NETWORKS AND HETEROGENEOUS MEDIA Website: http://aimSciences.org

Volume **1**, Number **4**, December **2006** pp. **537–568**

PERIODIC TRAVELING WAVES IN A TWO-DIMENSIONAL CYLINDER WITH SAW-TOOTHED BOUNDARY AND THEIR HOMOGENIZATION LIMIT

Hiroshi Matano

Graduate school of Mathematical Sciences, University of Tokyo
Komaba 3-8-1, Tokyo 153-8914, Japan

Ken-Ichi Nakamura

Department of Computer Science, University of Electro-Communications
Chofu, Tokyo 182-8585, Japan

Bendong Lou

Department of Mathematics, Tongji University
Siping Road 1239, Shanghai, China

Abstract. We study a curvature-dependent motion of plane curves in a two-dimensional cylinder with periodically undulating boundary. The law of motion is given by $V = \kappa + A$, where V is the normal velocity of the curve, κ is the curvature, and A is a positive constant. We first establish a necessary and sufficient condition for the existence of periodic traveling waves, then we study how the average speed of the periodic traveling wave depends on the geometry of the domain boundary. More specifically, we consider the homogenization problem as the period of the boundary undulation, denoted by ε, tends to zero, and determine the homogenization limit of the average speed of periodic traveling waves. Quite surprisingly, this homogenized speed depends only on the maximum opening angle of the domain boundary and no other geometrical features are relevant. Our analysis also shows that, for any small $\varepsilon > 0$, the average speed of the traveling wave is smaller than A, the speed of the planar front. This implies that boundary undulation always lowers the speed of traveling waves, at least when the bumps are small enough.

1. **Introduction.**

We discuss traveling waves for a curvature-driven motion of plane curves in a two-dimensional cylinder Ω_ε, whose boundaries undulate periodically with period $\varepsilon > 0$. The law of motion of the curve is given by

$$V = \kappa + A, \tag{1.1}$$

where V denotes the normal velocity of the curve, κ denotes the curvature and A is a positive constant representing a constant driving force. The domain Ω_ε is defined as follows: Let $g(y)$ be a 1-periodic smooth function satisfying

$$g(0) = g(1) = 0, \qquad g(y) \geq 0 \quad (y \in \mathbb{R})$$

2000 *Mathematics Subject Classification.* Primary: 35B27, 53C44; Secondary: 35B40, 35K55.
Key words and phrases. front propagation, periodic traveling wave, curve-shortening, homogenization.

and

$$\max_y g'(y) = \tan\alpha, \qquad \min_y g'(y) = -\tan\beta \tag{1.2}$$

for some α, $\beta \in (0, \pi/2)$ (see Figure 1(a)). Then we define

$$\Omega_\varepsilon := \left\{(x,y) \in \mathbb{R}^2 \mid -H - g_\varepsilon(y) < x < H + g_\varepsilon(y)\right\},$$

where $H > 0$ is a given constant and $g_\varepsilon(y) := \varepsilon g(y/\varepsilon)$ (see Figure 1(b)). We call α the *maximum opening angle* of the boundary. Denote the left (resp. right) boundary of Ω_ε by $\partial_-\Omega_\varepsilon$ (resp. $\partial_+\Omega_\varepsilon$).

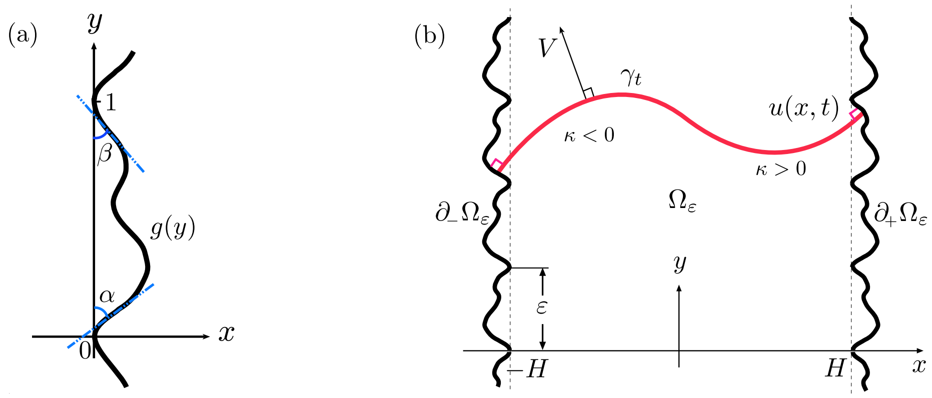

FIGURE 1. (a) The function g. (b) Ω_ε and the curve γ_t.

In this paper, by a solution of (1.1) we mean a time-dependent simple curve γ_t in Ω_ε which satisfies (1.1) and contacts $\partial_\pm\Omega_\varepsilon$ perpendicularly. Equation (1.1) appears as a certain singular limit of an Allen-Cahn type nonlinear diffusion equation under the Neumann boundary conditions. The curve γ_t represents the interface between two different phases. See, e.g., [11, 1] for details.

To avoid sign confusion, the normal to the curve γ_t will always be chosen toward the upper region, and the sign of the normal velocity V and the curvature κ will be understood in accordance with this choice of the normal direction. Consequently, κ is negative at those points where the curve is concave while it is positive where the curve is convex (see Figure 1(b)).

We will mainly consider the case where γ_t is expressed as a graph of a certain function $y = u(x,t)$ at each time t. Let $\zeta_-(t)$, $\zeta_+(t)$ be the x-coordinates of the end points of γ_t lying on $\partial_-\Omega_\varepsilon$, $\partial_+\Omega_\varepsilon$, respectively. In other words, $(\zeta_\pm(t), u(\zeta_\pm(t), t)) \in \partial_\pm\Omega_\varepsilon$. Now (1.1) is equivalent to

$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2}, \quad \zeta_-(t) < x < \zeta_+(t), \quad t > 0, \tag{1.3}$$

with the boundary conditions

$$u_x(\zeta_\pm(t), t) = \mp g_\varepsilon'(u(\zeta_\pm(t), t)), \qquad (\zeta_\pm(t), u(\zeta_\pm(t), t)) \in \partial_\pm\Omega_\varepsilon. \tag{1.4}$$

Throughout this paper we will assume that the boundary angles α, β satisfy

$$0 < \alpha,\ \beta < \frac{\pi}{4} \qquad \text{(slope condition)}$$

or equivalently,

$$G := \max_{y\in\mathbb{R}} |g'(y)| < 1. \tag{1.5}$$

We impose the above condition in order to prevent γ_t from developing singularities near the boundary $\partial\Omega_\varepsilon$. As we will show in Section 3 (Theorem 3.19), under the condition (1.5), the equation (1.3)-(1.4) has a time-global classical solution for any smooth initial data u_0 satisfying $|u_0'(x)| \le G$. Here the term "classical solution" is understood in the following sense:

Definition 1.1. A function $u(x,t)$ defined for $\zeta_-(t) \le x \le \zeta_+(t)$, $0 \le t < T$ is called a classical solution of (1.3)-(1.4) in the time interval $[0,T)$ if

(a) u, u_x are continuous for $\zeta_-(t) \le x \le \zeta_+(t)$, $0 \le t < T$, and u_{xx}, u_t are continuous for $\zeta_-(t) < x < \zeta_+(t)$, $0 < t < T$;
(b) u satisfies (1.3)-(1.4) for $\zeta_-(t) < x < \zeta_+(t)$, $0 < t < T$.

It is called a time-global classical solution if $T = +\infty$.

Now if $\varepsilon = 0$ then $\Omega_0 = \{(x,y) \in \mathbb{R}^2 \mid -H < x < H\}$ is a straight cylinder. In this case (1.1) has a *planar* traveling wave, namely a solution of the form $\gamma_t = \{(x, At) \mid -H \le x \le H\}$, which has a flat profile and moves at a constant speed A. On the other hand, if $\varepsilon > 0$, traveling waves in the usual sense cannot exist. In fact, as the front γ_t propagates, its shape and speed fluctuate due to undulation of the boundaries $\partial_\pm\Omega_\varepsilon$. Therefore we have to adopt a generalized notion of traveling waves. In the case of spatially periodic environments, such a generalized notion of traveling waves is well-established in the literature, which, in the present context, can be stated as follows:

Definition 1.2. A solution $U^\varepsilon(x,t)$ of (1.3)-(1.4) is called a *periodic traveling wave* — or simply a *traveling wave* — if it satisfies

$$U^\varepsilon(x, t+T_\varepsilon) = U^\varepsilon(x,t) + \varepsilon \tag{1.6}$$

for some $T_\varepsilon > 0$. Its *average speed* — or the effective speed — is defined by

$$c_\varepsilon = \frac{\varepsilon}{T_\varepsilon}.$$

Note that, despite its apparent notational difference, the above definition is equivalent to the standard definition of traveling waves in periodic media; see for example, [2, 4, 5, 12, 14]. In order to emphasize the difference from the classical notion of traveling waves, in the present paper we have adopted the term "periodic traveling wave" as in [12]. As is clear from the definition, periodic traveling waves change their profile periodically in time (see Figure 2).

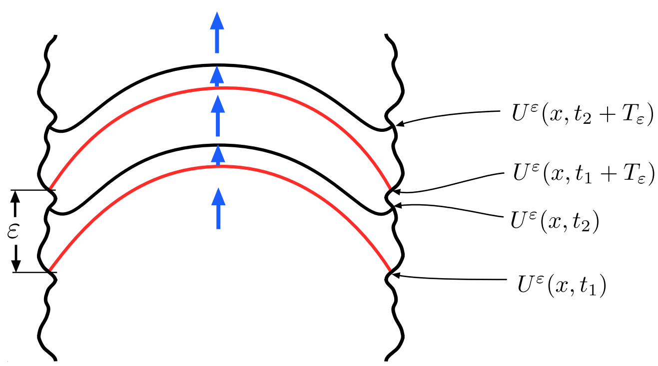

Figure 2. Periodic traveling wave.

The first aim of the present paper is to prove the existence and uniqueness of the periodic traveling wave, and discuss its stability. The second aim is to study the homogenization limit of the periodic traveling wave U^ε as $\varepsilon \to 0$ and determine its homogenized speed and profile. As we will see later in Subsection 2.2 and Section 5, the periodic traveling wave $U^\varepsilon(x,t)$ converges to a function of the form $\varphi(x) + c_0 t$ as $\varepsilon \to 0$, and determining the homogenized speed c_0 is equivalent to determining the contact angle θ^* of the homogenized profile φ (see Figure 3).

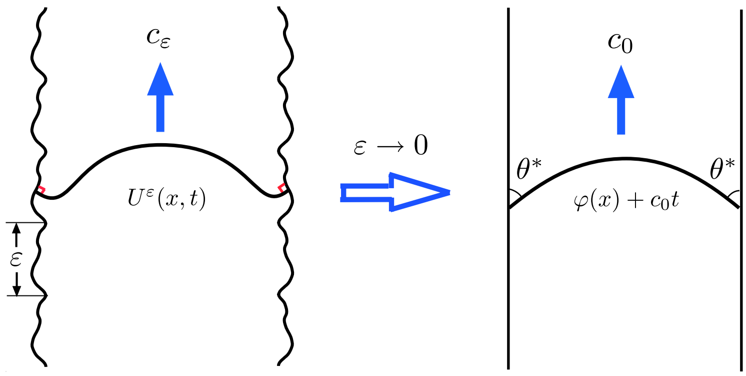

FIGURE 3. Homogenization limit of the periodic traveling wave.

This paper is organized as follows. In Section 2, we state our main theorems, Theorems 2.1 to 2.3. Theorem 2.1 gives a sharp criterion for the existence of periodic traveling waves. More precisely, a periodic traveling wave exists if $AH \geq \sin\alpha$, while it does not if $AH < \sin\alpha$, where α is the maximum opening angle of the boundary as defined in (1.2). In the latter case, propagation is always blocked (or, so to say, "pinned") and every solution of (1.3)-(1.4) converges to a stationary solution as $t \to \infty$. Theorem 2.2 asserts that the periodic traveling wave is asymptotically stable. Theorem 2.3 is concerned with the estimate of the speed of periodic traveling wave and its homogenization limit as $\varepsilon \to 0$.

In Section 3, we discuss the local and global existence of solutions of the initial-boundary value problem for (1.3)-(1.4). In doing so, we introduce a new coordinate system (iso-thermal coordinates) that converts (1.3)-(1.4) into a problem on a straight cylinder while preserving the contact angles.

In Section 4, we construct an *entire solution* — namely a solution that is defined for $-\infty < t < \infty$ — by using a renormalization argument. Then we prove the uniqueness (up to time shift) of the entire solution. The uniqueness implies that this entire solution must satisfy (1.6) for some $T_\varepsilon \in \mathbb{R}$, hence it is a periodic traveling wave. This establishes the existence and uniqueness of the periodic traveling wave, completing the proof of Theorem 2.1 for the case $AH \geq \sin\alpha$. Stability of the periodic traveling wave is also discussed (proof of Theorem 2.2).

In Section 5 we estimate the average speed of the periodic traveling wave and discuss the homogenization limit of the speed and the profile as $\varepsilon \to 0$ (proof of Theorem 2.3). The estimate of the speed is given by constructing a suitable upper and a lower solution. Once the limit speed $c_0 = \lim_{\varepsilon\to 0} c_\varepsilon$ is determined, the limit profile φ along with the contact angle θ^* will follow rather straightforwardly.

Finally, in Section 6, we consider the pinning case $AH < \sin\alpha$ and prove that every global solution of (1.3)-(1.4) converges to a stationary solution. This proof is based on the comparison principle, the energy functional and the uniqueness results of ω-limit points for one-dimensional parabolic equations.

2. **Main theorems.** In this section we present our main results on the periodic traveling wave and its homogenization limit. More basic questions such as the well-posedness of equation (1.3)-(1.4) will be discussed in Section 3. In the rest of the paper we impose the slope condition (1.5), which guarantees the existence of classical solutions for equation (1.3)-(1.4) (see Subsection 3.6).

2.1. **Existence and stability of periodic traveling wave.**

Theorem 2.1. (Existence). *Assume the slope condition* (1.5).

(i) *If $AH \geq \sin\alpha$, then there exists a periodic traveling wave of* (1.3)-(1.4)*, and it is unique up to time shift. Moreover, this periodic traveling wave U^ε satisfies*

$$U^\varepsilon(-x,t) = U^\varepsilon(x,t), \quad -\tan\alpha \leq \operatorname{sgn} x \cdot U^\varepsilon_x(x,t) \leq \tan\beta, \quad U^\varepsilon_t(x,t) > 0$$

for all (x,t) with $(x, U^\varepsilon(x,t)) \in \overline{\Omega}_\varepsilon$ and $t \in \mathbb{R}$.

(ii) *If $AH < \sin\alpha$ and if ε is sufficiently small, then there exists no periodic traveling wave. Moreover, every classical solution of* (1.3)-(1.4) *that is defined globally for $t \geq 0$ converges to a stationary solution as $t \to \infty$.*

Theorem 2.2. (Stability). *Assume the slope condition* (1.5) *and that $AH \geq \sin\alpha$. Then the periodic traveling wave $U^\varepsilon(x,t)$ is stable in the following sense:*

(i) [Stability] *Let Γ^ε_t be the solution curve of* (1.1) *associated with $U^\varepsilon(x,t)$. Then for any $\sigma > 0$ there exists $\delta > 0$ such that for any solution curve γ_t of* (1.1) *that is defined globally for $t \geq 0$ and satisfies $d_{\mathcal{H}}(\gamma_0, \Gamma^\varepsilon_\tau) < \delta$ for some $\tau \in \mathbb{R}$, it holds that $d_{\mathcal{H}}(\gamma_t, \Gamma^\varepsilon_{t+\tau}) < \sigma$ for all $t \geq 0$. Here $d_{\mathcal{H}}$ denotes the Hausdorff distance between two compact sets in $\mathbb{R}^2$.*

(ii) [Asymptotic stability] *Let $u(x,t)$ be a classical solution of* (1.3)-(1.4) *defined globally for $t \geq 0$ and let γ_t be the solution curve of* (1.1) *associated with $u(x,t)$. Then there exists a constant $\tau \in \mathbb{R}$ such that*

$$\lim_{t\to+\infty} d_{\mathcal{H}}(\gamma_t, \Gamma^\varepsilon_{t+\tau}) = 0.$$

Furthermore, γ_t approaches $\Gamma^\varepsilon_{t+\tau}$ in the C^2 sense as $t \to +\infty$.

