In case $r \in \mathcal{B}_2 \cup \mathcal{B}_3$, due to $\hat{r}(t-s) = r(t-s)$ we have $\eta^s(t, \hat{r}(t-s)) = u(t-s+T, r(t-s)) > 0$ for all $t > s$. By Lemma 2.1, $\mathcal{Z}_{[0,\hat{r}(t-s)]}[\eta^s(t,\cdot)] < \infty$ for all $t > s$, and $\eta^s(t,\cdot)$ has no degenerate zeros when t is sufficiently large. In case $r \in \mathcal{B}_4$, by taking $L > 1$ large with $\operatorname{spt}(u_0) \subset [0, L-1]$, we see that $\eta^s(t, L) > 0$ for t with $t-s>0$ small. By the maximum principle we have $\eta^s(t,x) > 0$ for $x \geq L$ and $t-s>0$ small. Using Lemma 2.1 on $[0, L]$ for small $t-s$, we conclude that $\mathcal{Z}_{[0,\infty)}[\eta^s(t,\cdot)] < \infty$. Using the zero-number properties in [2,9] we see that, for large t, $\eta^s(t,\cdot)$ has fixed number of zeros and all of them are non-degenerate. This proves Claim 1.

Claim 2. *There exists $Y(s)$ such that $u_x(nT-s,0)$ is monotone in large n and converges as $n \to \infty$ to $Y(s)$.*

Since $\eta^s(nT, 0) \equiv 0$, by Claim 1, there exists a positive integer $n_0 = n_0(s)$ such that when $n > n_0$, $\eta^s_x(nT, 0)$ does not change sign. This implies that $u_x(nT-s, 0)$ is monotone in $n > n_0$. Since $u_x(t-s, 0)$ is bounded by the assumption, we see that $u_x(nT-s, 0)$ converges to some number $Y(s)$. This proves Claim 2.

For each $s \in (-T, T)$, set

$$
u_n(t-s,x) := u(t-s+nT, x) \text{ in } \Omega(nT-s;T) := \{(t,x) \mid 0 \leq x \leq r(nT-s+t),\ t \in [-T,T]\}.
$$

In case $r \in \mathcal{B}_1 \cup \mathcal{B}_2$, from above we have estimate $\|u_n(\cdot - s, \cdot)\|_{C^{1+\alpha/2,2+\alpha}(\Omega(nT-s;T))} \leq C$. Hence, there exist a $C^{1,2}$ function $P(t,x)$ and a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$
u_{n_i}(\cdot - s, \cdot) \to P(\cdot - s, \cdot) \text{ as } i \to \infty, \qquad \text{in } C^{1,2} \text{ topology.}
$$

In case $r \in \mathcal{B}_3 \cup \mathcal{B}_4$, using the interior Schauder estimate and the Cantor's diagonal argument, we see that there is a subsequence $\{u_{n'_i}\}$ of $\{u_n\}$ and a $C^{1,2}$ function $P(t,x)$ such that, for any $X > 0$,

$$u_{n'_i}(\cdot - s, \cdot) \to P(\cdot - s, \cdot) \text{ as } i \to \infty, \quad \text{ in } C^{1,2}([-T,T] \times [0,X]) \text{ topology.} \tag{2.5}$$

In both cases, $P(t,x)$ satisfies (1.6) in $t \in [-T-s, T-s]$, except for $P(t,x) = P(t+T,x)$.

Let $\omega^s(u_0)$ be the ω-limit set, defined as the set of all accumulation points of $u(-s+nT,\cdot)$ as $n \to \infty$. Then $\omega^s(u_0)$ is not empty since $P(-s,x) \in \omega^s(u_0)$. By Claim 2, we have

$$\Psi_x(0) = Y(s) \text{ for any } \Psi(x) \in \omega^s(u_0).$$

In particular, $P_x(-s,0) = Y(s)$.

If $\omega^0(u_0)$ consists of a single point, then $u(nT,x) \to P(0,x)$ as $n \to \infty$, and so $u(nT + T, x) \to P(T,x) \equiv P(0,x)$ as $n \to \infty$. This implies that $P(t,x)$ is a periodic solution of (1.6) and $u(t,x)$ converges as $t \to \infty$ to $P(t,x)$ in $C^{1,2}$ topology. The theorem is then proved.

