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Recurrent traveling waves in a two-dimensional saw-toothed cylinder and their average speed

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ABSTRACT

We study a curvature-dependent motion of plane curves in a two-dimensional infinite cylinder with spatially undulating boundary. The law of motion is given by $V = \kappa + A$, where V is the normal velocity of the curve, κ is the curvature, and A is a positive constant. The boundary undulation is assumed to be almost periodic, or, more generally, recurrent in a certain sense. We first introduce the definition of recurrent traveling waves and establish a necessary and sufficient condition for the existence of such traveling waves. We then show that the traveling wave is asymptotically stable if it exists. Next we show that a regular traveling wave has a well-defined average speed if the boundary shape is strictly ergodic. Finally we study what we call "virtual pinning", which means that the traveling wave propagates over the entire cylinder with zero average speed. Such a peculiar situation can occur only in non-periodic environments and never occurs if the boundary undulation is periodic.

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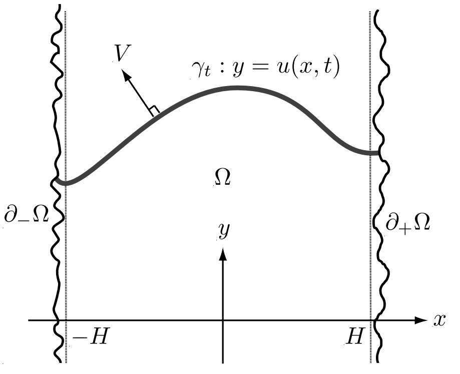

Fig. 1. Saw-toothed cylinder Ω and the curve γ_t.

1. Introduction

In this paper we study traveling waves for a curvature-driven motion of plane curves in a two-dimensional cylinder Ω with spatially undulating boundary. The law of motion of the curve is given by

$$V = \kappa + A, \tag{1.1}$$

where V denotes the normal velocity of the curve, κ denotes the curvature and A is a positive constant representing a constant driving force. The domain Ω is given in the form

$$\Omega := \{(x, y) \in \mathbb{R}^2 \mid -H - g_-(y) < x < H + g_+(y) \text{ for } y \in \mathbb{R}\},$$

where $H > 0$ is a given constant (see Fig. 1), and $g_\pm$ are bounded nonnegative $C^{3+\nu}$ functions for some $\nu \in (0, 1)$ satisfying the following conditions:

(G1) $\inf_{y\in\mathbb{R}} g_+(y) = \inf_{y\in\mathbb{R}} g_-(y) = 0$;
(G2) $\sup_{y\in\mathbb{R}} g'_\pm < 1$, $\inf_{y\in\mathbb{R}} g'_\pm > -1$;
(G3) $\|g_\pm\|_{C^{3+\nu}(\mathbb{R})} < \infty$;
(G4) $g := (g_-, g_+) : \mathbb{R} \to \mathbb{R}^2$ is recurrent in the sense to be defined in Definition 2.3.

Here and in what follows, the space $C^{3+\nu}(\mathbb{R})$ denotes, as in [3], the set of all functions $g = g(y)$ whose derivatives $g^{(j)}(y) := \partial_y^j g(y)$ $(0 \leqslant j \leqslant 3)$ are bounded, continuous and uniformly Hölder continuous with exponent ν in $\mathbb{R}$. Furthermore the symbol $\|\cdot\|_{C^{3+\nu}(\mathbb{R})}$ denotes the following norm:

$$\|g\|_{C^{3+\nu}(\mathbb{R})} = \sum_{j=0}^{3} \|g^{(j)}\|_{L^\infty(\mathbb{R})} + \sup_{y_1 \neq y_2} \frac{|g'''(y_1) - g'''(y_2)|}{|y_1 - y_2|^\nu}.$$

In this paper, by a solution of (1.1) we mean a time-dependent simple curve γ_t in Ω which satisfies (1.1) and whose endpoints contact $\partial_\pm\Omega$ perpendicularly, where $\partial_+\Omega$ (resp. $\partial_-\Omega$) denotes the right (resp. left) boundary of Ω. Eq. (1.1) appears as a certain singular limit of an Allen–Cahn type nonlinear diffusion equation under the Neumann boundary conditions. The curve γ_t represents the interface between two different phases. See, for example, [11,1] for details.

To avoid sign confusion, the normal to the curve γ_t will always be chosen toward the upper region, and the sign of the normal velocity V and the curvature κ will be understood in accordance with this choice of the normal direction. Consequently, κ is negative at those points where the curve is concave while it is positive where the curve is convex (see Fig. 1).

We will consider the case where γ_t is expressed as a graph of a certain function $y = u(x,t)$ at each time t. Let $\zeta_-(t)$, $\zeta_+(t)$ be the x-coordinates of the endpoints of γ_t lying on $\partial_-\Omega$, $\partial_+\Omega$, respectively. In other words, $(\zeta_\pm(t), u(\zeta_\pm(t),t)) \in \partial_\pm\Omega$. Then (1.1) is equivalent to

$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2}, \quad \zeta_-(t) < x < \zeta_+(t),\ t > 0, \tag{1.2}$$

under the boundary conditions

$$\begin{cases} u_x\big(\zeta_-(t),t\big) = g_-'\big(u(\zeta_-(t),t)\big), \quad u_x\big(\zeta_+(t),t\big) = -g_+'\big(u(\zeta_+(t),t)\big), \\ \big(\zeta_\pm(t), u(\zeta_\pm(t),t)\big) \in \partial_\pm\Omega. \end{cases} \tag{1.3}$$

We define the angles $\alpha_\pm$, $\beta_\pm \in [0, \pi/2)$ by

$$\sup_{y\in\mathbb{R}} g_\pm'(y) = \tan\alpha_\pm, \qquad \inf_{y\in\mathbb{R}} g_\pm'(y) = -\tan\beta_\pm. \tag{1.4}$$

Roughly speaking, $\alpha_\pm$ can be interpreted as the maximal opening angles of the channel Ω, and $\beta_\pm$ as the maximal closing angles. Then condition (G2) implies that $\alpha_\pm$, $\beta_\pm$ satisfy

$$0 \leqslant \alpha_\pm,\ \beta_\pm < \frac{\pi}{4} \quad \text{(slope condition)}$$

or, equivalently,

$$G := \max\big\{\|g_+'\|_{L^\infty(\mathbb{R})}, \|g_-'\|_{L^\infty(\mathbb{R})}\big\} < 1. \tag{1.5}$$

We impose the above condition in order to prevent γ_t from developing singularities near the boundary $\partial\Omega$. As we will show in Section 4 (Proposition 4.16), under the condition (1.5), the problem (1.2)–(1.3) has a time-global classical solution for any smooth data u_0 satisfying

$$\big|u_0'(x)\big| < \frac{1}{G} \quad (\text{or } +\infty \text{ if } G = 0). \tag{1.6}$$

Here the term “classical solution” is understood in the following sense:

Definition 1.1. A function $u(x,t)$ is called a *classical solution* of (1.2)–(1.3) in the time interval $[0,T)$ if

(a) $u(x,t)$ is defined for $\zeta_-(t) \leqslant x \leqslant \zeta_+(t)$, $0 \leqslant t < T$, where $\zeta_\pm(t)$ are some continuous functions on $0 < t < T$ that depend on each solution;
(b) u, u_x are continuous for $\zeta_-(t) \leqslant x \leqslant \zeta_+(t)$, $0 \leqslant t < T$, and u_{xx}, u_t are continuous for $\zeta_-(t) < x < \zeta_+(t)$, $0 < t < T$;
(c) u satisfies (1.2)–(1.3) for $\zeta_-(t) < x < \zeta_+(t)$, $0 < t < T$.