The above existence criteria $AH \geq \sin\alpha$ and $AH < \sin\alpha$ can be interpreted that the front propagates freely if the driving force A is large enough, while propagation is blocked (or pinned) if A is not large enough to push the front against the boundary bumps.

2.2. **Homogenization limit.** An important question concerning the periodic traveling wave is how its average speed depends on the geometrical shape of the boundaries. This problem is important in many physical phenomena and very little is known so far. We study this problem for the case where ε is very small. In other words, we determine the homogenization limit of the periodic traveling wave $U^\varepsilon(x,t)$ and the limit of its average speed c_ε as $\varepsilon \to 0$. Since $U^\varepsilon(x, t+\tau)$ is also a periodic traveling wave for any constant $\tau \in \mathbb{R}$, this may create ambiguity in the definition of U^ε. In order to avoid such ambiguity, hereafter we normalize $U^\varepsilon(x,t)$ so that

$$U^\varepsilon(0,0) = 0 \tag{2.1}$$

for every $\varepsilon > 0$. Our result is the following:

Theorem 2.3. (Homogenization limit) *Assume the slope condition* (1.5) *and that* $AH \geq \sin\alpha$. *Let* $U^\varepsilon(x,t)$ *be the periodic traveling wave of* (1.3)-(1.4).

(i) *For small* ε, *the average speed* c_ε *of* U^ε *satisfies*

$$c_0 < c_\varepsilon < c_0 + M\sqrt{\varepsilon}, \tag{2.2}$$

where c_0 *is the constant determined uniquely by*

$$H = \int_0^\alpha \frac{\cos s}{A - c_0 \cos s} ds, \tag{2.3}$$

and M *is a positive constant independent of* ε. *Moreover* c_0 *satisfies*

$$0 < c_0 < A, \quad \frac{\partial c_0}{\partial \alpha} < 0, \quad \frac{\partial c_0}{\partial A} > 0, \quad \frac{\partial c_0}{\partial H} > 0 \tag{2.4}$$

if $AH > \sin\alpha$, *and* $c_0 = 0$ *if* $AH = \sin\alpha$.

(ii) $U^\varepsilon(x,t)$ *converges to* $\varphi(x;c_0) + c_0 t$ *in* $C^{2,1}_{\mathrm{loc}}((-H,H)\times\mathbb{R})$ *as* $\varepsilon \to 0$, *where* $\varphi(x;c_0)$ *is the solution of*

$$c_0 = \frac{\varphi_{xx}}{1+\varphi_x^2} + A\sqrt{1+\varphi_x^2}, \tag{2.5}$$

$$\varphi(0) = 0, \quad \varphi_x(0) = 0. \tag{2.6}$$

Remark 2.4. Equation (2.5) is derived by setting $u = \varphi(x;c_0) + c_0 t$ in equation (1.3). Therefore the function $\varphi(x;c_0) + c_0 t$ is a traveling wave of (1.3) in Ω_0 with constant speed c_0 and constant profile $\varphi(x;c_0)$. As we will see in Lemma 5.1, if $c_0 > 0$, by introducing a parameter $\theta := -\arctan\varphi_x$, the problem (2.5)-(2.6) can be solved as follows:

$$\varphi(x(\theta;c_0);c_0) = -\frac{1}{c_0}\log\left(\frac{A - c_0\cos\theta}{A - c_0}\right), \tag{2.7}$$

$$x(\theta;c_0) = \int_0^\theta \frac{\cos s}{A - c_0\cos s} ds. \tag{2.8}$$

On the other hand, (2.5)-(2.6) with $c_0 = 0$ can be solved as

$$\varphi(x;0) = -\frac{1}{A}\left(1 - \sqrt{1 - A^2x^2}\right), \tag{2.9}$$

which coincides with the limit of $\varphi(x(\theta;c_0);c_0)$ as $c_0 \to 0$. Thus the condition (2.3) can be expressed as $x(\alpha;c_0) = H$, or, equivalently, that

$$\varphi_x(H;c_0) = -\tan\alpha.$$

This means that $\varphi(x;c_0) + c_0 t$ is a traveling wave in Ω_0 (or a stationary front if $c_0 = 0$) with contact angle $\theta^* = \frac{\pi}{2} - \alpha$ (see Figure 3). In other words,

$$(2.3) \iff \theta^* = \frac{\pi}{2} - \alpha.$$

Summarizing, Theorem 2.3 states that U^ε converges to a traveling wave in Ω_0 whose contact angle is $\theta^* = \frac{\pi}{2} - \alpha$, and that its speed c_0 is determined uniquely by this contact angle and the constants A, H through the identity (2.3).

3. **Local and global existence.** In this section we present basic existence results for solutions of (1.3)-(1.4) and derive uniform bounds on the derivatives of solutions.

3.1. **Change of variables.** In studying the existence of solutions for equation (1.3)-(1.4), it is convenient to introduce new coordinates that convert the domain into a flat cylinder. More precisely, we make a change of variables $(x, y) \mapsto (\xi, \eta)$, which gives a diffeomorphism $\overline{\Omega}_\varepsilon \to \overline{D}$, where

$$D := \{(\xi, \eta) \in \mathbb{R}^2 \mid -H < \xi < H,\ -\infty < \eta < \infty\}.$$

Here the functions $\xi(x, y)$ and $\eta(x, y)$ are to be specified later. With these new coordinates, the function $y = u(x, t)$ is expressed as $\eta = v(\xi, t)$, where the new unknown $v(\xi, t)$ is determined by the relation

$$\eta\,(x, u(x, t)) = v\,(\xi(x, u(x, t)), t) \tag{3.1}$$

for (x, t) with $(x, u(x, t)) \in \overline{\Omega}_\varepsilon$ and $t \geq 0$. The function $v(\xi, t)$ is well-defined by (3.1) provided that $x \mapsto \xi(x, u(x, t))$ is strictly monotone for each fixed t. We will see later that this monotonicity condition always holds for the class of solutions which we consider. Indeed, there exists a positive constant δ such that

$$\frac{\partial}{\partial x}\xi\,(x, u(x, t)) = \xi_x + \xi_y u_x \geq \delta > 0. \tag{3.2}$$

Once $v(\xi, t)$ is defined, then substituting it into the relation $y = u(x, t)$ yields

$$Y\,(\xi, v(\xi, t)) = u\,(X(\xi, v(\xi, t)), t)\,, \tag{3.3}$$

where the map $(\xi, \eta) \mapsto (X(\xi, \eta), Y(\xi, \eta)) : \overline{D} \to \overline{\Omega}_\varepsilon$ is the inverse map of $(x, y) \mapsto (\xi(x, y), \eta(x, y))$. The expression (3.3) gives a formula for recovering the original solution $u(x, t)$ from $v(\xi, t)$. In order for u to be smoothly dependent on v, we need the map $\xi \mapsto X(\xi, v(\xi, t))$ to be one-to-one for each fixed t. As we will see later, this is true since we have

$$\frac{\partial}{\partial \xi}X\,(\xi, v(\xi, t)) = X_\xi + X_\eta v_\xi \geq \delta_1 > 0 \tag{3.4}$$

for some constant $\delta_1 > 0$. See Lemma 3.6 for details.

3.2. **Iso-thermal coordinates.** Now we specify ξ and η. We adopt the so-called iso-thermal coordinates. First, $\xi(x, y)$ is given as a solution of the following boundary value problem:

$$\begin{cases} \Delta\xi = 0 & \text{in } \Omega_\varepsilon, \\ \quad \xi = -H & \text{on } \partial_-\Omega_\varepsilon, \\ \quad \xi = H & \text{on } \partial_+\Omega_\varepsilon. \end{cases} \tag{3.5}$$

It is easily seen that the problem (3.5) has a unique solution within the class of bounded functions. Next, we define $\eta(x, y)$ to be the conjugate harmonic function of ξ. More precisely, η is characterized by the Cauchy-Riemann relation

$$\begin{cases} \xi_x = \eta_y \\ \xi_y = -\eta_x \end{cases} \quad \text{in } \Omega_\varepsilon. \tag{3.6}$$

Such a function η exists since Ω_ε is a simply connected domain, and it is unique up to addition of a constant. Thus, η is uniquely determined by (3.6) under the following normalization condition:

$$\eta(0, 0) = 0. \tag{3.7}$$

By a moving plane argument with respect to lines that are parallel to the y-axis, we see that

$$\xi_x > 0 \qquad \text{in } \overline{\Omega}_\varepsilon. \tag{3.8}$$

Next we apply a similar reflection argument to a line ℓ_θ that is slightly tilted from the y-axis by an angle θ. More precisely, choose an arbitrary point $(x_0, y_0) \in \Omega_\varepsilon$ and consider a line ℓ_θ which passes through (x_0, y_0) and whose unit normal vector is $\boldsymbol{n}_\theta := (\cos\theta, \sin\theta)$. Denote by Ω_θ^L the portion of Ω_ε lying on the left-hand side of ℓ_θ, and let $(\Omega_\theta^L)^*$ be its reflection with respect to ℓ_θ. Also, denote by $\xi^*(x, y)$ the reflection of $\xi(x, y)$ with respect to ℓ_θ.

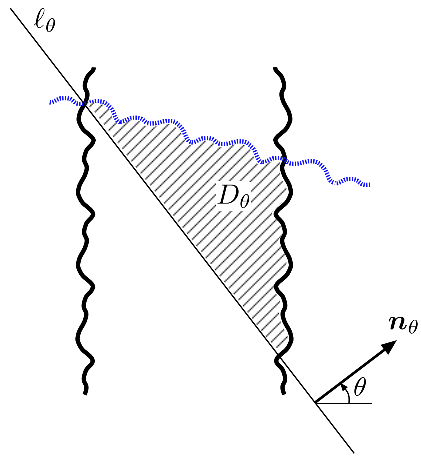

FIGURE 4. Reflection with respect to line ℓ_θ.

If θ is sufficiently small, namely, if $|\theta| < \pi/2 - \max\{\alpha, \beta\}$, then the reflection of the boundary curve $\partial_-\Omega_\varepsilon$ with respect to ℓ_θ does not intersect with $\partial_-\Omega_\varepsilon$ itself except on the line ℓ_θ, and the same is true of the curve $\partial_+\Omega_\varepsilon$. So long as θ has this property, one can see by the maximum principle that

$$\xi^* < \xi \qquad \text{in} \ \ D_\theta := (\Omega_\theta^L)^* \cap \Omega_\varepsilon,$$

since $\xi - \xi^* \geq 0, \ \not\equiv 0$ on the boundary of D_θ. Furthermore, $\xi = \xi^*$ on $\ell_\theta \cap \Omega_\varepsilon$, which is a portion of the boundary of D_θ. Hence, by the Hopf boundary lemma, the normal derivative of ξ on the line segment $\ell_\theta \cap \Omega_\varepsilon$ does not vanish. Thus we have $\boldsymbol{n}_\theta \cdot \nabla\xi(x_0, y_0) > 0$ for $|\theta| < \pi/2 - \max\{\alpha, \beta\}$. Consequently, the angle between the vector $\nabla\xi$ and the x-axis lies in the interval $[-\max\{\alpha, \beta\}, \ \max\{\alpha, \beta\}]$. This implies the following estimate:

$$\left|\frac{\xi_y}{\xi_x}\right| \leq \max\{\tan\alpha, \tan\beta\} = G \qquad \text{in} \ \ \Omega_\varepsilon. \tag{3.9}$$

This and (3.6) yield

$$\left|\frac{\eta_x}{\eta_y}\right| \leq G \qquad \text{in} \ \ \Omega_\varepsilon. \tag{3.10}$$

Before closing this subsection, let us list up basic properties and some useful identities concerning the map $(\xi(x, y), \eta(x, y))$ and its inverse $(X(\xi, \eta), Y(\xi, \eta))$. First we note that the map $(x, y) \mapsto (\xi(x, y), \eta(x, y)) : \overline{\Omega}_\varepsilon \to \overline{D}$ is one-to-one, since $\xi_x > 0$, $\eta_y > 0$ in $\overline{\Omega}_\varepsilon$ (see (3.8)). It is also easily seen that this map is onto, since the level curves of $\xi(x, y)$ and those of $\eta(x, y)$ are smooth curves stretching vertically and horizontally, respectively.

Next we recall that the Cauchy-Riemann relation (3.6) implies

$$\det \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} = \xi_x^2 + \xi_y^2 = \eta_x^2 + \eta_y^2.$$

This, together with the fact that $\xi_x > 0$ in $\overline{\Omega}_\varepsilon$, implies

$$\det \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} > 0 \qquad \text{in } \overline{\Omega}_\varepsilon. \tag{3.11}$$

Therefore $(\xi(x,y), \eta(x,y))$ is a diffeomorphism from $\overline{\Omega}_\varepsilon$ to $\overline{D}$, so its inverse map $(X(\xi,\eta), Y(\xi,\eta))$ is well-defined. The same Cauchy-Riemann relation implies

$$\nabla\xi \cdot \nabla\eta = 0 \qquad \text{in } \Omega_\varepsilon. \tag{3.12}$$

This means that the level curves of ξ and those of η intersect orthogonally everywhere. In particular, the level curves of η meet the boundary curves $\partial_-\Omega_\varepsilon$ and $\partial_+\Omega_\varepsilon$ perpendicularly.

As regards the derivatives of X, Y, clearly the following identity holds:

$$\begin{pmatrix} X_\xi & X_\eta \\ Y_\xi & Y_\eta \end{pmatrix} \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.13}$$

Consequently X and Y also satisfy the Cauchy-Riemann relation:

$$\begin{cases} X_\xi = Y_\eta \\ X_\eta = -Y_\xi \end{cases} \qquad \text{in } D. \tag{3.14}$$

We also note that the following holds for the quantities in (3.2) and (3.4):

$$(\xi_x + \xi_y\, u_x)(X_\xi + X_\eta\, v_\xi) = 1. \tag{3.15}$$

To see this, let us differentiate (3.1) by x, and (3.3) by ξ:

$$\begin{aligned} \eta_x + \eta_y u_x &= v_\xi(\xi_x + \xi_y u_x), \\ Y_\xi + Y_\eta v_\xi &= u_x(X_\xi + X_\eta v_\xi). \end{aligned}$$

Consequently

$$X_\xi + X_\eta v_\xi = X_\xi + X_\eta \frac{\eta_x + \eta_y u_x}{\xi_x + \xi_y u_x} = \frac{X_\xi(\xi_x + \xi_y u_x) + X_\eta(\eta_x + \eta_y u_x)}{\xi_x + \xi_y u_x}.$$

By (3.13), the numerator of the right-hand side is equal to 1.

3.3. **Equation in the new coordinates.** Let us now rewrite the equation (1.3) using the new coordinates ξ, η and the new unknown $v(\xi,t)$. Differentiating the expression

$$Y\left(\xi, v(\xi,t)\right) = u\left(X(\xi, v(\xi,t)), t\right)$$

twice by ξ and once by t, and using (3.14), we obtain

$$\begin{aligned} u_x &= \frac{-X_\eta + X_\xi v_\xi}{X_\xi + X_\eta v_\xi}, \\ u_{xx} &= \frac{1}{(X_\xi + X_\eta v_\xi)^3} \left\{ \frac{(1+v_\xi^2)(J_\eta - J_\xi v_\xi)}{2J^2} + \frac{v_{\xi\xi}}{J} \right\}, \\ u_t &= \frac{v_t}{(X_\xi + X_\eta v_\xi)J}, \end{aligned}$$

where

$$J(\xi,\eta) := \frac{1}{X_\xi^2(\xi,\eta) + X_\eta^2(\xi,\eta)} \ \left(= \xi_x^2(X,Y) + \xi_y^2(X,Y) \right).$$

Therefore, we find that (1.3) is converted into the following equation:

$$v_t = d(\xi, v, v_\xi) v_{\xi\xi} + f(\xi, v, v_\xi), \tag{3.16}$$

where

$$d(\xi, \eta, p) := \frac{J(\xi, \eta)}{1 + p^2},$$
$$f(\xi, \eta, p) := -\frac{1}{2} J_\xi(\xi, \eta) p + \frac{1}{2} J_\eta(\xi, \eta) + A\sqrt{J(\xi, \eta)(1 + p^2)}.$$

In deriving equation (3.16), we need to assume condition (3.4), the validity of which will be verified later in Lemma 3.6. Note also that, because of the orthogonality (3.12), the boundary condition (1.4) is converted to

$$v_\xi(-H, t) = v_\xi(H, t) = 0. \tag{3.17}$$

In accordance with Definition 1.1, a classical solution of this reduced problem is defined as follows:

Definition 3.1. A function $v(\xi, t)$ defined in $[-H, H] \times [0, T)$ is said to be a classical solution of (3.16)-(3.17) in the time interval $[0, T)$ if v, v_ξ are continuous in $[-H, H] \times [0, T)$, $v_{\xi\xi}$, v_t are continuous in $(-H, H) \times (0, T)$ and if v satisfies (3.16)-(3.17) in $(-H, H) \times (0, T)$. It is called a time-global classical solution if $T = +\infty$.