In what follows we assume that $\omega^0(u_0)$ consists of more than one point. Then there exist $P_1(t,x), P_2(t,x) \in C^{1,2}(\Omega_\infty)$, with $\Omega_\infty := \{(t,x) \mid 0 \le x \le \bar{r}(t),\ -T \le t \le T\}$, and two subsequences $\{n_j\}$ and $\{n'_j\}$ of $\{n\}$ such that

$$u_{n_j}(t,x) \to P_1(t,x) \text{ and } u_{n'_j}(t,x) \to P_2(t,x) \text{ as } j \to \infty \text{ in } C^{1,2}_{loc}(\Omega_\infty) \text{ topology },$$

$P_1(0,x), P_2(0,x) \in \omega^0(u_0)$ but they are different: $P_1(0,x) \not\equiv P_2(0,x)$. Without loss of generality we suppose that $P_1(0,x_1) > P_2(0,x_1) + \varepsilon$ for some $x_1 \in (0, \bar{r}(0))$ and some $\varepsilon > 0$. By continuity, there exists $\tau \in (0,T)$ such that

$$P_1(t,x_1) > P_2(t,x_1) \text{ for } t \in (-\tau,\tau). \tag{2.6}$$

Using Lemma 2.1 for $P_1 - P_2$ over $(t,x) \in (-\tau,\tau) \times [0,x_1]$ we conclude that $\mathcal{Z}_{[0,x_1]}[P_1(t,\cdot) - P_2(t,\cdot)] < \infty$ for $t \in (-\tau,\tau)$, and $P_1 - P_2$ has degenerate zeros at finite many time moments in $(-\tau,\tau)$. This, however, contradicts the following facts

$$P_1(-s,0) - P_2(-s,0) = 0, \quad P_{1x}(-s,0) - P_{2x}(-s,0) = Y(s) - Y(s) = 0 \text{ for } s \in (-T,T).$$

This proves that $\omega^0(u_0)$ is a singleton, and so the limit function P obtained above is a time-periodic solution of (1.6). This completes the proof of the theorem. □

Remark 2.2. The hypotheses on d and f in (H), as well as the boundedness of u in Theorem 1.1, are mainly used to derive the Schauder estimates and boundedness of u, u_t, u_x, u_{xx}, etc., which are needed when we apply the maximum principle and the zero-number argument. If the uniform boundedness for u, u_t, u_x, u_{xx} are given separately (for example, like $\|u(t,\cdot)\|_{H^2} \le C$ in [4] for the case $r(t) \equiv 1$), then some hypotheses on d and f can be omitted.

3. A general reaction–diffusion equation

In this section we consider a general heterogeneous reaction–diffusion equation

$$u_t = u_{xx} + g(t,x,u)[m(t,x) - u], \tag{3.1}$$

where m and g are C^1 functions, T-periodic in t, for some $m^0 > m_0 > 0$ and $M > 0$, m satisfies

$$m_0 \le m(t,x) \le m^0, \quad |m_t(t,x)| \le M, \quad |m_x(t,x)| \le M \quad \text{for } t \in [0,T],\ x \ge 0. \tag{3.2}$$

Moreover, to guarantee the boundedness of the solution we require that, for some continuous function $K(u) > 0$ and some C^1 function $g_0(u)$ without degenerate zeros,

$$\begin{cases} g_0(u) \le g(t,x,u) \le K(u), \ \ |g_t(t,x,u),\ g_x(t,x,u),\ g_u(t,x,u)| \le K(u) \text{ for } t,x,u \ge 0, \\ g_0(0) = 0, \quad g_0(u) > 0 \text{ for } u \ge m_0, \ \ G(u,m_0) := \int_u^{m_0} [g_0(s)(m_0 - s)]ds > 0 \text{ for } u \in [0,m_0). \end{cases} \tag{3.3}$$

Clearly, the typical monostable, bistable and combustion types of nonlinearities all satisfy the above assumptions. A multi-stable homogeneous example is $m(t,x) \equiv 16$ and

$$g(t,x,u)[16-u] = \begin{cases} -\frac{\sqrt{u}}{\pi}\sin(\pi\sqrt{u}), & 0 \le u \le 16, \\ 8 - \frac{u}{2}, & u \ge 16. \end{cases}$$

Using the general convergence result Theorem 1.1 we can study the asymptotic behavior for the solutions of the initial-boundary value problem of (3.1) in $[0,\infty)$, as well as the free boundary problem in a variable interval $[0, r(t)]$. Roughly speaking, one can show that any positive solution of these problems remains bounded and converges to a periodic one. Moreover, when the initial data is large enough, spreading happens in the sense that the limit periodic solution is a positive one, taking values in $[m_0, m^0]$ near $x = \infty$.