It is called a *time-global* classical solution if $T = +\infty$, and a stationary solution if it is independent of t. One can easily see that $v(x)$ is a stationary solution of (1.2)–(1.3) if and only if its graph is a (concave) circular arc of radius $1/A$ whose endpoints contact $\partial_\pm\Omega$ perpendicularly.

Any time global classical solution $u(x,t)$ of (1.2)–(1.3) satisfies

$$\left|u_x(x,t)\right| < \frac{1}{G} \tag{1.7}$$

for all large t (see Proposition 4.17 for more details). Thus, imposing the condition (1.6) — as we do later in this paper — does not restrict the class of solutions so much.

If $g_\pm(y) \equiv 0$, then $\Omega_0 = \{(x,y) \in \mathbb{R}^2 \mid -H < x < H\}$ is a straight cylinder. In this case (1.1) has a traveling wave whose profile moves at a constant speed A. On the other hand, if $g_\pm(y) \not\equiv 0$, traveling waves in the usual sense cannot exist. In fact, as the front γ_t propagates, its shape and speed fluctuate due to undulation of the boundaries $\partial_\pm\Omega$. Therefore we have to adopt a generalized notion of traveling waves.

In our earlier paper [9], we have studied the special case where $g_\pm(y)$ are periodic and $g_+ \equiv g_-$. In this case the notion of traveling waves is well-established. More precisely, a solution $U(x,t)$ of (1.2)–(1.3) is called a traveling wave (or a periodic traveling wave) if

$$U(x,t+T) = U(x,t) + L \tag{1.8}$$

for some $T \neq 0$, where L is the period of $g_\pm$. We have proved the existence, uniqueness and stability of such a traveling wave, and estimated the average speed $c = L/T$ near the homogenization limit.

In this paper we consider a much wider class of boundary undulations. Namely we simply assume that $g = (g_-, g_+)$ is a recurrent function (see Definition 2.3) and define the notion of traveling waves in such a domain (Definition 2.14). This class contains quasi-periodic and almost periodic functions as special cases. Note that the nature of traveling waves in a spatially non-periodic environment is far from being well understood compared with the periodic case.

The first aim of the present paper is to prove the existence and uniqueness of a traveling wave — or, so to say, a recurrent traveling wave — and discuss its stability. The second aim is to study the propagation speed of the recurrent traveling waves. We will show that a regular traveling wave has a well-defined average speed if the boundary shape is strictly ergodic. We also give, among other things, an example of a traveling wave with zero average speed for a special choice of g.

This paper is organized as follows. In Section 2, we recall basic notions of recurrence and almost periodicity and introduce an extended notion of traveling waves in a recurrent environment. The definition of a traveling wave which we adopt in this paper is basically the same as introduced in [7], which deals with the Allen–Cahn equation on $\mathbb{R}^1$. See Definition 2.14 and the subsequent remark.

In Section 3, we state our main results, Theorems 3.1, 3.2, 3.4, 3.5, 3.8, Remark 3.3, Proposition 3.6, Corollary 3.7. Theorems 3.1 and 3.2 give a sharp criterion for the existence of traveling waves. More precisely, a regular (recurrent) traveling wave exists if

$$2AH \geqslant \sin\alpha_+ + \sin\alpha_-. \tag{1.9}$$

On the other hand, if

$$2AH < \sin\alpha_+ + \sin\alpha_- \tag{1.10}$$

and if we consider (1.2)–(1.3) in the domain

$$\Omega^\varepsilon := \left\{(x,y) \;\middle|\; -H - g_-^\varepsilon(y) < x < H + g_+^\varepsilon(y)\right\}, \quad \text{where } g_\pm^\varepsilon(y) := \varepsilon g_\pm(y/\varepsilon) \tag{1.11}$$

for sufficiently small $\varepsilon > 0$, then no traveling wave exists. Theorem 3.4 asserts that the traveling wave is asymptotically stable if it exists. Theorem 3.5 asserts that a regular traveling wave has an average speed if g is "strictly ergodic" in the sense to be specified in Definition 2.6. The classification of the behavior of solutions for the general non-periodic case is given in Theorem 3.8. More precisely, their behavior is classified into: (a) convergence to a regular traveling wave; (b) virtual pinning; (c) pinning.

In Section 4, we present basic results on the local and global existence of solutions of the initial–boundary value problem for (1.2)–(1.3). In doing so, we introduce a new coordinate system (isothermal coordinates) that converts (1.2)–(1.3) into a problem on a straight cylinder while preserving the contact angles. This section also contains other basic results including the comparison principle, which will play an important role throughout the paper.

In Section 5 we construct an *entire solution* — namely a solution that is defined for $-\infty < t < \infty$ and prove its uniqueness up to the time shift. Our strategy is first to construct a monotone increasing solution $\bar{u}$ of (1.2)–(1.3) that is defined for all $t \geqslant 0$, then to construct an entire solution by considering the ω-limit point of the $\bar{u}$ in a certain sense that will be specified later. The assumption that Ω is recurrent plays a key role in this argument.

In Section 6, we first show that the entire solution $U(x,t)$ constructed in Section 5 is actually a traveling wave in the sense of Definition 2.14, thus completing the proof of Theorem 3.1. In order to prove this result one has to show that the profile of the curve $U(x,t)$ at each time moment t ("current profile") depends on the nearby boundary shape ("current landscape") continuously, but we will see that this continuous dependence is an immediate consequence of the uniqueness of the entire solution.

In the latter half of Section 6, we show that the traveling wave $U(x,t)$ has an average speed if g is strictly ergodic, thus establishing Theorem 3.5. We also prove the asymptotic stability of the recurrent traveling wave, thus establishing Theorem 3.4.

In Section 7, we consider the case where (1.10) holds and prove that every global solution of (1.2)–(1.3) converges to a stationary solution. The proof is based on the comparison principle, the energy functional and the uniqueness results on the ω-limit points for one-dimensional parabolic equations.

In Section 8, we discuss the above-mentioned "virtual pinning". This is the case where the front propagates at 0 average speed. As we will see, such a peculiar situation occurs only in a non-periodic environment.

2. Basic notations

2.1. Recurrence and almost periodicity

A set $A \subset \mathbb{R}$ is called *relatively dense* in $\mathbb{R}$ if there exists $M > 0$ such that any interval of the form $[b, b+M]$ contains a point in A.

Definition 2.1. A bounded continuous function $g : \mathbb{R} \to \mathbb{R}^m$ is called *almost periodic* in the sense of Bohr if, for any $\varepsilon > 0$, the following set is relatively dense in $\mathbb{R}$:

$$A_\varepsilon := \left\{ a \in \mathbb{R} \;\middle|\; \|\sigma_a g - g\|_{L^\infty(\mathbb{R};\mathbb{R}^m)} < \varepsilon \right\}.$$

Here σ_a denotes the shift operator:

$$\sigma_a : g(x) \mapsto g(x+a). \tag{2.1}$$

It is easily seen that any almost periodic function (in the sense of Bohr) is uniformly continuous on $\mathbb{R}$. It is also well known that the following conditions are equivalent (Bochner's criterion):

(a) g is almost periodic in the sense of Bohr;
(b) $\mathcal{H}_g$ is compact in $L^\infty(\mathbb{R};\mathbb{R}^m)$.