Remark 3.2. In what follows, when we say v is a classical solution in the closed time-interval $[0, T]$, we mean that v is a classical solution in $[0, T)$ and that v, v_ξ $v_{\xi\xi}$, v_t are continuous up to $t = T$.

Equation (3.16) is a quasilinear parabolic equation whose coefficients are smooth functions of (ξ, v, v_ξ). Furthermore, the coefficients $J(\xi, v)$, $J_\xi(\xi, v)$, $J_\eta(\xi, v)$ are bounded, thanks to the following lemma:

Lemma 3.3. *The functions $X(\xi, \eta)$ and $Y(\xi, \eta)$ are smooth in $\overline{D}$. Moreover, there exists a constant $p_\varepsilon \in (0, \varepsilon)$ such that*

$$X(\xi, \eta + p_\varepsilon) = X(\xi, \eta), \quad Y(\xi, \eta + p_\varepsilon) = Y(\xi, \eta) + \varepsilon \qquad \text{in } \overline{D}. \tag{3.18}$$

Consequently, the derivatives of X, Y, hence those of $J(\xi, \eta)$, are all periodic in η with period p_ε.

Proof. Since Ω_ε is invariant with respect to the translation $(x, y) \mapsto (x, y + \varepsilon)$, the function $\xi(x, y + \varepsilon)$ is also a solution of (3.5), hence

$$\xi(x, y + \varepsilon) = \xi(x, y) \qquad \text{in } \Omega_\varepsilon \tag{3.19}$$

by the uniqueness of the solution. Therefore both $\eta(x, y + \varepsilon)$ and $\eta(x, y)$ are the conjugate harmonic function of $\xi(x, y)$. Consequently

$$\eta(x, y + \varepsilon) = \eta(x, y) + p_\varepsilon \qquad \text{in } \Omega_\varepsilon \tag{3.20}$$

for some constant p_ε. Let us show that $0 < p_\varepsilon < \varepsilon$. Put

$$\Delta_\varepsilon := \Omega_\varepsilon \cap \{0 < y < \varepsilon\}, \quad \widetilde{\Delta}_\varepsilon := \Delta_\varepsilon \cap \{-H < x < H\}.$$

Since $\xi = -H$ on $\partial_-\Omega_\varepsilon$ and $\xi = H$ on $\partial_+\Omega_\varepsilon$, we obtain

$$\iint_{\Delta_\varepsilon} \xi_x \, dxdy = 2H\varepsilon.$$

Similarly, we have

$$\iint_{\widetilde{\Delta}_\varepsilon} \eta_y \, dxdy = 2Hp_\varepsilon.$$

Considering that $\eta_y = \xi_x > 0$ and that $\Delta_\varepsilon \supset \widetilde{\Delta}_\varepsilon$, we obtain $2H\varepsilon > 2Hp_\varepsilon > 0$, hence $\varepsilon > p_\varepsilon > 0$. From (3.19) and (3.20) one can easily deduce (3.18). $\square$

At the end of this subsection we derive useful gradient bounds which will be needed in the next subsection to prove the existence of a time-global solution. As mentioned in the introduction, we will assume (1.5) throughout the present paper. This condition is equivalent to

$$\max\{\tan\alpha,\, \tan\beta\} < 1.$$

Hereafter we denote by Q_T the space-time region on which u is defined:

$$Q_T := \{(x,t) \mid \zeta_-(t) < x < \zeta_+(t),\ 0 < t < T\}.$$

Lemma 3.4 (A priori gradient bound for u)**.** *Let u be a classical solution of* (1.3)-(1.4) *in the interval* $[0,T]$*, whose initial data* $u_0(x)$ *satisfies*

$$|u_0'(x)| \le G \qquad \text{for } \ \zeta_-(0) \le x \le \zeta_+(0). \tag{3.21}$$

Then

$$|u_x(x,t)| \le G \qquad \text{for } \ (x,t) \in \overline{Q}_T. \tag{3.22}$$

Furthermore, inequality (3.2) *holds everywhere in* $\overline{Q}_T$*.*

Proof. The function $w := u_x(x,t)$ satisfies the following equation:

$$\begin{cases} w_t = \dfrac{w_{xx}}{1+w^2} - \dfrac{2ww_x^2}{(1+w^2)^2} + A\dfrac{ww_x}{\sqrt{1+w^2}} & \text{in } Q_T, \\ w(\zeta_\pm(t),t) = \mp g'\left(\varepsilon^{-1}u(\zeta_\pm(t),t)\right), & \text{for } \ t \in (0,T). \end{cases}$$

This equation can be written in the form

$$w_t = a(x,t)w_{xx} + b(x,t)w_x,$$

with $a(x,t) > 0$. By the maximum principle, the maximum of $|w|$ on $\overline{Q}_T$ is attained on the parabolic boundary of Q_T. Thus (3.22) follows from (3.21) and (1.5). Next, combining this gradient estimate and (3.9), we obtain

$$\xi_x + \xi_y u_x \ge \xi_x\left(1 - \left|\frac{\xi_y}{\xi_x}\, u_x\right|\right) \ge \xi_x\,(1 - G^2).$$

Since $\min \xi_x > 0$ because of the periodicity of $\xi(x,y)$ in y, and since $1 - G^2 > 0$, the inequality (3.2) holds by setting $\delta := (1-G^2)\, \min_{\overline{\Omega}_\varepsilon} \xi_x$. $\square$

Definition 3.5. By an admissible function for equation (1.3)-(1.4), we mean a C^1 function $u_0(x)$ defined on some interval $\zeta_- \le x \le \zeta_+$ such that

(a) $(x, u_0(x)) \in \Omega_\varepsilon$ for all $\zeta_- < x < \zeta_+$;
(b) $(\zeta_\pm,\, u_0(\zeta_\pm)) \in \partial_\pm\Omega_\varepsilon$; and the graph of u_0 intersects $\partial_\pm\Omega_\varepsilon$ perpendicularly;
(c) $|u_0'(x)| \le G$ for all $\zeta_- \le x \le \zeta_+$.

We denote by C^1_{ad} the set of all admissible functions.

If $u_0 \in C^1_{\mathrm{ad}}$, then, as we have seen in the proof of the above lemma, $\xi_x(x,u_0) + \xi_y(x,u_0)u_0'(x) > 0$. Consequently, a function $v_0(\xi),\ -H \le \xi \le H$, is well-defined from the expression

$$\eta\,(x, u_0(x)) = v_0\,(\xi(x,u_0(x)))\,.$$

We denote by $\widetilde{C}^1_{\mathrm{ad}}$ the set of all the functions $v_0(\xi)$ obtained this way.

Lemma 3.6 (A priori gradient bound for v). *Let v be a classical solution of* (3.16)-(3.17) *in the time interval* $[0,T]$ *with initial data* $v_0 \in \widetilde{C}^1_{\mathrm{ad}}$. *Then inequality* (3.4) *holds everywhere in* $[-H,H]\times[0,T]$, *hence* $v(\cdot,t)\in\widetilde{C}^1_{\mathrm{ad}}$ *for every* $t\in[0,T]$. *Furthermore,*

$$|v_\xi(\xi,t)| \le \widetilde{G} := \frac{2G}{1-G^2} \qquad \text{for} \ \ (\xi,t)\in[-H,H]\times[0,T]. \tag{3.23}$$

Proof. Let us show that (3.4) holds with the following choice of δ_1:

$$\delta_1 := \frac{1}{(1+G^2)\max_{\overline{\Omega}_\varepsilon}\xi_x}.$$

Suppose the conclusion does not hold. Then there exist $t_0\in(0,T]$ such that

$$X_\xi + X_\eta v_\xi \begin{cases} >0 & \text{for} \ \ 0\le t\le t_0, \\ <\delta_1 & \text{for} \ \ t=t_0. \end{cases}$$

Since $X_\xi + X_\eta v_\xi > 0$ in the interval $[0,t_0]$, a function $u(x,t)$ is determined uniquely from (3.3), and it is a classical solution of (1.3)-(1.4). The assumption $v_0\in\widetilde{C}^1_{\mathrm{ad}}$ implies that the initial data $u(x,0)$ belongs to C^1_{ad}. Therefore, by Lemma 3.4, u satisfies (3.22). Combining this and (3.9), we obtain

$$\xi_x + \xi_y u_x = \xi_x\left(1+\frac{\xi_y}{\xi_x}u_x\right) \le (1+G^2)\max_{\overline{\Omega}_\varepsilon}\xi_x$$

for $0\le t\le t_0$. This and (3.15) yields

$$X_\xi + X_\eta v_\xi \ge \delta_1 \qquad \text{for} \ \ t=t_0,$$

contradicting our earlier assumption. This contradiction shows that (3.4) holds for all $t\in[0,T]$. Therefore, the solution $u(x,t)$ of (1.3)-(1.4) corresponding to $v(\xi,t)$ is defined for all $t\in[0,T]$ and satisfies (3.22). This means $v(\cdot,t)\in\widetilde{C}^1_{\mathrm{ad}}$ for all $t\in[0,T]$. Furthermore,

$$|v_\xi| = \left|\frac{\eta_x+\eta_y u_x}{\xi_x+\xi_y u_x}\right| = \left|\frac{u_x-\frac{\xi_y}{\xi_x}}{1+\frac{\xi_y}{\xi_x}u_x}\right| \le \frac{2G}{1-G^2}.$$

The lemma is proved. □

Corollary 3.7. *Let v_0 be an element of $\widetilde{C}^1_{\mathrm{ad}}$ which corresponds to $u_0\in C^1_{\mathrm{ad}}$. If there exists a time-global classical solution v for equation* (3.16)-(3.17) *with initial data v_0, then there exists a time-global classical solution u for equation* (1.3)-(1.4) *with initial data u_0. Moreover, $u(\cdot,t)$ belongs to C^1_{ad} for each $t>0$.*

3.4. **Comparison principles.**

Definition 3.8. A function $\hat{v}\in C^{2,1}([-H,H]\times[0,T])$ is called a *lower solution* of (3.16)-(3.17) on the interval $0\le t\le T$ if

$$\begin{cases} \hat{v}_t \le d(\xi,\hat{v},\hat{v}_\xi)\hat{v}_{\xi\xi} + f(\xi,\hat{v},\hat{v}_\xi), & (\xi,t)\in(-H,H)\times(0,T), \\ \pm\hat{v}_\xi(\pm H,t)\le 0, & t\in(0,T). \end{cases}$$

A function $\hat{v}\in C^{2,1}([-H,H]\times[0,T])$ is called an *upper solution* of (3.16)-(3.17) if the reversed inequalities hold.

The following proposition follows easily from the maximum principle:

Proposition 3.9 (Comparison principle for v)**.** *Let v^- and v^+ be a lower and an upper solution of* (3.16)-(3.17) *on the interval* $0 \le t \le T$*, respectively. Suppose that* $v^-(\xi,0) \le v^+(\xi,0)$ *for* $\xi \in [-H,H]$*. Then*

$$v^-(\xi,t) \le v^+(\xi,t) \quad \textit{for } (\xi,t) \in [-H,H] \times [0,T].$$

Furthermore, if $v^- \not\equiv v^+$ *then*

$$v^-(\xi,t) < v^+(\xi,t) \quad \textit{for } (\xi,t) \in [-H,H] \times (0,T].$$

Corollary 3.10 (Growth bound on v)**.** *There exists a constant $K_f > 0$, dependent only on A and $\|J\|_{L^\infty}$, $\|J_\eta\|_{L^\infty}$ such that for any classical solution v of* (3.16)-(3.17) *with initial data* $v_0 \in \widetilde{C}^1_{\rm ad}$,

$$\|v(\cdot,t)\|_{L^\infty} \le K_f t + \|v_0\|_{L^\infty}. \tag{3.24}$$

Proof. By the periodicity of J and J_η in η, we have

$$K_f := \sup_{(\xi,\eta)\in[-H,H]\times\mathbb{R}} |f(\xi,\eta,0)| < +\infty.$$

Consequently, $v^+(t) = K_f t + \|v_0\|_{L^\infty}$ is an upper solution of (3.16)-(3.17), while $v^-(t) = -K_f t - \|v_0\|_{L^\infty}$ is a lower solution of (3.16)-(3.17). Moreover $v^-(0) \le v_0(\xi) \le v^+(0)$ for $\xi \in [-H,H]$. Hence, by Proposition 3.9, we have

$$v^-(t) \le v(\xi,t) \le v^+(t) \quad \text{for } (\xi,t) \in [-H,H] \times [0,T].$$

This proves the corollary. □

Next we state the comparison principle for solution curves of (1.1).

Definition 3.11. Let $\hat{u}$ be a $C^{2,1}$-function defined for $\hat{\zeta}_-(t) \le x \le \hat{\zeta}_+(t)$, $0 \le t \le T$ such that $(\hat{\zeta}_\pm(t), \hat{u}(\hat{\zeta}_\pm(t),t)) \in \partial_\pm\Omega_\varepsilon$, respectively. Then $\hat{u}$ is called a *lower solution* of (1.3)-(1.4) on the interval $0 \le t \le T$ if

$$\begin{cases} \hat{u}_t \le \dfrac{\hat{u}_{xx}}{1+\hat{u}_x^2} + A\sqrt{1+\hat{u}_x^2}, & \hat{\zeta}_-(t) < x < \hat{\zeta}_+(t), \;\; 0 < t < T, \\ \hat{u}_x(\hat{\zeta}_+(t),t) \le -g_\varepsilon'(\hat{u}(\hat{\zeta}_+(t),t)), & 0 < t < T, \\ \hat{u}_x(\hat{\zeta}_-(t),t) \ge g_\varepsilon'(\hat{u}(\hat{\zeta}_-(t),t)), & 0 < t < T. \end{cases}$$

A function $\hat{u}$ is called an *upper solution* of (1.3)-(1.4) if the reversed inequalities hold.

We easily see that $\hat{u}$ is a lower solution of (1.3)-(1.4) if and only if $\hat{v}$, the expression of $\hat{u}$ in the coordinates (ξ,η,t), is a lower solution of (3.16)-(3.17), provided that $\hat{v}$ is well-defined by (3.1).

Notation. Let $\mathcal{C}$ denote the set of all simple (non-self-intersecting) C^1-curves γ in $\overline{\Omega}_\varepsilon$ such that

(a) The two endpoints of γ lie on $\partial_-\Omega_\varepsilon$ and on $\partial_+\Omega_\varepsilon$, respectively;
(b) Every point of γ except the endpoints lies in Ω_ε.

Each $\gamma \in \mathcal{C}$ divides Ω_ε into two open sets. We denote the one located above γ by $\mathcal{U}(\gamma)$. We then define an order relation in $\mathcal{C}$ by

$$\gamma \preceq \widetilde{\gamma} \;\overset{\text{def}}{\Longleftrightarrow}\; \mathcal{U}(\gamma) \supset \mathcal{U}(\widetilde{\gamma}).$$

We also write $\gamma \ll \widetilde{\gamma}$ if $\gamma \preceq \widetilde{\gamma}$ and $\gamma \cap \widetilde{\gamma} = \emptyset$.