3.1. Positive lower solutions

As preliminary, we present some solutions of $v'' + g_0(v)(m_0 - v) = 0$, which can be regarded as lower solutions of (3.1).

Lemma 3.1. *Assume (3.2) and (3.3). Then*

(i) *for each $h \in (h_0, m_0)$, the initial value problem*

$$v'' + g_0(v)(m_0 - v) = 0, \quad v(0) = h, \quad v'(0) = 0 \tag{3.4}$$

has a unique solution v_h over $[-L_h, L_h]$ for some $L_h > 0$, which is even, $v_h'(x) < 0$ for $x \in (0, L_h]$ and $v_h(\pm L_h) = 0$;

(ii) *the initial value problem*

$$v'' + g_0(v)(m_0 - v) = 0, \quad v(0) = 0, \quad v'(0) = [2G(0, m_0)]^{1/2}$$

has a unique solution V_0 *over* $[0, \infty)$*, with*

$$V_0'(x) > 0 \text{ for } x \geq 0, \quad V_0(x) \to m_0 \text{ as } x \to \infty;$$

(iii) *for each* $h \in (h_0, m_0)$ *and* v_h *in* (i)*, if* $\widetilde{V}$ *is a bounded solution of*

$$v'' + g_0(v)(m_0 - v) = 0 \ (x > 0), \quad v(0) = 0, \quad v(x) > 0 \ (x > 0), \tag{3.5}$$

and if $\widetilde{V}(x) \geq v_h(x - M)$ *in* $[-L_h + M, L_h + M]$ *for some* $M \geq L_h$*, then* $\widetilde{V}$ *is nothing but the solution* V_0 *in* (ii).

Proof. Multiplying the equation in (3.4) by $-2v'$, we obtain that

$$(v')^2 - 2G(v, m_0) = E$$

for some constant E. Therefore, any given solution v of $v'' + g_0(v)(m_0 - v) = 0$ is associated with a constant $E \geq -2G(v, m_0)$. By condition (3.3), we see that $-G(v, m_0) < 0$ for $v \in [0, m_0) \cup (m_0, \infty)$, and $-G(v, m_0) = 0$ for $v = m_0$. Therefore, $-G(v, m_0)$ has a global maximum 0 at $v = m_0$. Define

$$h_0 := \inf \left\{ h \in [0, m_0) \;\middle|\; \max_{v \in [0,h]} \big[-G(v, m_0) \big] \leq \min_{u \in [h, m_0]} \big[-G(u, m_0) \big] \right\}. \tag{3.6}$$

It is straightforward to find h_0 from the graph of the function $y = -2G(v, m_0)$ and we can see from $G_v(m_0, m_0) = 0,\ G_{vv}(m_0, m_0) = g_0(m_0) > 0$ that $h_0 \in [0, m_0)$. Since any solution v of $v'' + g_0(v)(m_0 - v) = 0$ is associated with a constant $E \geq -2G(v, m_0)$, and $E = -2G(v, m_0)$ for v equal to any maximum or minimum of v (i.e., where $v' = 0$), all solutions can be "read" from horizontal lines $y = E$ in the following cases:

(a) When $E \leq -2G(h_0, m_0)$, horizontal line segments of $y = E$ lying above the graph of $y = -2G(v, m_0)$ with both endpoints on the graph of $y = -2G(v, m_0)$ (which correspond to periodic solutions, or heteroclinic orbits, or homoclinic orbits of the solutions defined on $\mathbb{R}$).
(b) When $0 \geq E > -2G(h_0, m_0)$, horizontal line segments of $y = E$ issue from a point on the graph of $y = -2G(v, m_0)$ and lying above the graph of $y = -2G(v, m_0)$ (which correspond to solutions defined on $\mathbb{R}$ which are unbounded from below/above and bounded from above/below, respectively).
(c) When $E > 0$, horizontal lines $y = E$ lying above the graph of $y = -2G(v, m_0)$ (which correspond to solutions defined on $\mathbb{R}$ which are unbounded from both below and above).