Here $\mathcal{H}_g$ denotes the *hull* of the function $g(x)$, which is defined by

$$\mathcal{H}_g := \overline{\{\sigma_a g \mid a \in \mathbb{R}\}}^{L^\infty_{loc}(\mathbb{R};\mathbb{R}^m)}, \tag{2.2}$$

where $\overline{X}^{L^\infty_{loc}}$ stands for the closure of a set X in the L^∞_{loc} topology. In other words, $\mathcal{H}_g$ consists of functions g^* that can be written as the limit

$$g(x + a_n) \xrightarrow{n\to\infty} g^*(x) \quad \text{locally uniformly in } \mathbb{R}$$

for some sequence $\{a_n\} \subset \mathbb{R}$.

If $\mathcal{H}_g$ is compact in $L^\infty(\mathbb{R};\mathbb{R}^m)$, then the topology of $L^\infty(\mathbb{R};\mathbb{R}^m)$ and that of $L^\infty_{loc}(\mathbb{R};\mathbb{R}^m)$ are equivalent on this set, hence the following holds if g is almost periodic:

$$\mathcal{H}_g = \overline{\{\sigma_a g \mid a \in \mathbb{R}\}}^{L^\infty(\mathbb{R};\mathbb{R}^m)}.$$

It is well known that any almost periodic function g has the *mean value* in the sense that the following limit exists uniformly in $\alpha \in \mathbb{R}$:

$$\langle g\rangle := \lim_{L\to\infty} \frac{1}{L} \int_\alpha^{\alpha+L} g(y)\,dy. \tag{2.3}$$

See, for example, [4] for further properties of almost periodic functions.

Remark 2.2. In general, if a bounded continuous function g has the limit (2.3) uniformly in $\alpha \in \mathbb{R}$, then the uniformity of the convergence implies that the limit is independent of α. It is also clear from the uniformity that the following limit exists and is equal to (2.3):

$$\lim_{L\to\infty} \frac{1}{L} \int_{\alpha-L}^{\alpha} g(y)\,dy. \tag{2.4}$$

These general properties of the mean value will be useful when we discuss ergodicity later in this section.

Definition 2.3. A bounded continuous function $g : \mathbb{R} \to \mathbb{R}^m$ is called *recurrent* if, for any $M, \varepsilon > 0$, the following set is relatively dense in $\mathbb{R}$:

$$A_{\varepsilon,M} := \bigl\{a \in \mathbb{R} \bigm| \|\sigma_a g - g\|_{L^\infty((-M,M);\mathbb{R}^m)} < \varepsilon\bigr\}.$$

Clearly any almost periodic function is recurrent. From the point of view of dynamical systems, the above definitions can be restated as follows: g is almost periodic (resp. recurrent) if it is a recurrent point of the shift dynamics with respect to the $L^\infty(\mathbb{R};\mathbb{R}^m)$ (resp. $L^\infty_{loc}(\mathbb{R};\mathbb{R}^m)$) topology. The following lemma easily follows from the general theory of recurrent points, therefore we omit the proof.

Lemma 2.4. *For a bounded uniformly continuous function g on $\mathbb{R}$, the following conditions are equivalent*:

(a) *g is recurrent*;
(b) *if $\sigma_{a_n} g \to g^*$ in $L^\infty_{loc}(\mathbb{R};\mathbb{R}^m)$ as $n \to \infty$ for some sequence $\{a_n\}$, then there exists a sequence $\{b_n\}$ such that $\sigma_{b_n} g^* \to g$ in $L^\infty_{loc}(\mathbb{R};\mathbb{R}^m)$ as $n \to \infty$.*

Corollary 2.5. *If g is a recurrent function then $\|h\|_{L^\infty} = \|g\|_{L^\infty}$ for any $h \in \mathcal{H}_g$.*

From Lemma 2.4 we also see that any almost automorphic functions are recurrent. Here a bounded uniformly continuous function $g(y)$ is called *almost automorphic* if $\sigma_{a_n} g \to g^*$ in L^∞_{loc} implies $\sigma_{-a_n} g^* \to g$ (see [13]).

If g is a recurrent function, then from Definition 2.3 it is clear that there exists a sequence $\{a_n\}$ such that

$$a_n \to \infty, \qquad \sigma_{a_n} g \to g \quad \text{in } L^\infty_{loc}(\mathbb{R}; \mathbb{R}^m) \text{ as } n \to \infty. \tag{2.5}$$

In this paper we call such a sequence a *returning sequence*. A returning sequence satisfying $a_n \to -\infty$ also exists. Needless to say, if g is almost periodic, then the convergence $\sigma_{a_n} g \to g$ takes place in $L^\infty(\mathbb{R}; \mathbb{R}^m)$.

A recurrent function does not necessarily possess a mean value in the sense of (2.3). We now introduce the notion of ergodic functions as a subclass of recurrent functions.

Definition 2.6. A bounded uniformly continuous function $g : \mathbb{R} \to \mathbb{R}^m$ is called *uniquely ergodic* if, for any continuous map $F : \mathcal{H}_g \to \mathbb{R}$, the following limit exists uniformly in $\alpha \in \mathbb{R}$:

$$\lim_{L\to\infty} \frac{1}{L} \int_{\alpha}^{\alpha+L} F(\sigma_a g)\, da. \tag{2.6}$$

A function g is called *strictly* ergodic if it is recurrent and uniquely ergodic.

Let us show that the above definition is consistent with the usual notion of "unique ergodicity" in dynamical systems. As mentioned in Remark 2.2, the uniformity of the convergence in (2.6) implies that the limit is independent of α. It is also clear from the uniformity of the convergence that the limit

$$\lim_{L\to\infty} \frac{1}{L} \int_0^L F(\sigma_a h)\, da$$

exists for every $h \in \mathcal{H}_g$ and is independent of h. As one can easily see, this latter property is equivalent to saying that the shift dynamics on $\mathcal{H}_g$ — namely $h \mapsto \sigma_s h$ ($s \in \mathbb{R}$, $h \in \mathcal{H}_g$) — is uniquely ergodic in the usual sense. In other words, there exists a unique shift-invariant measure μ on $\mathcal{H}_g$ satisfying $\int_{\mathcal{H}_g} d\mu(z) = 1$. The limit (2.6) then equals the integral

$$\int_{\mathcal{H}_g} F(z)\, d\mu(z).$$

This is an interpretation of Definition 2.6 above.

Note that the limit (2.6) exists even if the range of F is $\mathbb{R}^m$ instead of $\mathbb{R}$, since the convergence takes place for each component of $\mathbb{R}^m$. In the special case where F is the $\mathbb{R}^m$-valued δ function $F : h \mapsto h(0) \in \mathbb{R}^m$, the above integral (2.6) reduces to the integral (2.3).

Note that any almost periodic function g is uniquely ergodic. Indeed, as we have remarked after Definition 2.3, g is recurrent. Furthermore, the almost periodicity of $g(y)$ implies the almost periodicity of $F(\sigma_a g)$ in a; hence $F(\sigma_a g)$ has uniform mean. Therefore g is uniquely ergodic; hence strictly ergodic.