Let u be a C^1 function defined on some interval $\zeta_- \le x \le \zeta_+$ such that $(x, u(x)) \in \Omega_\varepsilon$ for $\zeta_- < x < \zeta_+$ and that $(\zeta_\pm, u(\zeta_\pm)) \in \partial_\pm\Omega_\varepsilon$. Then the graph of u, say $\mathcal{G}(u)$, belongs to $\mathcal{C}$. For such functions u_1 and u_2, we define an order relation between them by

$$u_1 \preceq u_2 \stackrel{\text{def}}{\iff} \mathcal{G}(u_1) \preceq \mathcal{G}(u_2),$$
$$u_1 \ll u_2 \stackrel{\text{def}}{\iff} \mathcal{G}(u_1) \ll \mathcal{G}(u_2).$$

Proposition 3.12 (Comparison principle for u)**.** *Let u_1 and u_2 be a lower solution and an upper solution of* (1.3)-(1.4) *on the interval* $0 \le t \le T$*, respectively. Suppose that* $u_1(\cdot, 0) \preceq u_2(\cdot, 0)$*. Then*

$$u_1(\cdot, t) \preceq u_2(\cdot, t) \quad \textit{for } 0 \le t \le T.$$

Furthermore, if $u_1 \not\equiv u_2$ *then*

$$u_1(\cdot, t) \ll u_2(\cdot, t) \quad \textit{for } 0 < t \le T.$$

Proof. If u_1 and u_2 satisfy the condition (3.22), then they have expressions v_1 and v_2 in the coordinates (ξ, η, t), respectively. In this case, it is clear that $u_1(\cdot, t) \preceq u_2(\cdot, t)$ (resp. $u_1(\cdot, t) \ll u_2(\cdot, t)$) if and only if $v_1(\xi, t) \le v_2(\xi, t)$ (resp. $v_1(\xi, t) < v_2(\xi, t)$) for $\xi \in [-H, H]$. Therefore the conclusion follows from Proposition 3.9. The general case is basically the same: the conclusion follows from the strong maximum principle except that the Hopf boundary lemma has to be applied after an appropriate local change of coordinates near the boundary. We omit the details. □

The above proposition remains true even if the solution curve γ_t is not necessarily the graph of a function u. More precisely, we have:

Proposition 3.13 (Comparison principle for solutions of (1.1))**.** *Let* $\{\gamma_t\}_{t\in[0,T]}$ *and* $\{\widetilde{\gamma}_t\}_{t\in[0,T]} \subset \mathcal{C}$ *be solutions of* (1.1) *which contact* $\partial_\pm\Omega_\varepsilon$ *perpendicularly for* $0 \le t \le T$ *with initial data* γ_0 *and* $\widetilde{\gamma}_0 \in \mathcal{C}$*, respectively. Suppose that* $\gamma_0 \preceq \widetilde{\gamma}_0$*. Then*

$$\gamma_t \preceq \widetilde{\gamma}_t \quad \textit{for } 0 \le t \le T.$$

Furthermore, if $\gamma_0 \neq \widetilde{\gamma}_0$ *then*

$$\gamma_t \ll \widetilde{\gamma}_t \quad \textit{for } 0 < t \le T.$$

The proof of this proposition is similar to that of Proposition 3.9. In fact, by using local coordinates, one can express (1.1) locally as a quasilinear parabolic equation; one can then apply the maximum principle. The details are omitted.

3.5. **Uniform Hölder estimates.** In this subsection we derive uniform Hölder estimates for solutions of (3.16)-(3.17). More precisely, we show that if a classical solution v of (3.16)-(3.17) exists for $0 \le t \le T$ and if $v \in C^{2+\nu, 1+\nu/2}([-H, H] \times (0, T])$, then for any $\delta \in (0, T)$ there exists a constant $C_\delta > 0$ such that

$$\|\tilde{v}(\cdot, t)\|_{C^{2+\nu}([-H,H])} \le C_\delta \quad \text{for } \delta \le t \le T,$$

where $\nu \in (0, 1)$ is a constant to be specified later and C_δ depends on δ but is independent of v and T, and

$$\tilde{v}(\xi, t) := v(\xi, t) - \frac{1}{2H}\int_{-H}^{H} v(z, t)dz.$$

In the rest of this subsection, C_δ denotes a positive constant dependent on $\delta > 0$ but independent of v and T, whose actual value may differ in different contexts.

Lemma 3.14. *Let $v(\xi,t)$ be a classical solution of* (3.16)-(3.17) *in the time interval* $[0,T]$ *and define*

$$h(t) := \frac{1}{2H}\int_{-H}^{H} v(z,t)dz.$$

Then there exists a positive constant M independent of v and T such that

$$\sup_{0\le t\le T} |h'(t)| \le M.$$

Proof. Since equation (3.16) can be written in the following divergence form

$$v_t = \left(D(\xi,v,v_\xi)\right)_\xi + F(\xi,v,v_\xi),$$

with

$$\begin{aligned} D(\xi,\eta,p) &:= J(\xi,\eta)\arctan p,\\ F(\xi,\eta,p) &:= f(\xi,\eta,p) - (J_\xi(\xi,\eta) + J_\eta(\xi,\eta)p)\arctan p, \end{aligned}$$

we have

$$h'(t) = \frac{1}{2H}\int_{-H}^{H} F(z,v(z,t),v_\xi(z,t))dz.$$

Therefore the assertion of the lemma follows from Lemma 3.6 and the fact that F is periodic in η. □

Lemma 3.15. *Let $v(\xi,t)$ be a classical solution of* (3.16)-(3.17) *in the time interval* $[0,T]$. *Then there exists a positive constant $\nu\in(0,1)$ such that*

$$\|\tilde v\|_{C^{\nu,\nu/2}([-H,H]\times[\delta,T])} \le C_\delta,\;\; \|\tilde v_\xi\|_{C^{\nu,\nu/2}([-H,H]\times[\delta,T])} \le C_\delta \tag{3.25}$$

for any $\delta\in(0,T)$.

Proof. First we extend v to a function $\hat v$ defined on the whole line $\mathbb{R}$ by

$$\begin{aligned} \hat v(\xi,t) &:= \begin{cases} v(\xi,t), & -H\le\xi\le H,\\ v(2H-\xi,t), & H<\xi\le 3H, \end{cases}\\ \hat v(\xi+4H,t) &= \hat v(\xi,t), \quad (\xi,t)\in\mathbb{R}\times[0,T]. \end{aligned}$$

Then $\hat v$ solves

$$\hat v_t = \hat d(\xi,\hat v,\hat v_\xi)\hat v_{\xi\xi} + \hat f(\xi,\hat v,\hat v_\xi), \quad (\xi,t)\in\mathbb{R}\times(0,T),$$

where

$$\begin{aligned} \hat d(\xi,\eta,p) &:= \begin{cases} d(\xi,\eta,p), & -H\le\xi\le H,\\ d(2H-\xi,\eta,-p), & H<\xi\le 3H, \end{cases}\\ \hat d(\xi+4H,\eta,p) &= \hat d(\xi,\eta,p), \end{aligned}$$

and $\hat f$ is the extension of f defined in the same way as $\hat d$. In what follows, we write v, d, f instead of $\hat v$, $\hat d$, $\hat f$ for simplicity. Note that, by the boundary conditions $v_\xi(\pm H,t)=0$, the function v_ξ is continuous in $\mathbb{R}\times(0,T)$. Moreover, $w := v_\xi$ is a weak solution of

$$w_t = \left(P(\xi,t,w,w_\xi)\right)_\xi,$$

where $P(\xi,t,w,p) := d(\xi,v(\xi,t),w)p + f(\xi,v(\xi,t),w)$. Recalling the fact that $|w|$ is uniformly bounded by the constant $\widetilde G$, we obtain

$$|P(\xi,t,w,p)| \le K_1|p| + K_2, \quad P(\xi,t,w,p)p \ge K_3|p|^2 - K_4$$

for some positive constants K_j $(j = 1,2,3,4)$ independent of T. Hence, applying the interior Hölder estimates for quasilinear parabolic equations of divergence form (see [13, Theorem 2.2]) to the above equation for w, we see that

$$\|v_\xi\|_{C^{\mu,\mu/2}(\mathbb{R}\times[\delta,T])} = \|\tilde{v}_\xi\|_{C^{\mu,\mu/2}(\mathbb{R}\times[\delta,T])} \le C_\delta$$

for any $\delta \in (0,T)$, where the constant $\mu \in (0,1)$ does not depend on δ, v and T.

Next we derive the Hölder estimates for $\tilde{v}$. We note that $|\tilde{v}(\xi,t)| \le 2\widetilde{G}H$ for $(\xi,t) \in \mathbb{R}\times[0,T]$ and that $\tilde{v}$ satisfies

$$\tilde{v}_t = \left(D(\xi,\tilde{v}+h(t),\tilde{v}_\xi)\right)_\xi + F(\xi,\tilde{v}+h(t),\tilde{v}_\xi) - h'(t).$$

Here we extend D and F for all $\xi\in\mathbb{R}$ in the following way:

$$D(\xi,\eta,p) = D(2H-\xi,\eta,p),\ \ F(\xi,\eta,p) = F(2H-\xi,\eta,-p), \quad \text{for } H<\xi\le 3H,$$
$$D(\xi+4H,\eta,p) = D(\xi,\eta,p),\ \ F(\xi+4H,\eta,p) = F(\xi,\eta,p).$$

By Lemmas 3.3, 3.6 and 3.14, there exist positive constants M_j $(j=1,2,3)$ independent of T such that

$$\begin{aligned} |D(\xi,\eta+h(t),p)| &\le M_1|p|,\\ D(\xi,\eta+h(t),p)p &\ge M_2|p|^2,\\ |F(\xi,\eta+h(t),p)-h'(t)| &\le M_3(1+|p|), \end{aligned}$$

for all $(\xi,t,\eta,p)\in\mathbb{R}\times[0,T]\times\mathbb{R}\times[-\widetilde{G},\widetilde{G}]$. Again, applying the interior Hölder estimates for quasilinear parabolic equations ([13, Theorem 2.2]), we see that

$$\|\tilde{v}\|_{C^{\tilde{\mu},\tilde{\mu}/2}(\mathbb{R}\times[\delta,T])} \le C_\delta$$

for any $\delta>0$, where the constant $\tilde{\mu}\in(0,1)$ does not depend on δ, v and T.

Letting $\nu = \min\{\mu,\tilde{\mu}\}\in(0,1)$, we obtain (3.25). □

Corollary 3.16. *Let the assumptions of Lemma* 3.15 *hold. Then*

$$\|v(\cdot,t)\|_{C^{1+\nu}([-H,H])} \le C_\delta + K_f t + \|v_0\|_{L^\infty}$$

for $\delta\le t\le T$, *where* K_f *is the constant in Corollary* 3.10.

Proof. The assertion follows immediately from (3.24) and (3.25). □

Lemma 3.17. *Let* $v(\xi,t)$ *be a classical solution of* (3.16)-(3.17) *in the time interval* $[0,T]$ *and let* $\nu\in(0,1)$ *be the constant in Lemma* 3.15. *If* $v\in C^{2+\nu,1+\nu/2}([-H,H]\times(0,T])$, *then for any* $\delta\in(0,T)$ *we have*

$$\|\tilde{v}\|_{C^{2+\nu,1+\nu/2}([-H,H]\times[\delta,T])} \le C_\delta, \tag{3.26}$$
$$\|v\|_{C^{2+\nu,1+\nu/2}([-H,H]\times[\delta,T])} \le C_\delta + K_f T + \|v_0\|_{L^\infty}, \tag{3.27}$$

where K_f *is the constant in Corollary* 3.10 *and* C_δ *is a constant independent of the solution* v.

Proof. We note that $\tilde{v}$ satisfies the equation of the form

$$\tilde{v}_t = a(\xi,t)\tilde{v}_{\xi\xi} + b(\xi,t),$$

where

$$a(\xi,t) = d(\xi,\tilde{v}(\xi,t)+h(t),\tilde{v}_\xi(\xi,t)),$$
$$b(\xi,t) = f(\xi,\tilde{v}(\xi,t),\tilde{v}_\xi(\xi,t)) - \frac{1}{2H}\int_{-H}^{H} F(z,\tilde{v}(z,t)+h(t),\tilde{v}_\xi(z,t))dz.$$

Since $v_\xi(\pm H, t) = 0$, the functions a and b are continuous in $\mathbb{R} \times [0, T]$. Moreover, by Lemmas 3.14 and 3.15, for any $\delta \in (0, T)$, the quantities $\|a\|_{C^{\nu,\nu/2}(\mathbb{R}\times[\delta/2,T])}$ and $\|b\|_{C^{\nu,\nu/2}(\mathbb{R}\times[\delta/2,T])}$ are bounded by some positive constant that depends on δ. Hence the interior a priori estimates ([6, Theorem 8.11.1]) implies

$$\|\tilde{v}\|_{C^{2+\nu,1+\nu/2}(\mathbb{R}\times[\delta,T])} \le C_\delta.$$

By Corollary 3.10, Lemmas 3.14, 3.15 and the fact that

$$h'(t) = \frac{1}{2H}\int_{-H}^{H} F(z, \tilde{v}(z,t) + h(t), \tilde{v}_\xi(z,t))dz,$$

we have

$$\|h\|_{C^{1+\nu/2}([\delta,T])} \le K_f T + \|v_0\|_{L^\infty} + C_\delta.$$

Therefore,

$$\begin{aligned}\|v\|_{C^{2+\nu,1+\nu/2}([-H,H]\times[\delta,T])} &\le \|\tilde{v}\|_{C^{2+\nu,1+\nu/2}([-H,H]\times[\delta,T])} + \|h\|_{C^{1+\nu/2}([\delta,T])}\\ &\le C_\delta + K_f T + \|v_0\|_{L^\infty}.\end{aligned}$$

The lemma is proved. □

3.6. **Existence of time-global solutions.** Now we are ready to prove the existence of global solutions of (1.3)-(1.4).

Theorem 3.18. *Let $0 < \lambda < 1$. For any $v_0 \in \widetilde{C}^1_{\rm ad} \cap C^{1+\lambda}([-H, H])$, there exists a time-global classical solution $v(\xi, t)$ of (3.16)-(3.17) with initial data v_0. Moreover, $v \in C^{2+\nu,1+\nu/2}([-H, H] \times [\delta, T])$ for any $0 < \delta < T$, where $\nu \in (0, 1)$ is the constant in Lemma 3.15. If, in addition, $v_0 \in \widetilde{C}^1_{\rm ad} \cap C^2([-H, H])$, then $v_{\xi\xi}$, v_t are continuous up to $t = 0$.*

Proof. By the theory of abstract quasilinear parabolic equations ([7, Theorem 2.1]), we see that for any initial data $v_0 \in C^{1+\lambda}([-H, H])$ with $\lambda \in (0, 1)$ satisfying the compatibility condition $v_0'(\pm H) = 0$ there exists a unique local-in-time classical solution v of (3.16)-(3.17) in some time interval $0 \le t < T_1$. We set

$$T^* := \sup\{T_1 > 0 \mid v \text{ can be continued as a classical solution up to } t = T_1\}.$$

By Lemma 3.15, $v(\cdot, t) \in C^{1+\nu}([-H, H])$ for some $\nu \in (0, 1)$ and for every $t \in [\delta, T^*)$, where δ is any number in $(0, T^*)$.

In order to prove that $T^* = +\infty$, suppose the contrary: $T^* < +\infty$. Then, using again the local existence result of [7, Theorem 2.1] in the space $C^{1+\nu}([-H, H])$, we see that

$$\lim_{t\nearrow T^*} \|v(\cdot, t)\|_{C^{1+\nu}([-H,H])} = +\infty.$$

This, however, contradicts Corollary 3.16, hence $T^* = +\infty$.

Since $d(\xi, v, v_\xi), f(\xi, v, v_\xi) \in C^{\nu,\nu/2}([-H, H]\times[\delta, T])$ for any $0 < \delta < T$, standard regularity results for linear parabolic equations imply $v \in C^{2+\nu,1+\nu/2}([-H, H] \times [\delta, T])$. Moreover, [7, Theorem 2.1] also implies that $v_{\xi\xi}$, v_t are continuous up to $t = 0$ if $v_0 \in \widetilde{C}^1_{\rm ad} \cap C^2([-H, H])$. □

As mentioned in Corollary 3.7, the above theorem implies the existence of global classical solutions of (1.3)-(1.4).