In particular, when $0 > E > -2G(h_0, m_0)$, there is a horizontal line segment connecting the point $(0, E)$ with a point $(h, E) = (h, -2G(h, m_0))$ on the graph of $y = -2G(v, m_0)$, and $h \in (h_0, m_0)$. It corresponds to a solution of (3.4), and (i) is proved. When $E = 0$, the horizontal line segment connecting $(0, 0)$ with $(m_0, 0)$ corresponds to the solution V_0 in (ii). Finally, any solution V of (3.5) can be given by the horizontal line segments of $y = E$ with $E \geq 0$, as in cases (b) and (c). Among them only V_0 is a bounded solution. This proves (iii). □

3.2. *Positive time-periodic solutions*

We continue to consider positive time-periodic solutions of (3.1) over $x \in (0, \infty)$, that is, solutions of the following problem

$$\begin{cases} v_t = v_{xx} + g(t,x,v)[m(t,x) - v], & x > 0, t \in [0,T], \\ v(t,0) = 0, & t \in [0,T], \\ v(0,x) = v(T,x), & x \geq 0. \end{cases} \tag{3.7}$$

Lemma 3.2. *Assume* (3.2) *and* (3.3)*. Then the problem* (3.7) *has a solution* $\mathcal{V}^*(t,x)$*, and*

$$V_0(x) \leq \mathcal{V}^*(t,x) \leq m^0, \quad t \in [0,T],\ x \geq 0, \tag{3.8}$$

where V_0 *is that given in Lemma 3.1* (ii).

Proof. For any large integer n, we consider the initial-boundary value problem

$$\begin{cases} u_t = u_{xx} + g(t,x,u)[m(t,x) - u], & 0 < x < n, t > 0, \\ u(t,0) = 0, \ \ u(t,n) = m^0, & t > 0, \\ u(0,x) = m^0, & 0 \leq x \leq n. \end{cases} \tag{3.9}$$

By the parabolic theory, it has a time-global classical solution $u_n(t,x)$, with

$$V_0(x) \leq u_n(t,x) \leq m^0, \quad x \in [0,n],\ t > 0. \tag{3.10}$$

By the comparison principle, $u_n(t,x)$ is decreasing in n. Using Theorem 1.1 we see that $u_n(t,x)$ converges as $t \to \infty$ to a T-periodic solution $v_n(t,x)$, which also satisfies the inequalities (3.10), and decreases in n. Therefore, v_n converges as $n \to \infty$ to a time-periodic function $\mathcal{V}^*(t,x)$, which is a solution of (3.7) satisfying (3.8). □

Remark 3.3. From our approach we see that $\mathcal{V}^*$ is the largest solution of (3.7) with range in $[0, m^0]$. The uniqueness of such a solution is open. Since the nonlinearity we are considering is a temporal-spatial heterogeneous multi-stable one rather than a Fisher-KPP one (like the logistic nonlinearity), we guess that the uniqueness may be not true.

3.3. *The problem on the half line*

In this subsection we consider the initial-boundary value problem for (3.1) over the half line:

$$\begin{cases} u_t = u_{xx} + g(t,x,u)[m(t,x) - u], & x > 0,\ t > 0, \\ u(t,0) = 0, & t \geq 0, \\ u(x,0) = u_0(x), & x \geq 0. \end{cases} \tag{3.11}$$

On the asymptotic behavior for the solutions of this problem we have the following result.

Theorem 3.4. *Assume* (3.2) *and* (3.3). *If* $u_0 \in C(\mathbb{R})$ *is a compactly supported function with* $u_0(x) \geq, \not\equiv 0$, *then the problem* (3.11) *has a time global solution* $u(t,x)$ *with* $u(t,x) > 0$ *for all* $x > 0, t > 0$, *and there exists a nonnegative solution* $\mathcal{V}$ *of* (3.7) *such that* $u(t,\cdot) - \mathcal{V}(t,\cdot) \to 0$ *as* $t \to \infty$ *in* $L^\infty_{loc}([0,\infty))$.