The following well-known lemma follows easily from the definition:

Lemma 2.7. *If $\{g_n\}$ is a sequence of recurrent (resp. almost periodic) functions and if $g_n \to g^*$ in $L^\infty(\mathbb{R};\mathbb{R}^m)$, then g^* is also recurrent (resp. almost periodic).*

Applying the above lemma to the sequence $g_n(y) := n(g(y+\frac{1}{n}) - g(y))$, we immediately obtain the following lemma:

Lemma 2.8. *If g is a recurrent (resp. almost periodic) function and if g' is uniformly continuous, then g' is also recurrent (resp. almost periodic).*

Corollary 2.9. *If $g \in C^{3+\nu}(\mathbb{R})$ is recurrent, then $\|h\|_{C^{3+\nu}(\mathbb{R})} = \|g\|_{C^{3+\nu}(\mathbb{R})}$ for any $h \in \mathcal{H}_g$.*

Example 2.10. Let us give some examples of recurrent functions that are not almost periodic. For simplicity we consider only $\mathbb{R}$-valued functions.

(a) The following example due to Veech [15] is known to be almost automorphic (hence recurrent) but not almost periodic:

$$h(y) = \left(\frac{\sin \pi y}{\pi}\right)^m \sum_{n\in\mathbb{Z}} \frac{\operatorname{sgn}(\cos(2\pi n\theta))}{(y-n)^m},$$

where m is an even integer, θ is an irrational number and sgn denotes the sign function. Furthermore, the function h is uniformly continuous on $\mathbb{R}$.

(b) The second example is only a slight modification of example (a) above. It is a one-dimensional analogue of Penrose tiling. Let

$$\ldots \text{ABAABABAA} \ldots \tag{2.7}$$

be a sequence of letters A, B constructed by aligning the word elements

$$\ldots X_{-2}X_{-1}X_0X_1X_2\ldots,$$

where

$$X_n = \begin{cases} \text{“A”} & \text{if } n\alpha - [n\alpha] \in [0, 1-\alpha), \\ \text{“BA”} & \text{if } n\alpha - [n\alpha] \in [1-\alpha, 1). \end{cases}$$

Here α is an irrational number satisfying $0 < \alpha < 1$ and $[z]$ denotes the largest integer not exceeding z. Obviously the letters A, B are “uniformly” (more precisely “ergodically”) distributed over the sequence (2.7) with frequency ratio $1 : \alpha$. We partition the line $\mathbb{R}$ according to the sequence (2.7) by associating the letters A, B with intervals of length 1, α, respectively. We then number the letters B in (2.7) as

$$\ldots, B_{-2}, B_{-1}, B_0, B_1, B_2, \ldots$$

and denote by x_j the center point of the interval corresponding to B_j ($j \in \mathbb{Z}$). Finally we define a smooth function $h(x)$ by setting

$$h(x) = \sum_{j=-\infty}^{\infty} \phi(x - x_j),$$

where $\phi(x)$ is a compactly supported smooth nonnegative function with $\phi \not\equiv 0$, or more generally, ϕ is a uniformly continuous function satisfying $\sum_{n\in\mathbb{Z}} \phi(n) < \infty$. It is not difficult to see that the function $n \mapsto X_n$ is almost automorphic on $\mathbb{Z}$. Therefore, by a generalized version of Veech's theorem [15, Theorem 6.2.1], the function h is almost automorphic (hence recurrent) on $\mathbb{R}$.

One can further show that both of the above examples (a), (b) are strictly ergodic functions. See [8] for details.

2.2. Solution triples

Now we come back to the problem (1.2)–(1.3). The symbol g represents the pair (g_-, g_+), which is an $\mathbb{R}^2$-valued recurrent function. In order to study the behavior of solutions in a recurrent environment, it is useful to compare a solution with its spatial shift. A shifted solution no longer satisfies the equation in the same domain Ω, but only in a shifted domain. Thus it is important to keep in mind where the solution is defined. For this reason we introduce the notion of a solution triple.

Given an element $h = (h_-, h_+) \in \mathcal{H}_g$, we define a domain Ω_h by

$$\Omega_h := \bigl\{(x, y) \in \mathbb{R}^2 \bigm| -H - h_-(y) < x < H + h_+(y) \text{ for } y \in \mathbb{R}\bigr\}. \tag{2.8}$$

Clearly $\Omega = \Omega_g$. We denote by $(1.2)_h$–$(1.3)_h$ the problem (1.2)–(1.3) with Ω replaced by Ω_h.

Definition 2.11. Let $h \in \mathcal{H}_g$ and let $\mathcal{I}$ be an interval in $\mathbb{R}$. We call $(u, \Omega_h, \mathcal{I})$ a *solution triple* if $u(x,t)$ satisfies $(1.2)_h$–$(1.3)_h$ on the time-interval $\mathcal{I}$.

If $(U, \Omega_h, \mathbb{R})$ is a solution triple, we call $U(x,t)$ an *entire solution* in Ω_h. If $\Omega_h = \Omega$, then we simply call U an entire solution. We define the shift of Ω_h by

$$\sigma_a \Omega_h := \Omega_{\sigma_a h} \quad \bigl(= \Omega_h - (0, a)\bigr).$$

The shift of a solution $y = u(x,t)$ is defined by $y + a = u(x,t)$, thus

$$\sigma_a u(x,t) = u(x,t) - a.$$

The following lemma is an obvious consequence of the definition.

Lemma 2.12. *If $(u, \Omega_h, \mathcal{I})$ is a solution triple, then $(\sigma_a u(\cdot, \cdot + \tau), \sigma_a \Omega_h, \mathcal{I} - \tau)$ is a solution triple for any $a, \tau \in \mathbb{R}$.*

Next we define the convergence of a solution triple. Denote by $\gamma(v)$ the graph of a function $v(x)$ defined on some interval, say, $I \subset \mathbb{R}$, namely

$$\gamma(v) := \bigl\{\bigl(x, v(x)\bigr) \bigm| x \in I\bigr\}. \tag{2.9}$$

Here I is always chosen to be the entire domain of definition of v, which should be clear from the context. For example, in the case of the solution $u(x,t)$ of (1.2)–(1.3), we have $I = [\zeta_-(t), \zeta_+(t)]$.

Definition 2.13. By a *locally uniform convergence* of a sequence of solution triples $(u_n, \Omega_{h_n}, \mathcal{I}_n)$ to a triple $(u_\infty, \Omega_{h_\infty}, \mathcal{I}_\infty)$, we mean that $h_n \to h_\infty$ in $L^\infty_{loc}(\mathbb{R}; \mathbb{R}^2)$ as $n \to \infty$, $\mathcal{I}_\infty = \lim_{n\to\infty} \mathcal{I}_n$ and

$$\lim_{n\to\infty} d_{\mathcal{H}}\bigl(\gamma\bigl(u_n(\cdot, t)\bigr), \gamma\bigl(u_\infty(\cdot, t)\bigr)\bigr) = 0 \quad \text{locally uniformly in } t \in \mathcal{I}_\infty, \tag{2.10}$$

where $d_{\mathcal{H}}$ denotes the Hausdorff distance.

2.3. Definition of traveling waves

In the classical notion, a solution is called a traveling wave if its spatial profile does not change in time and moves at a constant speed. Such a traveling wave exists only if the environment is spatially uniform at least in the direction of the propagation of the wave.

For problem (1.2)–(1.3), such a classical traveling wave exists only if $g_\pm(y)$ are constants, and in that case, a traveling wave is given in the form $\varphi(x) + ct$, where φ represents the profile and c the speed. However, if $g_\pm(y)$ are nonconstant, one needs an extended notion of traveling waves that propagate in spatially inhomogeneous environment.