Theorem 3.19. *Let $0 < \lambda < 1$. For any $u_0 \in C^1_{\rm ad} \cap C^{1+\lambda}$, there exists a time-global classical solution $u(x, t)$ of (1.3)-(1.4) with initial data u_0. Moreover, if $u_0 \in C^1_{\rm ad} \cap C^2$ then u_{xx} and u_t are continuous up to $t = 0$.*

4. **Existence of periodic traveling waves.**

4.1. **Existence of entire solutions.** First we explain the geometric meaning of the assumption $AH \geq \sin\alpha$ in Theorems 2.1 to 2.3:

Lemma 4.1. *If $AH \geq \sin\alpha$, then problem* (1.3)-(1.4) *has no stationary solution for any $\varepsilon > 0$.*

Proof. Suppose that there exists a stationary solution $w(x)$ of (1.3)-(1.4). Then the graph of w is a circular arc of constant curvature $-A$ whose endpoints meet the boundaries $\partial_\pm\Omega_\varepsilon$ perpendicularly. Naturally the radius of this arc is $1/A$. Let $(x_\pm, w(x_\pm))$ be the endpoints on $\partial_\pm\Omega_\varepsilon$, respectively. Then we have $x_+ - x_- > 2H$ by the definition of g.

On the other hand, since $\mp w'(x_\pm) \leq \max g'(y) = \tan\alpha$, a simple geometric observation shows that the central angle of this arc is less than or equal to 2α, which implies

$$x_+ - x_- \leq \frac{2\sin\alpha}{A} \leq 2H.$$

This contradiction proves the lemma. □

Let $\overline{u}$ be the global solution of (1.3)-(1.4) with initial data $\overline{u}(x,0) \equiv 0$. Since problem (1.3)-(1.4) has reflection symmetry with respect to the y-axis, $\overline{u}$ is an even function in x. We denote by $[-\overline{\zeta}(t), \overline{\zeta}(t)]$ the horizontal span of the solution curve $\overline{u}(x,t)$ at each time $t \geq 0$. Then, since $-\tan\beta \leq \overline{u}_x(-\overline{\zeta}(t), t) \leq \tan\alpha$ and $\overline{u}_x(0,t) = 0$ for $t \geq 0$, we see from the maximum principle that

$$-\tan\beta \leq \overline{u}_x(x,t) \leq \tan\alpha \quad \text{ for } \ -\overline{\zeta}(t) \leq x \leq 0,\ t \geq 0.$$

The same inequality holds for $-\overline{u}_x(x,t)$ for $0 \leq x \leq \overline{\zeta}(t)$, $t \geq 0$, hence

$$-\tan\alpha \leq \operatorname{sgn} x \cdot \overline{u}_x(x,t) \leq \tan\beta \tag{4.1}$$

for all $x \in [-\overline{\zeta}(t), \overline{\zeta}(t)]$ and $t \geq 0$. Moreover, since $\overline{u}_t(x,0) \equiv A > 0$, the strong maximum principle yields that

$$\overline{u}_t(x,t) > 0 \quad \text{ for all } x \in [-\overline{\zeta}(t), \overline{\zeta}(t)],\ t \geq 0.$$

Let $\overline{v}$ be the expression of $\overline{u}$ in the coordinates (ξ, η, t). Then

$$\overline{v}_t(\xi,t) > 0, \quad |\overline{v}_\xi(\xi,t)| \leq \widetilde{G} \quad \text{ for all } \ (\xi,t) \in [-H,H] \times [0,+\infty). \tag{4.2}$$

For $t \geq 0$ we set $m(t) := \max_{|\xi| \leq H} \overline{v}(\xi,t)$.

Lemma 4.2. *For each $t \geq 0$, let $k(t)$ be a nonnegative integer satisfying*

$$k(t)p_\varepsilon \leq m(t) - m(0) < (k(t)+1)p_\varepsilon,$$

where p_ε is the period of $J(\xi,\eta)$ in η. Then the following hold:

(i) *$k(t)$ is nondecreasing and diverges to $+\infty$ as $t \to +\infty$.*

(ii) *For any $t, s \geq 0$,*

$$k(t+s) - k(t) \leq \frac{K_f}{p_\varepsilon} s + 1, \tag{4.3}$$

where K_f is the constant in Corollary 3.10.

Proof. (i) Since $\overline{v}_t > 0$, it is easily seen that $k(t)$ is nondecreasing in t. Suppose that $k(t)$ is bounded. Then $\|\overline{v}(\cdot,t)\|_{L^\infty}$ is also bounded. By the same argument as in the proof of Lemma 3.17, we see that for any $\delta > 0$, $\|\overline{v}(\cdot,t)\|_{C^{2+\nu}([-H,H])}$ is bounded for $t \geq \delta$. Therefore, $\overline{v}$ converges to some smooth function as $t \to +\infty$, which is necessarily an stationary solution of (3.16)-(3.17). This contradicts Lemma 4.1.

(ii) By the definition of k and the comparison principle, we have

$$\{k(t+s) - k(t) - 1\}p_\varepsilon \leq m(t+s) - m(t) \leq K_f s$$

for all $t, s \geq 0$. These inequalities imply (4.3). □

Proposition 4.3. *Problem* (3.16)-(3.17) *has an entire solution $V(\xi,t)$ with the following properties:*

(i) $V \in C^{2+\nu,1+\nu/2}([-H,H]\times[-T,T])$ *for any* $T > 0$.
(ii) $V(\cdot,t) \in \widetilde{C}^1_{\mathrm{ad}}$ *for all* $t \in \mathbb{R}$, *hence* $|V_\xi(\xi,t)| \leq \widetilde{G}$ *for all* $(\xi,t) \in [-H,H]\times\mathbb{R}$.
(iii) $\delta_0 := \inf_{(\xi,t)\in[-H,H]\times\mathbb{R}} V_t(\xi,t) > 0$.

Proof. (i) We construct an entire solution V by using a standard renormalization argument. For $n \in \mathbb{N}$ we define $v_n(\xi,t) := \overline{v}(\xi,t+n) - k(n)p_\varepsilon$. Then v_n solves (3.16)-(3.17) for $t \geq -n$ and satisfies

$$\frac{\partial v_n}{\partial t}(\xi,t) > 0, \qquad \left|\frac{\partial v_n}{\partial \xi}\right| \leq \widetilde{G},$$

for all $\xi \in [-H,H]$ and $t \geq -n$.

We fix $T > 0$. Then for any $n > T$ and $t \in [-T,T]$, we have

$$v_n(\xi,-T) \leq v_n(\xi,t) \leq v_n(\xi,T), \quad \xi \in [-H,H].$$

By (4.3),

$$\begin{aligned} v_n(\xi,-T) &= \overline{v}(\xi,n-T) - k(n)p_\varepsilon \geq m(n-T) - 2\widetilde{G}H - k(n)p_\varepsilon \\ &\geq m(0) - \{k(n) - k(n-T)\}p_\varepsilon - 2\widetilde{G}H \geq m(0) - p_\varepsilon - 2\widetilde{G}H - K_f T, \end{aligned}$$

and

$$\begin{aligned} v_n(\xi,T) &= \overline{v}(\xi,n+T) - k(n)p_\varepsilon \leq m(n+T) - k(n)p_\varepsilon \\ &\leq m(0) + p_\varepsilon + \{k(n+T) - k(n)\}p_\varepsilon \leq m(0) + 2p_\varepsilon + K_f T. \end{aligned}$$

Thus there exists a positive constant $\widetilde{K}$ which does not depend on n such that $\|v_n(\cdot,t)\|_{L^\infty} \leq K_f T + \widetilde{K}$ for all $n > T$ and $t \in [-T,T]$. Therefore, the same argument as in Subsection 3.5 implies

$$\|v_n\|_{C^{2+\nu,1+\nu/2}(R_T)} \leq M_T$$

for some positive constant M_T that depends on T but is independent of n. Here we set $R_T := [-H,H]\times[-T,T]$. Consequently, for any $T > 0$, there exist a sequence $n_j(T) \to \infty$ $(j \to \infty)$ and a function $V_T \in C^{2+\nu,1+\nu/2}(R_T)$ such that

$$\lim_{j\to\infty} \|v_{n_j(T)} - V_T\|_{C^{2,1}(R_T)} = 0. \tag{4.4}$$

Hence, by Cantor's diagonal argument, one can find a subsequence $\{l_j\}_j$ and a function $V(\xi,t)$ defined on $[-H,H]\times\mathbb{R}$ such that v_{l_j} converges to V as $j \to \infty$ in $C^{2,1}(R_T)$ for any $T > 0$. This implies that V is an entire solution of (3.16)-(3.17) and that $V \in C^{2+\nu,1+\nu/2}(R_T)$ for any $T > 0$.

(ii) Let $t \in \mathbb{R}$ be fixed. As in the proof of Lemma 3.6,

$$X_\xi(\xi, v_n(\xi,t)) + X_\eta(\xi, v_n(\xi,t))(v_n)_\xi(\xi,t) \geq \delta_1$$

for all $n > |t|$ and $\xi \in [-H, H]$. Since v_{l_j} converges to V in $C^2([-H,H])$ as $j \to \infty$, we have

$$X_\xi(\xi, V(\xi,t)) + X_\eta(\xi, V(\xi,t))V_\xi(\xi,t) \geq \delta_1.$$

This implies that the function $U(x,t)$ defined on $[\zeta_U^-(t), \zeta_U^+(t)] \times \mathbb{R}$ by the expression

$$U(X(\xi, V(\xi,t)), t) = Y(\xi, V(\xi,t))$$

solves (1.3) for all $t \in \mathbb{R}$, where $\zeta_U^\pm(t) = X(\pm H, V(\pm H, t))$. In order to prove the statement we shall prove $U(\cdot,t) \in C^1_{\mathrm{ad}}$ for any $t \in \mathbb{R}$.

Let $u_n(x,t)$ be the expression of v_n in the coordinates (x,y,t). Then $u_n(\cdot,t) \in C^1_{\mathrm{ad}}$ for any $t \in \mathbb{R}$. Setting $x = X(\xi, V(\xi,t))$ and $x_j = X(\xi, v_{l_j}(\xi,t))$ for $(\xi,t) \in [-H,H] \times \mathbb{R}$, we see that $x_j \to x$ as $j \to \infty$ and that

$$\begin{aligned}
u_{l_j}(x_j,t) &= Y(\xi, v_{l_j}(\xi,t)) \\
&\to Y(\xi, V(\xi,t)) = U(x,t), \\
\frac{\partial u_{l_j}}{\partial x}(x_j,t) &= \frac{Y_\xi(\xi, v_{l_j}(\xi,t)) + Y_\eta(\xi, v_{l_j}(\xi,t))(v_{l_j})_\xi(\xi,t)}{X_\xi(\xi, v_{l_j}(\xi,t)) + X_\eta(\xi, v_{l_j}(\xi,t))(v_{l_j})_\xi(\xi,t)} \\
&\to \frac{Y_\xi(\xi, V(\xi,t)) + Y_\eta(\xi, V(\xi,t))V_\xi(\xi,t)}{X_\xi(\xi, V(\xi,t)) + X_\eta(\xi, V(\xi,t))V_\xi(\xi,t)} = \frac{\partial U}{\partial x}(x,t)
\end{aligned}$$

as $j \to \infty$. This implies $U(\cdot,t) \in C^1_{\mathrm{ad}}$ for any $t \in \mathbb{R}$.

(iii) It is easily seen that $V_t(\xi,t) \geq 0$ for all $(\xi,t) \in [-H,H] \times \mathbb{R}$. Suppose that there exist sequences $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ with $V_t(\xi_n, t_n) \to 0$ as $n \to \infty$. We may assume $\xi_n \to \xi_\infty$ for some $\xi_\infty \in [-H,H]$. For $n \in \mathbb{N}$, let $k_n \in \mathbb{Z}$ be an integer satisfying

$$k_n p_\varepsilon \leq \max_{\xi \in [-H,H]} V(\xi, t_n) < (k_n+1)p_\varepsilon,$$

and let $V_n(\xi,t) := V(\xi, t_n + t) - k_n p_\varepsilon$. Then V_n is also an entire solution of (3.16)-(3.17).

By a similar argument to the one in the proof of (i), there exists an entire solution $\widetilde{V}(\xi,t)$ of (3.16)-(3.17) such that a subsequence $\{V_{n_j}\}_{j\in\mathbb{N}}$ converges to $\widetilde{V}$ in the topology of $C^{2,1}(R_T)$ for any $T > 0$. Moreover, $\widetilde{V}_t(\xi,t) \geq 0$ for all $(\xi,t) \in [-H,H] \times \mathbb{R}$. Since

$$\widetilde{V}_t(\xi_\infty, 0) = \lim_{j\to\infty} \frac{\partial V_{n_j}}{\partial t}(\xi_{n_j}, 0) = \lim_{j\to\infty} \frac{\partial V}{\partial t}(\xi_{n_j}, t_{n_j}) = 0,$$

the strong maximum principle yields $\widetilde{V}_t \equiv 0$, in other words, $\widetilde{V}$ is an equilibrium solution of (3.16)-(3.17). This contradicts Lemma 4.1. □

4.2. **Uniqueness of entire solution.** In this subsection, we prove that the entire solution $V(\xi,t)$ of (3.16)-(3.17) is unique up to time shift. Suppose that $W(\xi,t)$ is another entire solution of (3.16)-(3.17) satisfying the same properties as in Proposition 4.3 (i)-(iii). Define

$$\Lambda_{V,W}(t) := \inf \left\{ \Lambda > 0 \;\middle|\; \begin{array}{l} \text{there exists } a \in \mathbb{R} \text{ such that} \\ \quad V(\xi, t+a) \leq W(\xi,t) \leq V(\xi, t+a+\Lambda) \\ \text{for all } \xi \in [-H,H] \end{array} \right\}.$$

The function $\Lambda_{V,W}(t)$ has the following properties:

Lemma 4.4.

(i) *The function* $\Lambda_{V,W}(t)$ *is monotone decreasing and is bounded in* $t \in \mathbb{R}$.

(ii) *If* $\Lambda_{V,W}(t_0) = 0$ *for some* t_0*, then there exists* $a \in \mathbb{R}$ *such that* $V(\cdot, t+a) \equiv W(\cdot, t)$ *for* $t \geq t_0$*. If* $\Lambda_{V,W}(t_0) > 0$ *for some* t_0*, then* $\Lambda_{V,W}(t)$ *is positive and is strictly decreasing in* $t < t_0$.

Proof. (i) By the definition of $\Lambda_{V,W}$, for each fixed $t \in \mathbb{R}$, there exists $a(t) \in \mathbb{R}$ such that

$$V(\xi, t + a(t)) \leq W(\xi, t) \leq V(\xi, t + a(t) + \Lambda_{V,W}(t)), \quad \xi \in [-H, H]. \tag{4.5}$$

Therefore, it follows from the comparison principle that for any $s > 0$,

$$V(\xi, t + s + a(t)) \leq W(\xi, t + s) \leq V(\xi, t + s + a(t) + \Lambda_{V,W}(t)), \quad \xi \in [-H, H].$$

This implies $\Lambda_{V,W}(t + s) \leq \Lambda_{V,W}(t)$ for $s > 0$.

Next we note that, by Proposition 4.3,

$$\max_{|\xi| \leq H} V(\xi, t) - \min_{|\xi| \leq H} V(\xi, t) \leq 2\widetilde{G}H, \quad \max_{|\xi| \leq H} W(\xi, t) - \min_{|\xi| \leq H} W(\xi, t) \leq 2\widetilde{G}H. \tag{4.6}$$

By the definition of $\Lambda_{V,W}$, there exist $\xi_1, \xi_2 \in [-H, H]$ and $a \in \mathbb{R}$ satisfying

$$V(\xi_1, t + a) = W(\xi_1, t), \quad W(\xi_2, t) = V(\xi_2, t + a + \Lambda_{V,W}(t)).$$

In view of this and (4.6), we have

$$\begin{aligned} V(\xi, t + a + \Lambda_{V,W}(t)) - V(\xi, t + a) &\leq V(\xi_2, t + a + \Lambda_{V,W}(t)) - V(\xi_1, t + a) + 4\widetilde{G}H \\ &= W(\xi_2, t) - W(\xi_1, t) + 4\widetilde{G}H \leq 6\widetilde{G}H. \end{aligned}$$

On the other hand, Proposition 4.3 (iii) implies that

$$V(\xi, t + a + \Lambda_{V,W}(t)) - V(\xi, t + a) \geq \delta_0 \Lambda_{V,W}(t).$$

Hence

$$0 \leq \Lambda_{V,W}(t) \leq \frac{6\widetilde{G}H}{\delta_0}.$$

(ii) The former statement is obvious. Suppose that $\Lambda_{V,W}(t_0) > 0$ for some t_0. Then by (i), $\Lambda_{V,W}(t) > 0$ for any fixed $t < t_0$. Therefore (4.5) and the comparison principle (Proposition 3.9) yield

$$V(\xi, t + s + a(t)) < W(\xi, t + s) < V(\xi, t + s + a(t) + \Lambda_{V,W}(t)), \quad \xi \in [-H, H]$$

for all $s > 0$. Consequently, by the continuity of $V(\xi, t)$ in t, there exists $\delta = \delta(t, s) > 0$ such that

$$V(\xi, t + s + a(t) + \delta) < W(\xi, t + s) < V(\xi, t + s + a(t) + \Lambda_{V,W}(t) - \delta), \quad \xi \in [-H, H].$$

From this it follows that $\Lambda_{V,W}(t + s) \leq \Lambda_{V,W}(t) - 2\delta$. This proves statement (ii). $\square$

Remark 4.5. In fact, by the backward uniqueness theorem for linear parabolic equations (see [3]), we have $\Lambda_{V,W}(t) = 0$ for all $t \in \mathbb{R}$ in the former case of (ii), and $\Lambda_{V,W}(t) > 0$ for all $t \in \mathbb{R}$ in the latter case of (ii).