Moreover, let v_h, L_h *be given as in Lemma 3.1* (i) *for some* $h \in (h_0, m_0)$. *If* $u_0(x) \geq v_h(x - M)$ *in* $[-L_h + M, L_h + M]$ *for some* $M \geq L_h$, *then* spreading happens *in the sense that* $u(t,x) - \mathcal{V}(t,x) \to 0$ *as* $t \to \infty$ *for a solution* $\mathcal{V}$ *of* (3.7) *satisfying* (3.8).

Proof. Under the hypotheses (3.2) and (3.3), one can show by the maximum principle that $0 \leq u(t,x) < \|u_0\|_\infty + m^0$ for all $x \geq 0,\ t > 0$. The global existence of u then follows from the standard parabolic theory. The convergence of u to a nonnegative time-periodic solution $\mathcal{V}$ (i.e., a nonnegative solution of (3.7)) is a consequence of our Theorem 1.1.

Now we prove the spreading phenomena when $u_0 \geq v_h(x - M)$. We compare the solution u of (3.11) with the solution of

$$\begin{cases} w_t = w_{xx} + g_0(w)(m_0 - w), & x > 0, t > 0, \\ w(t,0) = 0, & t \geq 0, \\ w(0,x) = \tilde{v}_h(x), & x \geq 0, \end{cases} \tag{3.12}$$

where $\tilde{v}_h$ is a function defined in $[0,\infty)$ by extending $v_h(x - M)$ to be zero outside its support. Clearly, $0 \leq w(t,x) < m_0$ for all $x \geq 0, t > 0$. Hence

$$g_0(w)(m_0 - w) \leq g(t,x,w)(m(t,x) - w), \quad x,\ t \geq 0,\ w \in [0, m_0].$$

This implies that $w(t,x)$ is a lower solution of (3.11), and so

$$u(t,x) \geq w(t,x) > \tilde{v}_h(x), \quad x \geq 0,\ t > 0. \tag{3.13}$$

By Theorem 1.1, $u(t,x) \to \mathcal{V}(t,x)$ and $w(t,x) \to W(x)$ as $t \to \infty$, for some stationary solution W of (3.12) (the latter convergence can also be derived in the same way as in [8,16]). From Lemma 3.1 (iii) we see that W is nothing but the function V_0 in Lemma 3.1 (ii). Taking limit as $t \to \infty$ in (3.13) we have

$$\mathcal{V}(t,x) \geq V_0(x), \quad x \geq 0,\ t \in [0,T].$$

Hence $\mathcal{V}$ satisfies the first inequality in (3.8).

Now we study the upper bound of $\mathcal{V}$. Note that $u \leq \|u_0\|_\infty + m^0$, we choose $\delta > 0$ such that $g(t,x,u) \geq g_0(u) \geq \delta$ for all $t, x \geq 0$ and $u \in [m^0, m^0 + \|u_0\|_\infty]$, and consider the ordinary differential equation

$$\zeta_t = \delta(m^0 - \zeta), \quad \zeta(0) = m^0 + \|u_0\|_\infty.$$

It is easily seen that the solution $\zeta(t) = m^0 + \|u_0\|_\infty e^{-\delta t}$ of this problem is an upper solution of (3.11), and so $u(t,x) \leq \zeta(t)$. This reduces to $\mathcal{V}(t,x) \leq m^0$. So we obtain the spreading result. □

3.4. The problem with a free boundary

We continue to consider the following free boundary problem

$$\begin{cases} u_t = u_{xx} + g(t,x,u)[m(t,x) - u], & x \in (0, r(t)),\ t > 0, \\ u(t,0) = u(t,r(t)) = 0,\ r'(t) = -u_x(t,r(t)), & t > 0, \\ r(0) > 0, \quad u(0,x) = u_0(x), & x \in [0, r(0)]. \end{cases} \tag{3.14}$$

In 2010, this problem with diffusive logistic equation $u_t = u_{xx} + u(1-u)$ and Neumann boundary condition at $x = 0$ was studied by Du and Lin [14]. Since then, various special cases of the problem (3.14) (with Neumann, Dirichlet or Robin boundary condition at $x = 0$) have been studied extensively. For the background and known results for these problems, we refer the readers to [12–15,19,21,29] etc.