In the special case where $g = (g_-, g_+)$ is periodic, the problem has been discussed in our earlier paper [9]. In this case, an entire solution U is called a traveling wave (or a "periodic traveling wave") if there exists T such that

$$U(x, t + T) = U(x, t) + L, \tag{2.11}$$

where L denotes the period of g. The average speed of such a traveling wave is given by $c := L/T$. If g is non-periodic, the above definition does not work, and we need to introduce a more generalized notion. Following [7], we define the traveling wave as follows:

Definition 2.14. Let g be a recurrent function. An entire solution U of (1.2)–(1.3) is called a *traveling wave* if there exist a continuous map $\mathcal{F} : \mathcal{H}_g \to \mathcal{K}$ and a strictly monotone increasing function $p : \mathbb{R} \to \mathbb{R}$ such that

$$p(t) \to \pm\infty \quad \text{as } t \to \pm\infty, \tag{2.12}$$

$$\gamma\big(\sigma_{p(t)} U(\cdot, t)\big) = \mathcal{F}(\sigma_{p(t)} g) \quad \text{for } t \in \mathbb{R}. \tag{2.13}$$

Here $\mathcal{K}$ denotes the space of all continuous curves in $\mathbb{R}^2$ endowed with the Hausdorff distance, and γ is as defined in (2.9). A traveling wave is called *regular* if there exists a constant $\delta > 0$ such that

$$\frac{p(t) - p(s)}{t - s} \geqslant \delta \quad \text{for all } t > s. \tag{2.14}$$

Intuitively, the function $p(t)$ in the above definition indicates the position of the front at time t. (We will later set $p(t) = U(0, t)$ in Section 6.) Thus $\sigma_{p(t)} U(x, t) := U(x, t) - p(t)$ represents the profile of the solution observed from the current position $p(t)$ (the *current profile*) and $\sigma_{p(t)} g$ represents the spatial environment observed from the current position $p(t)$ (the *current landscape*). Thus (2.13) means that the current profile depends on the current profile continuously. The above definition of traveling waves is due to [7]. Though [7] deals with the Allen–Cahn equation in spatially heterogeneous media, the basic spirit is the same.

Note that further generalization of the notion of traveling waves that allow randomness of the media is given in [12]. Note also that [2] gives another generalized notion of front propagation called *generalized transition waves*. This notion applies not only to traveling waves in spatial and temporal heterogeneities but also to front propagation through an unbounded domain of arbitrary shape. However this notion is too general for our problem (1.2)–(1.3), since a generalized transition wave in this context simply means a solution $u(x, t)$ for which $\max_x u(x, t) - \min_x u(x, t)$ remains bounded as $t \to +\infty$. Thus the mildly more restrictive class of traveling waves as defined in Definition 2.14 is the right concept for our purposes.

If g is periodic, then $\mathcal{H}_g$ is homeomorphic to the circle, thus it is not difficult to see that Definition 2.14 is equivalent to the established definition (2.11) for periodic traveling waves. See [8] for further details.

2.4. Average speed

The *average speed* of a traveling wave $U(x,t)$ is defined by

$$c := \lim_{T\to\infty} \frac{U(0,t+T)-U(0,t)}{T} \quad \text{uniformly in } t\in\mathbb{R}, \tag{2.15}$$

provided that this limit exists. As we will see later in Theorem 3.5, the traveling wave for (1.2)–(1.3) has an average speed if g is strictly ergodic. However this is not necessarily the case if we do not assume unique ergodicity.

In the general case, for an entire solution U of (1.2)–(1.3) we define the *upper average speed* c_+ and the *lower average speed* c_- of U by

$$c_+ := \lim_{T\to\infty} \sup_{t\in\mathbb{R}} \frac{U(0,t+T)-U(0,t)}{T}, \tag{2.16}$$

$$c_- := \lim_{T\to\infty} \inf_{t\in\mathbb{R}} \frac{U(0,t+T)-U(0,t)}{T}. \tag{2.17}$$

Lemma 2.15. *For any entire solution U of (1.2)–(1.3) satisfying $0 < U_t \leqslant C$ for some $C > 0$, both of the limits (2.16), (2.17) exist. The limit (2.15) exists if and only if $c_+ = c_-$.*

The proof of this lemma will be given in Section 6.2. Clearly one has

$$0 \leqslant c_- \leqslant c_+.$$

If $U(x,t)$ is a regular traveling wave, then

$$0 < c_- \leqslant c_+.$$

For the convenience of later arguments we define $c_\pm$ for any entire solution $U(x,t)$ that is not necessarily a traveling wave.

3. Main theorems

In this section we present our main results on the recurrent traveling wave and its average speed. More basic questions such as the well-posedness of problem (1.2)–(1.3) will be discussed in Section 4. In this section and in the rest of the paper, (G1)–(G4) will be the standing hypotheses, which we will not repeat in each theorem. The symbol G denotes the constant in (1.5).

3.1. Existence and stability of recurrent traveling waves

Theorem 3.1 *(Existence). If (1.9) holds, then there exists a regular traveling wave $U(x,t)$ of (1.2)–(1.3) (see Definition 2.14), and it is unique up to time shift. Furthermore, there exists a constant $\rho > 0$ such that $U_t(x,t) \geqslant \rho$ and $|U_x(x,t)| \leqslant G$ for every $t \in \mathbb{R}$ and x for which $U(x,t)$ is defined.*

Theorem 3.2 *(Non-existence). Consider the problem (1.2)–(1.3) with $g_\pm$ replaced by $g_\pm^\varepsilon$ as in (1.11). If (1.10) holds, then for any sufficiently small $\varepsilon > 0$ there exist infinitely many stationary solutions in Ω^ε. Furthermore, every classical solution of (1.2)–(1.3) that is defined globally for $t \geqslant 0$ converges to one of these stationary solutions as $t \to +\infty$.*

The above existence criteria (1.9) and (1.10) can be interpreted that the front propagation occurs if and only if the driving force A is bigger than a certain threshold value, and that this threshold value depends only on the maximal opening angle $\alpha_\pm$ and the channel width H.

Remark 3.3. As we will see in Section 5, the condition (1.9) guarantees that the problem (1.2)–(1.3) has no stationary solution in Ω_h for any $h \in \mathcal{H}_g$ (Lemma 5.1 and Remark 5.2). In fact, as it is clear from the arguments in Section 5, the assumption (1.9) in Theorem 3.1 can be replaced by this latter condition. The same is true of the assumption (1.9) in Theorem 3.4 below. See also the assertion (A1) $\Leftrightarrow$ (A2) in Theorem 3.8.

Theorem 3.4 *(Stability). Assume that (1.9) holds, or simply that a regular traveling wave for (1.2)–(1.3) exists. Then the traveling wave $U(x,t)$ is stable in the following sense:*

(i) [Stability] *Let Γ_t be the solution curve of (1.1) associated with $U(x,t)$. Then for any $\sigma > 0$ there exists $\delta > 0$ such that for any solution curve γ_t of (1.1) that is defined globally for $t \geqslant 0$ and satisfies $d_{\mathcal{H}}(\gamma_0, \Gamma_\tau) < \delta$ for some $\tau \in \mathbb{R}$, it holds that $d_{\mathcal{H}}(\gamma_t, \Gamma_{t+\tau}) < \sigma$ for all $t \geqslant 0$, where $d_{\mathcal{H}}$ denotes the Hausdorff distance.*

(ii) [Asymptotic stability] *Let $u(x,t)$ be a classical solution of (1.2)–(1.3) defined globally for $t \geqslant 0$ and let γ_t be the solution curve of (1.1) associated with $u(x,t)$. Then there exists a constant $\tau \in \mathbb{R}$ such that*

$$\lim_{t\to+\infty} d_{\mathcal{H}}(\gamma_t, \Gamma_{t+\tau}) = 0.$$

Furthermore, γ_t approaches $\Gamma_{t+\tau}$ in the C^2 sense as $t \to +\infty$.