The following is the main result of this subsection:

Lemma 4.6. $W(\xi, t)$ *is a time-shift of* $V(\xi, t)$.

Proof. We only need to show that $\Lambda_{V,W}(t) = 0$ for all $t \in \mathbb{R}$. Suppose that $\Lambda_{V,W}(t_0) > 0$ for some $t_0 \in \mathbb{R}$. Then Lemma 4.4 (i) implies that $\Lambda_{V,W}(t)$ converges to some $\overline{\Lambda} > 0$ as $t \to -\infty$. Define

$$V_n := V(\xi, t-n) - l_n p_\varepsilon, \quad W_n := W(\xi, t-n) - l_n p_\varepsilon,$$

where l_n is the largest integer that does not exceed $\max_{\xi \in [-H,H]} V(\xi, -n)/p_\varepsilon$. Then V_n and W_n are solutions of (3.16)-(3.17) and have the same properties as in Proposition 4.3.

A diagonal argument similar to the one in the proof of Proposition 4.3 shows that there exist a sequence $n_j \to \infty$ $(j \to \infty)$ and two entire solutions V_∞ and W_∞ such that

$$\lim_{j\to\infty} \|V_{n_j} - V_\infty\|_{C^{2,1}(R_T)} = 0, \quad \lim_{j\to\infty} \|W_{n_j} - W_\infty\|_{C^{2,1}(R_T)} = 0$$

for any $T > 0$. Moreover, we see that $\Lambda_{V_{n_j},W_{n_j}}(t) \to \Lambda_{V_\infty,W_\infty}(t)$ as $j \to \infty$. On the other hand, $\Lambda_{V_{n_j},W_{n_j}}(t) = \Lambda_{V,W}(t-n_j) \to \overline{\Lambda}$ as $j \to \infty$. Therefore $\Lambda_{V_\infty,W_\infty}(t) \equiv \overline{\Lambda}$ for all $t \in \mathbb{R}$. This, however, contradicts Lemma 4.4 (ii) and the fact that $\overline{\Lambda} > 0$.

Thus we have $\Lambda_{V,W}(t) = 0$ for all $t \in \mathbb{R}$, and hence there exists a constant a such that $V(\xi, t+a) \equiv W(\xi, t)$ for all $(\xi, t) \in [-H, H] \times \mathbb{R}$. □

4.3. **Existence of periodic traveling wave.** Let V^ε be an entire solution of (3.16)-(3.17) obtained in Proposition 4.3. The following proposition shows that V^ε changes its shape periodically in time:

Proposition 4.7. *There exists a positive constant T_ε such that*

$$V^\varepsilon(\xi, t + T_\varepsilon) = V^\varepsilon(\xi, t) + p_\varepsilon \tag{4.7}$$

for all $(\xi, t) \in [-H, H] \times \mathbb{R}$.

Proof. Since $V^\varepsilon(\xi, t) + p_\varepsilon$ is also an entire solution of (3.16)-(3.17), Lemma 4.6 implies that $V^\varepsilon(\xi, t) + p_\varepsilon$ is a time-shift of V^ε. In other words, there exists a constant T_ε that satisfies (4.7) for all $(\xi, t) \in [-H, H] \times \mathbb{R}$. Moreover, the positivity of T_ε immediately follows from the fact that $V_t^\varepsilon > 0$. □

Proof of Theorem 2.1 (i). Let $U^\varepsilon(x, t)$ and $u_n(x, t)$ be the expression of $V^\varepsilon(\xi, t)$ and $v_n(\xi, t) = \overline{v}(\xi, t+n) - k(n)p_\varepsilon$ $(n \in \mathbb{N})$ in the coordinates (x, y, t), respectively. Then U^ε is a unique entire solution of (1.3)-(1.4) up to time shift. By (3.18), it holds that $u_n(x, t) = \overline{u}(x, t+n) - k(n)\varepsilon$ for every $n \in \mathbb{N}$.

As in the proof of Proposition 4.3(ii), for any (x, t) with $(x, U^\varepsilon(x, t)) \in \overline{\Omega}_\varepsilon$ and $t \in \mathbb{R}$, there exists a sequence $\{x_j\}_j$ with $x_j \to x$ as $j \to \infty$ such that

$$u_{l_j}(x_j, t) \to U^\varepsilon(x, t), \quad \frac{\partial u_{l_j}}{\partial x}(x_j, t) \to U_x^\varepsilon(x, t) \quad \text{as } j \to \infty,$$

where $\{l_j\}_j$ is the sequence in the proof of Proposition 4.3. Therefore, noting that $\overline{u}$ is an even function in x satisfying (4.1), we obtain

$$U^\varepsilon(-x, t) = U^\varepsilon(x, t), \quad -\tan\alpha \le \operatorname{sgn} x \cdot U_x^\varepsilon(x, t) \le \tan\beta,$$

for all (x, t) with $(x, U^\varepsilon(x, t)) \in \overline{\Omega}_\varepsilon$ and $t \in \mathbb{R}$. Moreover, since $V_t^\varepsilon(\xi, t) > 0$ for all $(\xi, t) \in [-H, H] \times \mathbb{R}$, it is easily seen that $U_t^\varepsilon(x, t) > 0$ for all (x, t) with $(x, U^\varepsilon(x, t)) \in \overline{\Omega}_\varepsilon$ and $t \in \mathbb{R}$.

Next we show that U^ε is a periodic traveling wave. By the expression (3.3), for any fixed (x,t) with $(x, U^\varepsilon(x,t)) \in \overline{\Omega}_\varepsilon$ and $t \in \mathbb{R}$, there exists a unique $\xi \in [-H, H]$ satisfying

$$x = X(\xi, V^\varepsilon(\xi,t)), \quad U^\varepsilon(x,t) = Y(\xi, V^\varepsilon(\xi,t)).$$

Then (3.18) and (4.7) imply

$$x = X(\xi, V^\varepsilon(\xi,t) + p_\varepsilon) = X(\xi, V^\varepsilon(\xi, t+T_\varepsilon)),$$

hence

$$\begin{aligned} U^\varepsilon(x, t+T_\varepsilon) &= Y(\xi, V^\varepsilon(\xi, t+T_\varepsilon)) = Y(\xi, V^\varepsilon(\xi,t) + p_\varepsilon) \\ &= Y(\xi, V^\varepsilon(\xi,t)) + \varepsilon = U^\varepsilon(x,t) + \varepsilon. \end{aligned}$$

This means, by definition (1.6), that $U^\varepsilon(x,t)$ is a periodic traveling wave of (1.3)-(1.4). $\square$

4.4. **Asymptotic stability of the periodic traveling wave.** In this subsection we show that the periodic traveling wave U^ε is stable and that any solution of (1.3)-(1.4) with $u_0 \in C^1_{\mathrm{ad}} \cap C^{1+\lambda}$ $(\lambda > 0)$ converges to a time-shift of U^ε as $t \to +\infty$. As a matter of fact, such properties of U^ε follow from the general result of [10, Theorem 2.4], which discusses asymptotic stability of periodic traveling waves in the framework of order-preserving discrete dynamical systems. (See also Theorems 8.12 and 8.15 of the same paper for related results.) For the self-containedness of the present paper, we give a complete proof here.

Lemma 4.8. *For any $t_0 \in \mathbb{R}$, $V^\varepsilon(\xi, t+t_0)$ is stable under the perturbation of initial data within the class $\widetilde{C}^1_{\mathrm{ad}} \cap C^{1+\lambda}([-H,H])$ with $\lambda > 0$. More precisely, for any $\sigma > 0$ there exists $\delta > 0$ such that for any $v_0 \in \widetilde{C}^1_{\mathrm{ad}} \cap C^{1+\lambda}([-H,H])$ satisfying $\|v_0 - V^\varepsilon(\cdot, t_0)\|_{L^\infty} < \delta$, the solution $v(\xi,t)$ of* (3.16)-(3.17) *with initial data v_0 satisfies*

$$\|v(\cdot,t) - V^\varepsilon(\cdot, t+t_0)\|_{L^\infty} < \sigma$$

for all $t \geq 0$.

Proof. We may assume $t_0 = 0$ without loss of generality. By (4.7), for any constant $s > 0$, the function $V^\varepsilon(\xi, t+s) - V^\varepsilon(\xi,t)$ is periodic in t with period T_ε. In view of this and the continuity of V^ε, we can choose sufficiently small $s > 0$ such that

$$\max_{(\xi,t) \in [-H,H] \times \mathbb{R}} |V^\varepsilon(\xi, t+s) - V^\varepsilon(\xi, t-s)| < \sigma. \tag{4.8}$$

Fix such $s > 0$. Now, since $V^\varepsilon_t > 0$, we have

$$V^\varepsilon(\xi, t-s) < V^\varepsilon(\xi,t) < V^\varepsilon(\xi,t+s), \quad (\xi,t) \in [-H,H] \times \mathbb{R}.$$

Again using the periodicity and the continuity of $V^\varepsilon(\xi,t+s) - V^\varepsilon(\xi,t)$, we see that

$$\min_{(\xi,t) \in [-H,H] \times \mathbb{R}} (V^\varepsilon(\xi,t+s) - V^\varepsilon(\xi,t)) = \min_{(\xi,t) \in [-H,H] \times \mathbb{R}} (V^\varepsilon(\xi,t) - V^\varepsilon(\xi,t-s)) > 0.$$

Denote by δ the left-hand side of the above inequality. Then $\|v_0 - V^\varepsilon(\cdot,0)\|_{L^\infty} < \delta$ implies

$$V^\varepsilon(\xi,-s) \leq v_0(\xi) \leq V^\varepsilon(\xi,s), \quad \xi \in [-H,H].$$

The comparison principle (Proposition 3.9) then yields

$$V^\varepsilon(\xi,t-s) \leq v(\xi,t) \leq V^\varepsilon(\xi,t+s), \quad \xi \in [-H,H], \ t \geq 0.$$

Since $V^\varepsilon_t > 0$, we obtain

$$|v(\xi,t) - V^\varepsilon(\xi,t)| \leq |V^\varepsilon(\xi,t+s) - V^\varepsilon(\xi,t-s)| < \sigma, \quad \xi \in [-H,H], \ t \geq 0.$$

The lemma is proved. □

Lemma 4.9. *Let* $v(\xi,t)$ *be the solution of* (3.16)-(3.17) *with initial value* $v_0 \in \widetilde{C}^1_{\rm ad} \cap C^{1+\lambda}([-H,H])$, $\lambda > 0$. *Then there exists a constant* τ *such that*

$$\lim_{t\to+\infty} \|v(\cdot,t) - V^\varepsilon(\cdot,t+\tau)\|_{C^2([-H,H])} = 0. \tag{4.9}$$

Proof. Let $a \in \mathbb{R}$ and $\Lambda > 0$ be constants such that

$$V^\varepsilon(\xi,a) \le v_0(\xi) \le V^\varepsilon(\xi,a+\Lambda), \quad \xi \in [-H,H].$$

For each $n \in \mathbb{N}$, we define $v_n(\xi,t) := v(\xi,t+nT_\varepsilon) - np_\varepsilon$ for $t \ge -nT_\varepsilon$. Then the comparison principle and (4.7) imply

$$V^\varepsilon(\xi,t+a) \le v_n(\xi,t) \le V^\varepsilon(\xi,t+a+\Lambda), \quad \xi \in [-H,H], \ t \ge -nT_\varepsilon.$$

Arguing as in the proof of Proposition 4.3 (i), we can choose a subsequence $\{v_{n_j}\}_{j\in\mathbb{N}}$ converging to an entire solution, say W, as $j \to \infty$ locally uniformly in (ξ,t). By Lemma 4.6 we have $W(\xi,t) = V^\varepsilon(\xi,t+\tau)$ for some $\tau \in \mathbb{R}$, hence, $\|v_{n_j}(\cdot,t) - V^\varepsilon(\cdot,t+\tau)\|_{L^\infty} \to 0$ as $j \to \infty$ for any $t \in \mathbb{R}$. This implies, in particular,

$$\lim_{j\to\infty} \|v(\cdot,n_jT_\varepsilon) - V^\varepsilon(\cdot,n_jT_\varepsilon+\tau)\|_{L^\infty} = 0.$$

Combining this and Lemma 4.8, we obtain

$$\lim_{t\to+\infty} \|v(\cdot,t) - V^\varepsilon(\cdot,t+\tau)\|_{L^\infty} = 0.$$

Letting

$$\tilde{v}(\xi,t) = v(\xi,t) - \frac{1}{2H}\int_{-H}^{H} v(z,t)dz, \quad \widetilde{V}^\varepsilon(\xi,t) = V^\varepsilon(\xi,t) - \frac{1}{2H}\int_{-H}^{H} V^\varepsilon(z,t)dz,$$

we see that $\{\tilde{v}(\cdot,t)\}_{t\ge\delta}$ and $\{\widetilde{V}^\varepsilon(\cdot,t)\}_{t\ge\delta}$ remain bounded in $C^{2+\nu}([-H,H])$ for some fixed $\delta > 0$ as in Section 3.5. Therefore, the above convergence takes place in the C^2 topology. The lemma is proved. □

The stability of the periodic traveling wave U^ε follows from the above lemmas, as we will see below.

Proof of Theorem 2.2 (i). Since the derivatives of ξ, η and those of X, Y are all bounded, there exists a constant $C > 1$ such that for any $(x_1,y_1),(x_2,y_2) \in \overline{\Omega}_\varepsilon$,

$$C^{-1}d((\xi_1,\eta_1),(\xi_2,\eta_2)) \le d((x_1,y_1),(x_2,y_2)) \le Cd((\xi_1,\eta_1),(\xi_2,\eta_2)), \tag{4.10}$$

where $\xi_j = \xi(x_j,y_j)$, $\eta_j = \eta(x_j,y_j)$ $(j = 1,2)$ and d denotes the Euclidean metric in $\mathbb{R}^2$.

First we assume that γ_t is given by the graph of $u(x,t)$, a classical solution of (1.3)-(1.4). Let v be the expression of u in the coordinates (ξ,η,t). By (4.10) and the fact that $|v_\xi| \le \widetilde{G}$, $|V^\varepsilon_\xi| \le \widetilde{G}$, there exists a constant $\widetilde{C} > 1$ satisfying

$$\widetilde{C}^{-1}\|v(\cdot,t) - V(\cdot,t+\tau)\|_{L^\infty} \le d_{\mathcal{H}}(\gamma_t,\Gamma^\varepsilon_{t+\tau}) \le \widetilde{C}\|v(\cdot,t) - V(\cdot,t+\tau)\|_{L^\infty} \tag{4.11}$$

for any fixed $t \ge 0$ and $\tau \in \mathbb{R}$. In view of this and Lemma 4.8, we obtain the statement of (ii) for the case where γ_t is the graph of a classical solution of (1.3)-(1.4).