Theorem 3.5. *Assume* (3.2) *and* (3.3). *Assume further that* $u_0 \in C(\mathbb{R})$, $u_0 \geq, \not\equiv 0$ *and it has a support in* $[0, r(0)]$. *Then the problem* (3.14) *has a time global solution pair* $(u(t,x), r(t))$ *with* $u(t,x) > 0$ *for all* $x \in (0, r(t)), t > 0$, *and, as* $t \to \infty$,

$$r(t) \to \bar{r}, \quad u(t,\cdot) \to \mathcal{V}(t,\cdot) \text{ in } L^\infty_{loc}([0,\bar{r})), \tag{3.15}$$

for some number $\bar{r} \in (0, \infty]$ *and some nonnegative solution* $\mathcal{V}$ *of* (3.7).

Moreover, let v_h *and* L_h *be given as in Lemma 3.1* (i) *for some* $h \in (h_0, m_0)$. *If, for some* $M \geq L_h$, $u_0(x) \geq v_h(x - M)$ *in* $[-L_h + M, L_h + M] \subset [0, r(0)]$, *then* spreading happens *in the sense that* $r(t) \to \infty$ *and* $u(t,x) - \mathcal{V}(t,x) \to 0$ *as* $t \to \infty$, *for a solution* $\mathcal{V}$ *of* (3.7) *satisfying* (3.8).

Proof. As in [14,15], one can show that the problem (3.14) has a time global solution pair $(u(t,x), r(t))$, where u is defined in $\{(t,x) \mid x \in [0, r(t)],\ t \geq 0\}$. By the maximum principle, $u(t,x) > 0$ in $x \in (0, r(t))$ and $t > 0$, $u_x(r(t), t) < 0$ for all t. Hence $r(t)$ is increasing in t, and $\bar{r} := \lim_{t\to\infty} r(t)$ exists. The convergence of $u(t,\cdot)$ as $t \to \infty$ is a consequence of our Theorem 1.1.

Now we consider the case where $u_0 \geq v_h(x - M)$. By comparison we have $u(t,x) \geq v_h(x - M)$. Hence the ω-limit $\mathcal{V}$ of u also satisfies $\mathcal{V}(t,x) \geq v_h(x - M)$. If $\bar{r}$ is a finite number, then using the Hopf lemma as in [14,15] we conclude that $u(t,x) \to 0$, a contradiction. The rest proof is similar as that in the proof of Theorem 3.4. □

4. A (mean) curvature flow equation

The phenomenon of nonlinear waves is one of the most fundamental processes occurring in reaction–diffusion excitable systems. Examples of such waves include oscillating chemical waves in the Belousov–Zhabotinsky reactions [33]; waves in heart muscle which are involved in cardiac arrhythmias [1,11]. One important aspect in the study of such waves is the motion in the presence of obstacles. This is particularly important in cardiac excitable tissue (see more background in [30] and reference therein).

In this section we consider such a wave on the xy-plane. We suppose that there is an obstacle in the field, which is a bar along x-axis with a gap in the middle, that is, the obstacle is $\{(x,y) \in \mathbb{R}^2 \mid x \in (-\infty, -L(t)] \cup [L(t), \infty),\ |y| \leq h\}$ for some small $h > 0$ and some positive, T-periodic and smooth function $L(t)$. We consider the motion of a wave (denoted by a family of simple curves $\Gamma(t)$) while it touches the ends of the obstacle. The latter, when the obstacle is sufficiently thin (that is, when $h \ll 1$), can be roughly regarded as the points $(\pm L(t), 0)$. In many cases the motion of a nonlinear wave depends on its mean curvature, and the law is a mean curvature flow. We also consider a (mean) curvature flow equation

$$\mathbf{V} = A(t, \mathbf{n})\kappa + B(t, \mathbf{n}) \text{ on } \Gamma(t)$$

(cf. [7,17,20,26,28]), where, $\mathbf{V}$ denotes the normal velocity of $\Gamma(t)$, $\mathbf{n}$ denotes its (upward) unit normal vector, and κ denotes its curvature, $A > 0$ and B are C^1 smooth functions, T-periodic in t.