The assertion (i) of the above theorem is an easy consequence of the comparison principle, which we will state in Section 4.4. The assertion (ii), on the other hand, needs a more refined argument based on the dynamical system theory and the strong maximum principle. We remark that no spectral analysis nor energy estimates will be involved in the proof of (ii). Therefore, the convergence rate in (ii) is so far unknown.

The proof of Theorems 3.1, 3.2 and 3.4 will be carried out in Section 6.1, Section 7 and Section 6.3, respectively.

3.2. Existence of average speed

In general, a traveling wave may not have an average speed as defined in (2.15). However the following theorem holds:

Theorem 3.5 *(Average speed). If g is strictly ergodic (see Definition 2.6), then a regular recurrent traveling wave has an average speed in the sense that the limit in (2.15) exists.*

As a special case of the above theorem, regular traveling waves have an average speed if g is periodic, quasi-periodic or almost periodic.

The proof of the above theorem will be given in Section 6.2.

3.3. Pinning and virtual pinning

In a spatially non-periodic environment, some traveling waves may have zero average speed under certain special circumstances. Such a peculiar situation never occurs in periodic environments. Let us first see what happens in the periodic case.

Proposition 3.6. *Instead of* (G4), *assume that g is periodic. Then the following conditions are equivalent*:

(a_1) *There exists a regular traveling wave in Ω_g;*
(a_2) *There exists no stationary solution in Ω_g.*

Corollary 3.7. *Instead of* (G4), *assume that g is periodic. Then either of the following occurs*:

(a) *There exists a regular traveling wave in Ω_g. Any time-global solution approaches this traveling wave or its time shift as $t \to +\infty$.*
(b) *There exists no traveling wave in Ω_g. Any time-global solution converges to a stationary solution as $t \to +\infty$.*

The situation differs if g is non-periodic. The following theorem classifies the behavior of solution curves for the general non-periodic case.

Theorem 3.8. *Let g be recurrent. Then one of the following holds*:

(A1) *There exists a regular traveling wave in Ω_g and any time-global solution approaches this traveling wave or its time shift as $t \to +\infty$.*
(B1) *There exists an entire solution $U(x,t)$ in Ω_g such that $U(x,t) \to \pm\infty$ as $t \to \pm\infty$ and such that its lower average speed c_- equals* 0 (*see* (2.17)).
(C1) *There exists no entire solution $U(x,t)$ in Ω_g such that $U(x,t) \to \pm\infty$ as $t \to \pm\infty$.*

Furthermore,

$$(\mathrm{A1}) \Leftrightarrow (\mathrm{A2}), \qquad (\mathrm{B1}) \Leftrightarrow (\mathrm{B2}), \qquad (\mathrm{C1}) \Leftrightarrow (\mathrm{C2}),$$

where

(A2) *There exists no stationary solution in Ω_h for any $h \in \mathcal{H}_g$.*
(B2) *There exists no stationary solution in Ω_g, but there exists a stationary solution in Ω_h for some $h \in \mathcal{H}_g$, $h \neq g$.*
(C2) *There exists a stationary solution in Ω_g.*

The case (B1) above can occur only in non-periodic environments. We call this situation *virtual pinning*, while the case (C1) is called *pinning*. We note that the entire solution $U(x,t)$ that appears in statement (B1) above can be shown to be a traveling wave in the sense of Definition 2.14. It is therefore an example of a non-regular traveling wave. See the forthcoming paper [10] for details.

The proof of the above results, along with some examples, will be given in Section 8.

4. Local and global existence

In this section we present basic existence results for solutions of (1.2)–(1.3) and derive uniform bounds on the derivatives of solutions. The main existence results will be stated in Section 4.5. We also prove the comparison principle in Section 4.4.

4.1. Change of variables

In studying the existence of solutions for problem (1.2)–(1.3), it is convenient to introduce new coordinates that convert the domain into a flat cylinder. More precisely, we make a change of variables $(x, y) \mapsto (\xi, \eta)$, which gives a diffeomorphism $\overline{\Omega} \to \overline{D}$, where

$$D := \{(\xi, \eta) \in \mathbb{R}^2 \mid -H < \xi < H, \ -\infty < \eta < \infty\}.$$

Here the functions $\xi(x, y)$ and $\eta(x, y)$ are to be specified later. With these new coordinates, the function $y = u(x,t)$ is expressed as $\eta = v(\xi, t)$, where the new unknown $v(\xi, t)$ is determined by the relation

$$\eta\big(x, u(x,t)\big) = v\big(\xi\big(x, u(x,t)\big), t\big) \tag{4.1}$$

for (x,t) with $(x, u(x,t)) \in \overline{\Omega}$ and $t \geqslant 0$. The function $v(\xi, t)$ is well-defined by (4.1) provided that $x \mapsto \xi(x, u(x,t))$ is strictly monotone for each fixed t. We will see later that this monotonicity condition always holds for the class of solutions which we consider. Indeed, there exists a positive constant δ such that

$$\frac{\partial}{\partial x}\xi\big(x, u(x,t)\big) = \xi_x + \xi_y u_x \geqslant \delta > 0. \tag{4.2}$$

Once $v(\xi,t)$ is defined, then substituting it into the relation $y = u(x,t)$ yields

$$Y\big(\xi, v(\xi,t)\big) = u\big(X\big(\xi, v(\xi,t)\big), t\big), \tag{4.3}$$

where the map $(\xi,\eta) \mapsto (X(\xi,\eta), Y(\xi,\eta)) : \overline{D} \to \overline{\Omega}$ is the inverse map of $(x,y) \mapsto (\xi(x,y), \eta(x,y))$. The expression (4.3) gives a formula for recovering the original solution $u(x,t)$ from $v(\xi,t)$. In order for u to be smoothly dependent on v, we need the map $\xi \mapsto X(\xi, v(\xi,t))$ to be one-to-one for each fixed t. As we will see later, this is true since we have

$$\frac{\partial}{\partial \xi}X\big(\xi, v(\xi,t)\big) = X_\xi + X_\eta v_\xi \geqslant \delta_1 > 0 \tag{4.4}$$

for some constant $\delta_1 > 0$. See Lemma 4.8 for details.

4.2. Isothermal coordinates

Now we specify ξ and η. We adopt the so-called isothermal coordinates. First, $\xi(x,y)$ is given as a solution of the following boundary value problem:

$$\begin{cases} \Delta\xi = 0, & \text{in } \Omega, \\ \xi = -H, & \text{on } \partial_-\Omega, \\ \xi = H, & \text{on } \partial_+\Omega. \end{cases} \tag{4.5}$$

It is easily seen that the problem (4.5) has a unique solution within the class of bounded functions, and the maximum principle — more precisely, the Phragmèn–Lindelöf principle — implies

$$-H < \xi(x,y) < H \quad \text{for } (x,y) \in \Omega. \tag{4.6}$$

Furthermore, the a priori estimates for Laplace's equation yield

$$\|\xi\|_{C^{3+\nu}(\overline{\Omega})} \leqslant C, \tag{4.7}$$

where C is a positive constant depending on ν, H and $\|g_\pm\|_{C^{3+\nu}(\mathbb{R})}$.