Next we consider the general case. We may assume $\tau = 0$ without loss of generality. By the above argument for the graph case, for any $\sigma > 0$, we can find $\delta > 0$ and $t_0 > 0$ such that $d_{\mathcal{H}}(\Gamma^\varepsilon_{t_0},\Gamma^\varepsilon_{-t_0}) < \delta$ implies $d_{\mathcal{H}}(\Gamma^\varepsilon_{t+t_0},\Gamma^\varepsilon_{t-t_0}) < \sigma$ for all $t \ge 0$. Since $\Gamma^\varepsilon_{-t_0} \ll \Gamma^\varepsilon_0 \ll \Gamma^\varepsilon_{t_0}$, there exists a sufficiently small $\delta' > 0$ such that $d_{\mathcal{H}}(\gamma_0,\Gamma^\varepsilon_0) < \delta'$

implies $\Gamma^\varepsilon_{-t_0} \ll \gamma_0 \ll \Gamma^\varepsilon_{t_0}$. Therefore, the comparison principle (Proposition 3.13) implies

$$\Gamma^\varepsilon_{t-t_0} \ll \gamma_t \ll \Gamma^\varepsilon_{t+t_0}, \quad \Gamma^\varepsilon_{t-t_0} \ll \Gamma^\varepsilon_t \ll \Gamma^\varepsilon_{t+t_0},$$

hence,

$$d_{\mathcal{H}}(\gamma_t, \Gamma^\varepsilon_t) \leq d_{\mathcal{H}}(\Gamma^\varepsilon_{t+t_0}, \Gamma^\varepsilon_{t-t_0}) < \sigma$$

for all $t \geq 0$. □

Proof of Theorem 2.2 (ii). Let v be the expression of u in the coordinates (ξ, η, t). In view of (4.11) and Lemma 4.9, we obtain

$$\lim_{t\to+\infty} d_{\mathcal{H}}(\gamma_t, \Gamma^\varepsilon_{t+\tau}) = 0,$$

where τ is the constant satisfying (4.9) for v. Moreover, Lemma 4.9 also implies that the above convergence takes place in the C^2 topology. The theorem is proved. □

5. **Homogenization limit.** Let $U^\varepsilon(x,t)$ be the periodic traveling wave of (1.3)-(1.4) obtained in the previous section and let $c_\varepsilon = \varepsilon/T_\varepsilon$ be the average speed of U^ε. In this section we discuss the homogenization limit of U^ε and c_ε as $\varepsilon \to 0$ by constructing suitable upper and lower solutions of (1.3).

5.1. **Preliminaries.** In this subsection we study basic properties of a traveling wave solution of (1.3) that is defined in $[-H, H] \times \mathbb{R}$ and contacts the boundaries $x = \pm H$ with a given angle $\theta^* := \frac{\pi}{2} - \alpha$. We will see later that this traveling wave coincides with the homogenization limit of U^ε and plays an important role in the construction of suitable upper and lower solutions.

First we note that a traveling wave solution of (1.3) (with a constant speed and a constant profile) is generally written in the form $u(x,t) = \varphi(x) + ct$. Substituting this into (1.3) yields that (φ, c) satisfies

$$c = \frac{\varphi_{xx}}{1+\varphi_x^2} + A\sqrt{1+\varphi_x^2}. \tag{5.1}$$

In addition, considering the normalization and the symmetry of Ω_ε, we impose the following initial condition:

$$\varphi(0) = 0, \quad \varphi_x(0) = 0. \tag{5.2}$$

We denote by $\varphi(x;c)$ the solution of (5.1)-(5.2).

Lemma 5.1. *If $AH \geq \sin\alpha$, then there exists a unique $c_0 \in [0, A)$ such that*

$$\varphi_x(H; c_0) = -\tan\alpha. \tag{5.3}$$

The constant $c_0 = c_0(\alpha, A, H)$ is determined by

$$H = \int_0^\alpha \frac{\cos s}{A - c_0 \cos s} ds, \tag{5.4}$$

and satisfies

$$c_0 = 0 \ \text{ if } \ AH = \sin\alpha, \quad 0 < c_0 < A \ \text{ if } \ AH > \sin\alpha, \tag{5.5}$$

$$\frac{\partial c_0}{\partial \alpha} < 0, \ \frac{\partial c_0}{\partial A} > 0, \ \frac{\partial c_0}{\partial H} > 0 \ \text{ if } \ AH > \sin\alpha. \tag{5.6}$$

Moreover, if $c_0 = 0$ then

$$\varphi(x; 0) = -\frac{1}{A}\left(1 - \sqrt{1 - A^2x^2}\right), \tag{5.7}$$

while if $c_0 > 0$ *then by introducing a parameter* $\theta \in (-\alpha, \alpha)$, $\varphi(x; c_0)$ *can be expressed as*

$$\varphi(x(\theta; c_0); c_0) = -\frac{1}{c_0} \log\left(\frac{A - c_0 \cos\theta}{A - c_0}\right), \tag{5.8}$$

$$x(\theta; c_0) = \int_0^\theta \frac{\cos s}{A - c_0 \cos s} ds. \tag{5.9}$$

Proof. Set $\theta(x; c) = -\arctan \varphi_x(x; c)$. Then θ solves the initial value problem:

$$\begin{cases} \theta_x = \dfrac{A}{\cos\theta} - c, \\ \theta(0) = 0. \end{cases} \tag{5.10}$$

By the uniqueness of the solution of (5.10), $\theta(-x; c) = -\theta(x; c)$ for all $x \in I_c$, where I_c is the maximal interval of existence for $\theta(x; c)$. Since $c \in [0, A)$, we have $\theta_x \geq A - c > 0$ for all $x \in I_c$. Therefore, there exists $x_{\max}(c) > 0$ such that $I_c = (-x_{\max}(c), x_{\max}(c))$ and that $\theta(x; c) \to \pm\frac{\pi}{2}$ as $x \to \pm x_{\max}(c)$.

By (5.10), the solution $\theta(x; c)$ is implicitly defined in I_c by

$$x = \int_0^{\theta(x;c)} \frac{\cos s}{A - c \cos s} ds. \tag{5.11}$$

Since θ is strictly monotone increasing in x and since $\alpha \in (0, \pi/2)$, there exists a unique $x_\alpha(c) \in (0, x_{\max}(c))$ such that $\theta(x_\alpha(c); c) = \alpha$. In view of (5.11), we have

$$x_\alpha(c) = \int_0^\alpha \frac{\cos s}{A - c \cos s} ds,$$

and hence $x_\alpha(c)$ is strictly monotone increasing in c. Since

$$\lim_{c \searrow 0} x_\alpha(c) = \frac{\sin\alpha}{A}, \quad \lim_{c \nearrow A} x_\alpha(c) = +\infty,$$

and since $AH \geq \sin\alpha$, there exists a unique $c_0 = c_0(\alpha, A, H)$ satisfying (5.5) such that $x_\alpha(c_0) = H$. This means that $\varphi_x(H; c_0) = -\tan\alpha$ for the unique solution c_0 of (5.4). Moreover, differentiating (5.4) by α, A and H, and noting that

$$\frac{\partial}{\partial c}\left(\int_0^\alpha \frac{\cos s}{A - c\cos s} ds\right) = \int_0^\alpha \frac{\cos^2 s}{(A - c\cos s)^2} > 0,$$

we obtain (5.6) by the implicit function theorem.

In the case where $c_0 = 0$, we easily see that $\varphi(x; 0)$ defined in (5.7) is the solution of (5.1)-(5.2) with $c = 0$. In the case where $c_0 > 0$, we obtain

$$\begin{aligned} \varphi(x; c_0) &= -\int_0^x \tan\theta(z; c_0) dz = -\int_0^{\theta(x;c_0)} \frac{\sin s}{A - c_0 \cos s} ds \\ &= -\frac{1}{c_0} \log\left(\frac{A - c_0 \cos\theta(x; c_0)}{A - c_0}\right). \end{aligned} \tag{5.12}$$

Putting $\theta = \theta(x; c_0)$ in (5.11) with $c = c_0$ and (5.12), we obtain (5.8)-(5.9). $\square$

5.2. **Lower solution.** Set $\varphi_0(x) = \varphi(x; c_0) - \varphi(-H; c_0)$. Then φ_0 satisfies (5.1) with $c = c_0$ and $\varphi_0(\pm H) = 0$.

Lemma 5.2. *If $\varepsilon > 0$ is sufficiently small, then $c_0 < c_\varepsilon$.*

Proof. Let C_- be a constant satisfying $U^\varepsilon(x, 0) \geq \varphi_0(x) + C_-$ for all $x \in [-H, H]$ and $U^\varepsilon(x_-, 0) = \varphi_0(x_-) + C_-$ for some $x_- \in [-H, H]$. By (5.1) with $c = c_0$, the function $u^-(x, t) := \varphi_0(x) + c_0 t + C_-$ satisfies (1.3) in $I_{c_0} \times \mathbb{R}$, where $I_{c_0} = (-x_{\max}(c_0), x_{\max}(c_0))$ is the maximal interval of existence for $\varphi(x; c_0)$.

If ε is sufficiently small such that $H + \varepsilon \max g < x_{\max}(c_0)$, then

$$x_0(t) := \min\{x \geq H \mid (x, u^-(x, t)) \in \partial_+\Omega_\varepsilon\}$$

is well-defined for every $t \in \mathbb{R}$. Since

$$u_x^-(x, t) \leq -\tan\alpha \quad \text{for} \quad x \geq H, \quad u_x^-(x, t) \geq \tan\alpha \quad \text{for} \quad x \leq -H,$$

we see that $w^-(\cdot, 0) \preceq U^\varepsilon(\cdot, 0)$, $w^-(\cdot, 0) \not\equiv U^\varepsilon(\cdot, 0)$ and that

$$u_x^-(x_0(t), t) \leq -g'_\varepsilon(u^-(x_0(t), t)), \quad u_x^-(-x_0(t), t) \geq g'_\varepsilon(u^-(x_0(t), t)).$$

Hence u^- is a lower solution of (1.3)-(1.4). By the comparison principle, we have $u^-(\cdot, t) \ll U^\varepsilon(\cdot, t)$ for all $t > 0$, in particular, $u^-(x_-, T_\varepsilon) < U^\varepsilon(x_-, T_\varepsilon)$. Noting that

$$\begin{aligned} u^-(x_-, T_\varepsilon) &= \varphi_0(x_-) + c_0 T_\varepsilon + C_-, \\ U^\varepsilon(x_-, T_\varepsilon) &= U^\varepsilon(x_-, 0) + \varepsilon = \varphi_0(x_-) + C_- + \varepsilon, \end{aligned}$$

we obtain $c_0 < c_\varepsilon$. $\square$

5.3. **Upper solution.** Let $t_0 \in \mathbb{R}$ be such that $U^\varepsilon(\pm H, t_0) = 0$ and let C_+ be a constant satisfying $U^\varepsilon(x, t_0) \leq \varphi_0(x) + C_+$ for all $x \in [-H, H]$ and $U^\varepsilon(x_+, t_0) = \varphi_0(x_+) + C_+$ for some $x_+ \in [-H, H]$. We define

$$u^+(x, t) = \psi(x, t) + \varphi_0(x) + c_0 t + C_+$$

for $x \in [-H, H]$ and $t \geq 0$, where $\psi(x, t) = L\sqrt{\varepsilon}(1 - e^{-\rho^2 t}\cos\rho x)$, $\rho = \pi/(2H)$ and L is a positive constant. Note that ψ satisfies the heat equation $\psi_t = \psi_{xx}$.

Lemma 5.3. *If*

$$L \geq \left(\frac{A}{\cos^3\alpha} + 1\right) H e^{\rho^2} \tag{5.13}$$

and if $\varepsilon > 0$ is sufficiently small, then

$$U^\varepsilon(x, t + t_0) \leq u^+(x, t), \quad x \in [-H, H], \ t \in [0, 1]. \tag{5.14}$$

Proof. To prove the lemma, it suffices to show that

$$u_t^+ \geq \frac{u_{xx}^+}{1 + (u_x^+)^2} + A\sqrt{1 + (u_x^+)^2}, \quad x \in [-H, H], \ t \geq 0, \tag{5.15}$$

and

$$U^\varepsilon(\pm H, t + t_0) < u^+(\pm H, t), \quad t \in [0, 1]. \tag{5.16}$$

We denote by $\mathcal{U}(t)$ the region in Ω_ε above the graph of $U^\varepsilon(\cdot, t)$.

We take a sufficiently small $\varepsilon > 0$ such that

$$\psi_{xx} + 2\varphi_{0xx} \leq L\rho^2\varepsilon^{1/2} - 2(A - c_0) \leq 0, \quad x \in [-H, H], \ t \geq 0.$$

Then $(u_x^+)^2 = \varphi_{0x}^2 + \psi_x(\psi_x + 2\varphi_{0x}) \leq \varphi_{0x}^2$. Since $\varphi_{0xx} < 0$ and $\psi_{xx} \geq 0$, we have

$$u_t^+ = \psi_{xx} + \frac{\varphi_{0xx}}{1 + \varphi_{0x}^2} + A\sqrt{1 + \varphi_{0x}^2} \geq \frac{u_{xx}^+}{1 + (u_x^+)^2} + A\sqrt{1 + (u_x^+)^2}$$

for all $x \in [-H, H]$ and $t \geq 0$.

Since $U^\varepsilon(\pm H, t_0) < u^+(\pm H, 0)$,

$$\tau_0 = \sup\{\tau \geq 0 \mid U^\varepsilon(\pm H, t + t_0) < u^+(\pm H, t) \text{ for } t \in [0, \tau]\}$$

is well-defined and is positive. Suppose $\tau_0 < 1$. Then we may assume $U^\varepsilon(-H, \tau_0 + t_0) = u^+(-H, \tau_0)$ since the other case where $U^\varepsilon(H, \tau_0 + t_0) = u^+(H, \tau_0)$ can be treated similarly. Note that $U^\varepsilon(x, t + t_0) \leq u^+(x, t)$ for $(x, t) \in [-H, H] \times [0, \tau_0]$ by the comparison principle.

Let $y_0 \in (0, 1)$ be such that $g'(y_0) = \tan\alpha$ and let $\chi(x)$ be an arc with constant curvature $-A$ satisfying $\chi(-H - \vartheta) = 0$ and $\chi'(-H - \vartheta) = \tan\alpha$, where $\vartheta = g_\varepsilon(\varepsilon y_0)$. Then we have

$$\chi(x) = -\frac{1}{A}\cos\alpha + \frac{1}{A}\sqrt{\cos^2\alpha + 2A\sin\alpha \cdot (H + \vartheta + x) - A^2(H + \vartheta + x)^2}.$$

Set $I = [-H - \vartheta, -H + \sqrt{\varepsilon}]$. Since $\chi''(x) = -A(1 + \chi'(x)^2)^{3/2} \geq -A/\cos^3\alpha := -K$ for $x \in I$,

$$\begin{aligned}
\chi(-H + \sqrt{\varepsilon}) &= (\sqrt{\varepsilon} + \vartheta)\tan\alpha + (\sqrt{\varepsilon} + \vartheta)^2 \int_0^1 (1 - s)\chi''(-H - \vartheta + s(\sqrt{\varepsilon} + \vartheta))ds \\
&\geq (\sqrt{\varepsilon} + \vartheta)\tan\alpha - \frac{K}{2}(\sqrt{\varepsilon} + \vartheta)^2 \geq \varphi_0(-H + \sqrt{\varepsilon}) + \vartheta\tan\alpha - K\varepsilon
\end{aligned}$$

for sufficiently small $\varepsilon > 0$. Here we used the fact that $\varphi_{0xx} < 0$ and $\vartheta = O(\varepsilon)$.

Take a constant h such that the arc $\chi(x) + C_+ + h$ intersects the graph of $u^+(x, \tau_0)$ at $x = -H + \sqrt{\varepsilon}$. Then, if L satisfies (5.13), we obtain

$$\begin{aligned}
h &= u^+(-H + \sqrt{\varepsilon}, \tau_0) - \chi(-H + \sqrt{\varepsilon}) - C_+ \\
&= \psi(-H + \sqrt{\varepsilon}, \tau_0) + \varphi_0(-H + \sqrt{\varepsilon}) + c_0\tau_0 - \chi(-H + \sqrt{\varepsilon}) \\
&\leq L\sqrt{\varepsilon}\left(1 - \frac{e^{-\rho^2}}{H}\sqrt{\varepsilon}\right) + c_0\tau_0 - \vartheta\tan\alpha + K\varepsilon \\
&\leq u^+(-H, \tau_0) - \vartheta\tan\alpha - \varepsilon - C_+.
\end{aligned}$$

On the other hand, since $\varphi_0(x) - \chi(x)$ is strictly monotone increasing in I, we have $\varphi_0(x) - \chi(x) < h$ for all $x \in I$. This implies that $U^\varepsilon(x, t_0) < \chi(x) + C_+ + h$ for $x \in [-H, -H + \sqrt{\varepsilon}]$.