In the special case where $\Gamma(t)$ is the graph of a function $y = u(t, x)$, we have

$$\mathbf{V} = \frac{u_t}{\sqrt{1+u_x^2}}, \quad \mathbf{n} = \frac{(-u_x, 1)}{\sqrt{1+u_x^2}}, \quad \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

and so our problem is converted into

$$\begin{cases} u_t = a(t, u_x)\dfrac{u_{xx}}{1+u_x^2} + b(t, u_x)\sqrt{1+u_x^2}, & x \in (-L(t), L(t)),\ t > 0, \\ u(t, \pm L(t)) = 0, & t > 0, \end{cases} \tag{4.1}$$

where $a(t, u_x) := A(t, \mathbf{n})$, $b(t, u_x) := B(t, \mathbf{n})$ are C^1 smooth functions, T-periodic in t and $a > 0$. Denote

$$\begin{cases} A_0 := \min\{A(t, \mathbf{n}) \mid \mathbf{n} \in \mathbb{S}^1,\ t \in [0, T]\}, \quad B^0 := \max\{|B(t, \mathbf{n})| \mid \mathbf{n} \in \mathbb{S}^1,\ t \in [0, T]\}, \\ L^0 := \max\limits_{t \in [0,T]} L(t), \quad K := \max\limits_{t \in [0,T]} |L'(t)| \quad \text{and} \quad \varphi(z) := \frac{1}{2}(1 - z^2) \text{ for } |z| \le 1. \end{cases} \tag{4.2}$$

Theorem 4.1. *Let $A_0, B^0, L^0, K, \varphi$ be defined as in (4.2). Assume*

$$A_0 \ge 2L^0(\sqrt{2}B^0 + K). \tag{4.3}$$

If $u_0 \in C^2([-L(0), L(0)])$ satisfies

$$|u_0(x)| \le L(0)\varphi\Big(\frac{x}{L(0)}\Big) \text{ for } x \in [-L(0), L(0)], \tag{4.4}$$

then the problem (4.1) with initial data $u(0, x) = u_0(x)$ has a bounded time global classical solution $u(t, x)$, and it converges as $t \to \infty$ to a time-periodic one.

Proof. Suppose that $u(t, x)$ is a classical solution of (4.1) in the time interval $(0, \tau)$. First, we present the a priori bound for $u_x(t, \pm L(t))$ in $t \in (0, \tau)$ by constructing lower and upper solutions. Set

$$U(t, x) := L(t) \cdot \varphi\Big(\frac{x}{L(t)}\Big), \quad x \in [-L(t), L(t)],\ t > 0.$$

Then, with $z := x/L(t)$, we have

$$\begin{aligned} U_t - A(t,\mathbf{n})\frac{U_{xx}}{1+U_x^2} - B(t,\mathbf{n})\sqrt{1+U_x^2} &\geq L'[\varphi(z) - z\varphi'(z)] + \frac{A_0}{L^0}\cdot\frac{1}{1+z^2} - B^0\sqrt{1+z^2} \\ &= \frac{1}{2}L'(1+z^2) + \frac{A_0}{L^0}\cdot\frac{1}{1+z^2} - B^0\sqrt{1+z^2} \\ &\geq -K + \frac{A_0}{2L^0} - B^0\sqrt{2} \geq 0. \end{aligned}$$

The last inequality follows from (4.2). Hence $U(t,x)$ is an upper solution of (4.1). Since the inequality (4.4) implies that $U(0,x) \geq u(0,x)$, by comparison we have

$$u(t,x) \leq U(t,x) \text{ for } x \in [-L(t), L(t)],\ t \in (0,\tau).$$

Noting $u(t,\pm L(t)) \equiv U(t,\pm L(t)) \equiv 0$ we have

$$u_x(t,-L(t)) \leq U_x(t,-L(t)) = 1, \quad u_x(t,L(t)) \geq U_x(t,L(t)) = -1 \text{ for all } t \in (0,\tau).$$

Similarly, one can show that $-U(t,x)$ is a lower solution of (4.1) and so

$$u_x(t,-L(t)) \geq -U_x(t,-L(t)) = -1, \quad u_x(t,L(t)) \leq -U_x(t,L(t)) = 1 \text{ for all } t \in (0,\tau).$$