Next, we define $\eta(x,y)$ to be the conjugate harmonic function of ξ. More precisely, η is characterized by the Cauchy–Riemann relation

$$\begin{cases} \xi_x = \eta_y, \\ \xi_y = -\eta_x, \end{cases} \quad \text{in } \Omega. \tag{4.8}$$

Such a function η exists since Ω is a simply connected domain, and it is unique up to addition of a constant. Thus, η is uniquely determined by (4.8) under the following normalization condition:

$$\eta(0,0) = 0. \tag{4.9}$$

Since we will consider the problem (1.2)–(1.3) with Ω replaced by Ω_h for $h \in \mathcal{H}_g$ where Ω_h is as in (2.8), we study here the dependence of ξ and η on h. We denote by ξ_h and η_h the solution of (4.5) with Ω replaced by Ω_h and its conjugate harmonic function normalized by $\eta_h(0,0) = 0$, respectively. Then we easily see by the uniqueness that

$$\xi_{\sigma_a g}(x,y) = \xi(x, y+a), \qquad \eta_{\sigma_a g}(x,y) = \eta(x,y+a) - \eta(0,a). \tag{4.10}$$

Lemma 4.1. *If a sequence $h_n \in \mathcal{H}_g$ converges to $h^* \in \mathcal{H}_g$ in $L^\infty_{loc}(\mathbb{R};\mathbb{R}^2)$ then*

$$\xi_{h_n} \to \xi_{h^*}, \qquad \eta_{h_n} \to \eta_{h^*} \quad \text{in } C^3_{loc}(\Omega_{h^*}).$$

Furthermore, for any sequence of points $(x_n, y_n) \in \overline{\Omega}_{h_n}$ converging to $(x_*, y_*) \in \partial\Omega_{h^*}$, we have

$$\xi_{h_n}(x_n,y_n) \to \xi_{h^*}(x_*,y_*), \qquad \eta_{h_n}(x_n,y_n) \to \eta_{h^*}(x_*,y_*)$$

in the C^3 sense.

Proof. For $h \in \mathcal{H}_g$, we define a map $S_h : (x,y) \mapsto (\hat{x}_h, \hat{y}_h)$ by

$$\hat{x}_h = \frac{2}{2H + h_+(y) + h_-(y)}\left(x - \frac{h_+(y) - h_-(y)}{2}\right), \qquad \hat{y}_h = y. \tag{4.11}$$

By this map, the domain Ω_h is converted to $D_1 := \{(\hat{x}_h, \hat{y}_h) \in \mathbb{R}^2 \mid -1 < \hat{x}_h < 1\}$. Let $\hat{\xi}_h$ be the expression of ξ_h in the new coordinates $(\hat{x}_h, \hat{y}_h)$. Then $\hat{\xi}_h$ solves

$$(E_h): \quad \begin{cases} \mathcal{L}_h \hat{\xi} = 0, & \text{in } D_1, \\ \hat{\xi} = \pm H, & \text{at } \hat{x}_h = \pm 1. \end{cases}$$

Here $\mathcal{L}_h$ is a uniformly elliptic linear operator of second order whose coefficients are determined by $h_\pm(\hat{y}_h)$, $h'_\pm(\hat{y}_h)$, $h''_\pm(\hat{y}_h)$ and $\hat{x}_h$.

By Corollary 2.9, (4.7) implies $\|\xi_{h_n}\|_{C^{3+\nu}(\overline{\Omega}_{h_n})} \leqslant C$; hence $\|\hat{\xi}_{h_n}\|_{C^{3+\nu}(\overline{D_1})}$ is bounded by a positive constant independent of n. Note also that the convergence $h_n \to h^*$ in $L^\infty_{loc}(\mathbb{R};\mathbb{R}^2)$ and the bound on $\|h_n\|_{C^{3+\nu}(\mathbb{R})}$ imply that $h_n \to h^*$ in $C^3_{loc}(\mathbb{R};\mathbb{R}^2)$; hence the coefficients of the operator $\mathcal{L}_{h_n}$ converge to those of $\mathcal{L}_{h^*}$ locally uniformly in $\overline{D}_1$. Combining these observations and considering that the solution of (E_{h^*}) is unique, we see that $\hat{\xi}_{h_n} \to \hat{\xi}_{h^*}$ in $C^3_{loc}(\overline{D}_1)$. The statement of the lemma for ξ_h immediately follows from this.

The statement for η_h follows from the Cauchy–Riemann relation and the following formula:

$$\eta_h(x,y) = \int\limits_C -(\xi_h)_y\,dx + (\xi_h)_x\,dy,$$

where C is any simple curve in Ω_h connecting $(0,0)$ and (x,y). $\square$

By the Hopf boundary lemma and a reflection argument with respect to lines that are parallel to the y-axis, we see that

$$\xi_x > 0 \quad \text{in } \overline{\Omega}. \tag{4.12}$$

To prove this, let $(x_0, y_0) \in \Omega$ be an arbitrary point and let $\xi^*(x, y) := \xi(2x_0 - x, y)$ be the reflection of ξ with respect to the line $\ell_0 := \{x = x_0\}$. The portion of Ω lying on the left-hand side of ℓ_0 is denoted by Ω_0^L and its reflection with respect to ℓ_0 by $(\Omega_0^L)^*$. Then we have

$$\Delta\xi = 0 = \Delta\xi^* \quad \text{in } D_0 := \left(\Omega_0^L\right)^* \cap \Omega,$$
$$\xi = \xi^* \quad \text{on } \partial D_0 \cap \ell_0, \qquad \xi > \xi^* \quad \text{on } \partial D_0 \setminus \ell_0.$$

Hence by the maximum principle, we have $\xi > \xi^*$ in D_0. Furthermore, since $(x_0, y_0) \in \partial D_0 \cap \ell_0$, the Hopf boundary lemma implies $\xi_x(x_0, y_0) > \xi_x^*(x_0, y_0) = -\xi_x(x_0, y_0)$. Thus we obtain $\xi_x > 0$ in Ω. The same inequality on $\partial\Omega$ follows from (4.6) and the Hopf boundary lemma. This proves (4.12).

Furthermore, the following lemma holds:

Lemma 4.2. $\inf_{\overline{\Omega}} \xi_x > 0$.

Proof. Suppose that $\inf_{\overline{\Omega}} \xi_x = 0$. Then there exists a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ in $\overline{\Omega}$ satisfying $\xi_x(x_n, y_n) \to 0$ as $n \to \infty$. Let $h_n := \sigma_{y_n} g$ and $\xi_n := \xi_{h_n}$. Then $(x_n, 0) \in \overline{\Omega}_{h_n}$ and $(\xi_n)_x(x_n, 0) \to 0$ as $n \to \infty$.