Let $\delta \in [0, \varepsilon)$ be such that the graph of $\chi_0(x) := \chi(x) + h + C_+ + \delta$ intersects $\partial_-\Omega_\varepsilon$ perpendicularly at $x = -H - \vartheta$. Then the arc $\chi_0(x)$ is a stationary curve of (1.1) and the graph of $\chi_0(x)$ for $x \in I$ is contained in $\mathcal{U}(t_0)$. Since

$$U^\varepsilon(-H + \sqrt{\varepsilon}, t + t_0) \leq u^+(-H + \sqrt{\varepsilon}, t) \leq \chi_0(-H + \sqrt{\varepsilon})$$

for all $t \in [0, \tau_0]$, the graph of $\chi_0(x)$ for $x \in I$ is contained in $\mathcal{U}(t + t_0)$ for all $t \in [0, \tau_0]$ by the comparison principle. Especially,

$$U^\varepsilon(-H, \tau_0 + t_0) \leq \chi_0(-H) < \vartheta\tan\alpha + h + C_+ + \varepsilon \leq u^+(-H, \tau_0).$$

This contradicts the supposition that $U^\varepsilon(-H, \tau_0 + t_0) = u^+(-H, \tau_0)$. Thus we obtain $\tau_0 \geq 1$. The lemma is proved. $\square$

5.4. **Homogenization limit of the speed.**

Proof of Theorem 2.3 (i). By Lemma 5.2 we only need the estimate of c_ε from above. Let n_ε be the largest integer that is less than or equal to $(L\sqrt{\varepsilon}+c_0)\varepsilon^{-1}$, where L is the constant in Lemma 5.3. Then we have

$$\begin{aligned} U^\varepsilon(x,t_0+1) &\le u^+(x,1) = \psi(x,1)+\varphi_0(x)+c_0+C_+ \\ &\le \varphi_0(x)+C_+ +(n_\varepsilon+1)\varepsilon \end{aligned}$$

for all $x\in[-H,H]$. On the other hand,

$$U^\varepsilon(x,t_0+(n_\varepsilon+1)T_\varepsilon) = U^\varepsilon(x,t_0)+(n_\varepsilon+1)\varepsilon \le \varphi_0(x)+C_+ +(n_\varepsilon+1)\varepsilon$$

for all $x\in[-H,H]$, and equality holds at $x_+\in[-H,H]$. Since $U^\varepsilon_t>0$, we obtain $1\le(n_\varepsilon+1)T_\varepsilon\le((L\sqrt{\varepsilon}+c_0)\varepsilon^{-1}+1)T_\varepsilon$, hence

$$c_\varepsilon \;=\; \frac{\varepsilon}{T_\varepsilon} \;\le\; c_0+L\sqrt{\varepsilon}+\varepsilon.$$

This proves (2.2). □

5.5. **Homogenization limit of the profile.**

Lemma 5.4. *Assume $AH\ge\sin\alpha$. Then there exist constants $k_1\ge0$, $k_2>0$ and C independent of ε such that*

$$k_1t-C\le U^\varepsilon(x,t+t_0)-U^\varepsilon(\tilde x,t_0)\le k_2t+C \tag{5.17}$$

for all x, $\tilde x\in[-H,H]$, $t\ge0$ and $t_0\in\mathbb{R}$.

Proof. (i) Since $|U^\varepsilon_x|\le G$, for any fixed $t_0\in\mathbb{R}$, we have

$$U^\varepsilon(0,t_0)-GH\le U^\varepsilon(\tilde x,t_0)\le U^\varepsilon(0,t_0)+GH$$

for all $\tilde x\in[-H,H]$. Let

$$r_1=\frac{H}{\sin\alpha},\quad r_2=\frac{H}{\sin\beta},\quad k_1=\frac{AH-\sin\alpha}{H},\quad k_2=\frac{AH+\sin\beta}{H\cos\beta}$$

and define

$$R_1(x)=-r_1+\sqrt{r_1^2-x^2},\quad R_2(x)=r_2-\sqrt{r_2^2-x^2}.$$

Then the graph of R_1 is a circular arc with curvature $-\sin\alpha/H$ and that of R_2 is a circular arc with curvature $\sin\beta/H$. Therefore,

$$w^-(x,t)=k_1t+R_1(x)+U^\varepsilon(0,t_0)-GH,\quad w^+(x,t)=k_2t+R_2(x)+U^\varepsilon(0,t_0)+GH$$

satisfy

$$w^-_t\le\frac{w^-_{xx}}{1+(w^-_x)^2}+A\sqrt{1+(w^-_x)^2},\quad w^+_t\ge\frac{w^+_{xx}}{1+(w^+_x)^2}+A\sqrt{1+(w^+_x)^2}$$

for all $x\in[-H,H]$, $t\ge0$ and that

$$w^-_x(\pm H,t)=\mp\tan\alpha,\quad w^+_x(\pm H,t)=\pm\tan\beta.$$

Therefore, by the comparison principle, $w^-(x,t)\le U^\varepsilon(x,t+t_0)\le w^+(x,t)$, and hence

$$k_1t+R_1(x)-2GH\le U^\varepsilon(x,t+t_0)-U^\varepsilon(\tilde x,t_0)\le k_2t+R_2(x)+2GH$$

for all $x\in[-H,H]$ and $t\ge0$. Letting $C=\max\{r_1,r_2\}+2GH$, we obtain (5.17). □

Proof of Theorem 2.3 (ii). By Lemma 5.4, we have

$$\|U^\varepsilon(\cdot,t)\|_{L^\infty([-H,H])} \leq k_2|t| + C, \quad t \in \mathbb{R}. \tag{5.18}$$

Since $|U^\varepsilon_x(x,t)| \leq G$ for all $(x,t) \in [-H,H] \times \mathbb{R}$, applying the interior Hölder estimates for quasilinear parabolic equations ([13, Theorem 2.3]) to (1.3), we see that there exists a constant $\tilde{\nu} \in (0,1)$ independent of ε such that for any fixed $\delta \in (0,H)$ and $T > 0$, we have

$$\|U^\varepsilon_x\|_{C^{\tilde{\nu},\tilde{\nu}/2}([-H+\delta,H-\delta]\times[-T,T])} \leq C_\delta,$$

where C_δ is a positive constant dependent on δ but independent of ε and T. Therefore, (5.18) and the interior a priori estimates for linear parabolic equations ([6, Theorem 8.11.1]) imply that there exists a constant $M = M(\delta,T)$ independent of ε satisfying

$$\|U^\varepsilon\|_{C^{2+\tilde{\nu},1+\tilde{\nu}/2}([-H+\delta,H-\delta]\times[-T,T])} \leq M.$$

Hence we can find a subsequence $\{U^{\varepsilon_j}\}_j$ which converges to a function U^0 in $C^{2,1}_{\mathrm{loc}}((-H,H)\times\mathbb{R})$.

By (1.6), we have

$$\frac{1}{T_\varepsilon}\int_t^{t+T_\varepsilon} U^\varepsilon_t(x,s)ds = \frac{\varepsilon}{T_\varepsilon} = c_\varepsilon$$

for any fixed $(x,t) \in (-H,H)\times\mathbb{R}$. Setting $\varepsilon = \varepsilon_j$ in the above equalities and letting $j \to \infty$, we obtain $U^0_t(x,t) = c_0$. This means that $U^0(x,t) = \Phi(x) + c_0 t$ for some $\Phi(x)$. Since U^ε satisfies (1.3) and since $U^\varepsilon(0,0) = U^\varepsilon_x(0,0) = 0$, the function Φ must satisfy

$$\begin{aligned} &c_0 = \frac{\Phi_{xx}}{1+\Phi_x^2} + A\sqrt{1+\Phi_x^2}, \quad x \in (-H,H),\\ &\Phi(0) = 0, \quad \Phi_x(0) = 0. \end{aligned}$$

Thus we obtain $\Phi(x) = \varphi(x;c_0)$, where φ is given in Subsection 5.1.

Since for any sequence $\{\tilde{\varepsilon}_j\}_j$ with $\tilde{\varepsilon}_j \to 0$ as $j \to \infty$, we can find a subsequence of $U^{\tilde{\varepsilon}_j}$ which converges to the same limit $\varphi(x;c_0) + c_0 t$, we conclude that

$$\lim_{\varepsilon\to 0}\|U^\varepsilon(x,t) - \varphi(x;c_0) - c_0 t\|_{C^{2,1}_{\mathrm{loc}}((-H,H)\times\mathbb{R})} = 0.$$

The theorem is proved. □

6. **The pinning case.** In this section we consider the case $AH < \sin\alpha$ and prove Theorem 2.1 (ii).

Lemma 6.1. *Assume the slope condition* (1.5) *and* $AH < \sin\alpha$. *Then* (1.3)-(1.4) *has a stationary solution for any small* $\varepsilon > 0$.

Proof. Recall that a stationary solution $w(x)$ of (1.3)-(1.4) is a circular arc of constant curvature $-A$ whose endpoints meet the boundaries $\partial_\pm\Omega_\varepsilon$ at $(x_\pm, w(x_\pm))$ perpendicularly.

Define

$$W(x) = \frac{1}{A}\left(\sqrt{1-A^2x^2} - \sqrt{1-A^2H^2}\right).$$

Clearly W solves (1.3) with $W(\pm H) = 0$. Choose $y_0 \in (0,1)$ such that $g'(y_0) = \tan\alpha$ and let ε be sufficiently small so that $A(H+\varepsilon g(y_0)) < \sin\alpha$. Then we have

$$W'(H+\varepsilon g(y_0)) = -\frac{A(H+\varepsilon g(y_0))}{\sqrt{1-A^2(H+\varepsilon g(y_0))^2}} > -\tan\alpha = -g'(y_0).$$

Since $W'(H+\varepsilon g(0)) = W'(H) < 0 = -g'(0)$, there exists $y_1 \in (0, y_0)$ satisfying $W'(H+\varepsilon g(y_1)) = -g'(y_1)$. Define

$$\begin{aligned} w(x) : &= W(x) + \varepsilon y_1 - W(H + \varepsilon g(y_1)) \\ &= W(x) + \varepsilon y_1 - W(H + g_\varepsilon(\varepsilon y_1)). \end{aligned}$$

Then the graph of w contacts $\partial\Omega_\varepsilon$ perpendicularly at $(x_\pm, w(x_\pm)) = (\pm(H + g_\varepsilon(\varepsilon y_1)), \varepsilon y_1)$. Moreover, the slope condition (1.5) assures that the graph of w does not touch $\partial\Omega_\varepsilon$ except at $(x_\pm, w(x_\pm))$. Consequently, w is a stationary solution of (1.3)-(1.4). □

Proof of Theorem 2.1 (ii). Let $w(x)$ be the stationary solution of (1.3)-(1.4) obtained in Lemma 6.1 and let $q(\xi)$ be the expression of w in the coordinates (ξ, η). Then, for any $n \in \mathbb{Z}$, $w(x)+n\varepsilon$ is a stationary solution of (1.3)-(1.4), while $q(\xi)+np_\varepsilon$ is a stationary solution of (3.16)-(3.17).

Let $u(x,t)$ be a classical solution of (1.3)-(1.4) with initial data $u_0 \in C^1_{\text{ad}}$ and let $v(\xi,t)$ be the solution of (3.16)-(3.17) which corresponds to u with initial data $v_0 \in \widetilde{C}^1_{\text{ad}}$. Then we can find $n_1, n_2 \in \mathbb{Z}$ satisfying $q(\xi) + n_1 p_\varepsilon < v_0(\xi) < q(\xi) + n_2 p_\varepsilon$ for all $\xi \in [-H, H]$. Hence by the comparison principle,

$$q(\xi) + n_1 p_\varepsilon < v(\xi,t) < q(\xi) + n_2 p_\varepsilon, \quad \xi \in [-H,H], \;\; t \ge 0.$$

In other words, $\|v(\cdot,t)\|_{L^\infty}$ is bounded for $t \ge 0$. Arguing as in Subsection 3.5, we see that $\|v(\cdot,t)\|_{C^{2+\nu}([-H,H])}$ is bounded for $t \ge \delta$ with some fixed $\delta > 0$.

Next we note that (3.16)-(3.17) has a Lyapunov functional. This follows from the general result of [15] or [9] on one-dimensional quasilinear parabolic equations. (A Lyapunov functional can also be constructed by using the energy functional associated with (1.3)-(1.4).) Therefore, a standard dynamical systems theory shows that the ω-limit set of v is non-empty and is contained in the set of stationary solutions. The uniqueness of the ω-limit point can be shown by the same zero-number argument as in [8], or it also follows from the result in [15]. (The result in [8] is given for semilinear equations, but the proof is virtually the same for a quasilinear equation.) Consequently there exists a stationary solution V^* of (3.16)-(3.17) satisfying

$$\lim_{t \to +\infty} \|v(\cdot,t) - V^*\|_{C^2([-H,H])} = 0.$$

Let $U^*(x)$ be the expression of V^* in the coordinates (x,y). Then U^* is a stationary solution of (1.3)-(1.4) and u converges to U^* as $t \to +\infty$ in the C^2 topology. □

Acknowledgments. This work was initiated when the third author was visiting the University of Tokyo, under the support of the 21st century COE program of the University of Tokyo.

REFERENCES

[1] M. Alfaro, D. Hilhorst and H. Matano, *The singular limit of the Allen-Cahn equation and the FitzHugh-Nagumo system*, preprint.

[2] H. Berestycki and F. Hamel, *Front propagation in periodic excitable media*, Comm. Pure Appl. Math., **55** (2002), 1–84.

[3] A. Friedman, "Partial Differential Equations of Parabolic Type," Prentice-Hall, Inc., Englewood Cliffs, N.J. 1964.

[4] S. Heinze, *Wave solutions to reaction-diffusion systems in perforated domains*, Z. Anal. Anwendungen, **20** (2001), 661–676.

[5] S. Heinze, G. Papanicolaou and A. Stevens, *Variational principles for propagation speeds in inhomogeneous media*, SIAM J. Appl. Math., **62** (2001), 129–148.

[6] N. V. Krylov, "Lectures on Elliptic and Parabolic Equations in Hölder Spaces," Graduate Studies in Mathematics Vol.12, Amer. Math. Soc., 1996.

[7] A. Lunardi, *Abstract quasilinear parabolic equations*, Math. Ann., **267** (1984), 395–415.

[8] H. Matano, *Convergence of solutions of one-dimensional semilinear parabolic equations*, J. Math. Kyoto Univ., **18** (1978), 221–227.

[9] H. Matano, *Asymptotic behavior of solutions of semilinear heat equations on* S^1, in "Nonlinear Diffusion Equations and Their Equilibrium States II" (eds. W.-M. Ni, L. A. Peletier and J. Serrin), Math. Sci. Res. Inst. Publ., 13, Springer, New York, 1988, 139–162.

[10] T. Ogiwara and H. Matano, *Monotonicity and convergence results in order-preserving systems in the presence of symmetry*, Discrete Contin. Dyn. Syst. **5** (1) (1999), 1–34.

[11] J. Rubinstein, P. Sternberg, J. B. Keller, *Fast reaction, slow diffusion, and curve shortening*, SIAM J. Appl. Math., **49** (1989), 116–133.

[12] N. Shigesada, K. Kawasaki and E. Teramoto, *Traveling periodic waves in heterogeneous environments*, Theor. Population Biol., **30** (1986), 143–146.

[13] N. S. Trudinger, *Pointwise Estimates and Quasilinear Parabolic Equations*, Comm. Pure Appl. Math., **21** (1968), 205–226.

[14] J. Xin, *Front Propagation in Heterogeneous Media*, SIAM Rev., **42** (2000), 161-230.

[15] T. I. Zelenyak, *Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable*, Differ. Equ., **4** (1968), 17–22.

Received for publication September 2006.

E-mail address: matano@ms.u-tokyo.ac.jp
E-mail address: nakamura@im.uec.ac.jp
E-mail address: blou@mail.tongji.edu.cn