Next we give the a priori bound for $\zeta := u_x(t,x)$. Clearly, ζ solves

$$\begin{cases} \zeta_t = \dfrac{a(t,\zeta)}{1+\zeta^2}\zeta_{xx} + \tilde{b}(t,x)\zeta_x, & x \in (-L(t), L(t)),\ t \in (0,\tau), \\ \zeta(t,\pm L(t)) = u_x(t,\pm L(t)) \in [-1,1], & t \in (0,\tau), \end{cases}$$

where $\tilde{b}$ is a continuous function in $\Omega_\tau := \{(t,x) \mid x \in [-L(t),L(t)],\ t \in (0,\tau)\}$. By the maximum principle we have

$$|u_x(t,x)| = |\zeta(t,x)| \leq M_1 := 1 + \|u_{0x}\|_\infty, \quad x \in [-L(t),L(t)],\ t \in (0,\tau). \tag{4.5}$$

Now, we straighten the variable boundary $x = \pm L(t)$ and transfer the problem into one on the fixed interval. Set

$$z := \frac{x}{L(t)}, \quad w(t,z) := u(t, L(t)z) = u(t,x). \tag{4.6}$$

Then

$$|w_z(t,z)| = L(t)|u_x(t,L(t)z)| \leq M_2 := L^0 M_1, \quad z \in [-1,1], t > 0, \tag{4.7}$$

and the problem (4.1) with initial data u_0 is converted into

$$\begin{cases} w_t = a\Big(t, \dfrac{w_z}{L(t)}\Big)\dfrac{w_{zz}}{L^2(t)+w_z^2} + b\Big(t, \dfrac{w_z}{L(t)}\Big)\dfrac{1}{L(t)}\sqrt{L^2(t)+w_z^2} + \dfrac{L'(t)z}{L(t)}w_z, \\ \qquad\qquad\qquad\qquad z \in (-1,1),\ t>0, \\ w(t,\pm 1) = 0, \qquad t>0, \\ w(0,z) = u_0(L(0)z), \qquad z\in[-1,1]. \end{cases} \tag{4.8}$$

Since the coefficients in this equation do not satisfy (H), we modify them in the following way. Set

$$d(t,z,w,p) := \begin{cases} a\Big(t, \dfrac{p}{L(t)}\Big)\dfrac{1}{L^2(t)+p^2}, & |p|\le M_2+1, z\in[-1,1], t, w\in\mathbb{R}, \\ 1, & |p|\ge M_2+2, z\in[-1,1], t, w\in\mathbb{R}, \\ \text{smooth function}, & |p|\in[M_2, M_2+3], z\in[-1,1], t, w\in\mathbb{R}, \end{cases}$$

and

$$f(t,z,w,p) := \begin{cases} b\Big(t, \dfrac{p}{L(t)}\Big)\dfrac{1}{L(t)}\sqrt{L^2(t)+p^2} + \dfrac{L'(t)z}{L(t)}p, \\ \qquad\qquad\qquad |p|\le M_2+1, z\in[-1,1], t, w\in\mathbb{R}, \\ 0, \qquad |p|\ge M_2+2, z\in[-1,1], t, w\in\mathbb{R}, \\ \text{smooth function}, \quad |p|\in[M_2, M_2+3], z\in[-1,1], t, w\in\mathbb{R}. \end{cases}$$

Then both d and f are smooth and bounded functions in $(t,z,w,p)\in[0,T]\times[-1,1]\times[-L^0, L^0]\times\mathbb{R}$. Moreover, thanks to (4.7), the following problem

$$\begin{cases} w_t = d(t,z,w,w_z)w_{zz} + f(t,z,w,w_z), & z\in(-1,1),\ t>0, \\ w(t,\pm 1) = 0, & t>0, \\ w(0,z) = u_0(L(0)z), & z\in[-1,1] \end{cases} \tag{4.9}$$

has the same solution as (4.8). Using the a priori bound for w_z and the standard parabolic theory as in [26], we see that the problem (4.9) has a bounded time global classical solution $w(t,x)$. Since d and f satisfy (H), we can apply Theorem 1.1 (i) to show that $w(t,z)$ converges as $t\to\infty$ to a periodic solution $P(t,z)$ of (4.9).

Therefore, the problem (4.1) with initial data u_0 satisfying (4.4) has a bounded time global classical solution $u(t,x)$, and it converges to the periodic function $\widetilde{P}(t,x) := P\big(t, \frac{x}{L(t)}\big)$, which is a time-periodic solution of (4.1). This completes the proof. □

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