Taking a subsequence if necessary, we may assume that $x_n \to x_0$ as $n \to \infty$ and that $h_n \to h^*$ in $L^\infty_{loc}(\mathbb{R}; \mathbb{R}^2)$ as $n \to \infty$ for some $x_0 \in \mathbb{R}$ and $h^* \in \mathcal{H}_g$. Clearly we have $(x_0, 0) \in \overline{\Omega}_{h^*}$. Therefore, Lemma 4.1 implies $(\xi_{h^*})_x(x_0, 0) = 0$. This, however, is impossible since (4.12) holds for ξ_{h^*} and Ω_{h^*}. This contradiction proves the lemma. □

Remark 4.3. By the above lemma, we obtain

$$\delta_g := \inf_{\overline{\Omega}_g} (\xi_g)_x(x, y) = \inf_{\overline{\Omega}_h} (\xi_h)_x(x, y) > 0 \quad (\forall h \in \mathcal{H}_g). \tag{4.13}$$

The latter equality follows from the fact that any $h \in \mathcal{H}_g$ can be given as a limit $\sigma_{a_n} g \to h$ $(n \to \infty)$ for some sequence $\{a_n\}$, hence $\xi_h = \lim_{n\to\infty} \xi_g(x, y + a_n)$ in C^3.

Next we apply a similar reflection argument to a line ℓ_θ that is slightly tilted from the y-axis by an angle θ (see Fig. 2). More precisely, choose an arbitrary point $(x_0, y_0) \in \Omega$ and consider a line ℓ_θ which passes through (x_0, y_0) and whose unit normal vector is $\boldsymbol{n}_\theta := (\cos\theta, \sin\theta)$. Denote by Ω_θ^L the portion of Ω lying on the left-hand side of ℓ_θ, and let $(\Omega_\theta^L)^*$ be its reflection with respect to ℓ_θ. Also, denote by $\xi^*(x, y)$ the reflection of $\xi(x, y)$ with respect to ℓ_θ.

If θ is sufficiently small, namely, if $|\theta| < \pi/2 - \max\{\alpha_\pm, \beta_\pm\}$, then the reflection of the boundary curves $\partial_\pm\Omega$ with respect to ℓ_θ do not intersect with $\partial_\pm\Omega$ themselves except on the line ℓ_θ. So long as θ has this property, one can see by the maximum principle that

$$\xi^* < \xi \quad \text{in } D_\theta := \left(\Omega_\theta^L\right)^* \cap \Omega,$$

since $\xi - \xi^* \geqslant 0$, $\not\equiv 0$ on the boundary of D_θ. Furthermore, $\xi = \xi^*$ on $\ell_\theta \cap \Omega$, which is a portion of the boundary of D_θ. Hence, by the Hopf boundary lemma, the normal derivative of ξ on the line segment $\ell_\theta \cap \Omega$ does not vanish. Thus we have $\boldsymbol{n}_\theta \cdot \nabla\xi(x_0, y_0) > 0$ for $|\theta| < \pi/2 - \max\{\alpha_\pm, \beta_\pm\}$. Consequently,

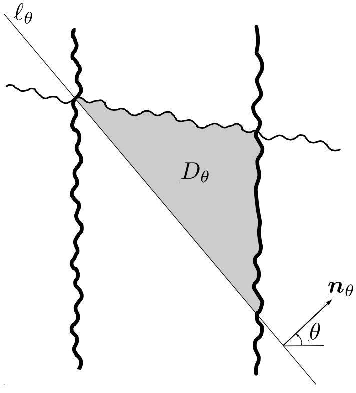

Fig. 2. Reflection with respect to line ℓ_θ.

the angle between the vector $\nabla\xi$ and the x-axis lies in the interval $[-\max\{\alpha_\pm, \beta_\pm\}, \max\{\alpha_\pm, \beta_\pm\}]$. This implies the following estimate:

$$\left|\frac{\xi_y}{\xi_x}\right| \leqslant \max\{\tan\alpha_\pm, \tan\beta_\pm\} = G \quad \text{in } \Omega. \tag{4.14}$$

This and (4.8) yield

$$\left|\frac{\eta_x}{\eta_y}\right| \leqslant G \quad \text{in } \Omega. \tag{4.15}$$

Before closing this subsection, let us list up basic properties and some useful identities concerning the map $(\xi(x, y), \eta(x, y))$ and its inverse $(X(\xi, \eta), Y(\xi, \eta))$. First we note that the map $(x, y) \mapsto (\xi(x, y), \eta(x, y)) : \overline{\Omega} \to \overline{D}$ is one-to-one, since $\inf_{\overline{\Omega}} \xi_x = \inf_{\overline{\Omega}} \eta_y > 0$ (see Lemma 4.2). It is also easily seen that this map is onto, since the level curves of $\xi(x, y)$ and those of $\eta(x, y)$ are smooth curves stretching vertically and horizontally, respectively.

Next we recall that the Cauchy–Riemann relation (4.8) implies

$$\det\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} = \xi_x^2 + \xi_y^2 = \eta_x^2 + \eta_y^2.$$

This, together with (4.13), implies

$$\det\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \geqslant \delta_g^2 \quad \text{in } \overline{\Omega}. \tag{4.16}$$

Therefore $(\xi(x, y), \eta(x, y))$ is a diffeomorphism from $\overline{\Omega}$ to $\overline{D}$, so its inverse map $(X(\xi, \eta), Y(\xi, \eta))$ is well-defined and is of class C^3. The same Cauchy–Riemann relation implies

$$\nabla\xi \cdot \nabla\eta = 0 \quad \text{in } \Omega. \tag{4.17}$$

This means that the level curves of ξ and those of η intersect orthogonally everywhere. In particular, the level curves of η meet the boundary curves $\partial_-\Omega$ and $\partial_+\Omega$ perpendicularly.

As regards the derivatives of X, Y, clearly the following identity holds:

$$\begin{pmatrix} X_\xi & X_\eta \\ Y_\xi & Y_\eta \end{pmatrix} \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.18}$$

Consequently X and Y also satisfy the Cauchy–Riemann relation:

$$\begin{cases} X_\xi = Y_\eta, \\ X_\eta = -Y_\xi, \end{cases} \quad \text{in } D. \tag{4.19}$$

Furthermore, by (4.7), (4.16), (4.18) and the boundedness of X, we see that

$$\|X\|_{C^{3+\nu}(\bar{D})} \leqslant \widetilde{C}, \qquad \|Y_\xi\|_{C^{2+\nu}(\bar{D})} \leqslant \widetilde{C}, \qquad \|Y_\eta\|_{C^{2+\nu}(\bar{D})} \leqslant \widetilde{C}, \tag{4.20}$$

where $\widetilde{C} > 0$ is a constant depending on ν, H, $\|g_\pm\|_{C^{3+\nu}(\mathbb{R})}$ and δ_g.

We also note that the following holds for the quantities in (4.2) and (4.4):

$$(\xi_x + \xi_y u_x)(X_\xi + X_\eta v_\xi) = 1. \tag{4.21}$$

To see this, let us differentiate (4.1) by x, and (4.3) by ξ:

$$\eta_x + \eta_y u_x = v_\xi(\xi_x + \xi_y u_x),$$

$$Y_\xi + Y_\eta v_\xi = u_x(X_\xi + X_\eta v_\xi).$$

Consequently

$$X_\xi + X_\eta v_\xi = X_\xi + X_\eta \frac{\eta_x + \eta_y u_x}{\xi_x + \xi_y u_x} = \frac{X_\xi(\xi_x + \xi_y u_x) + X_\eta(\eta_x + \eta_y u_x)}{\xi_x + \xi_y u_x}.$$

By (4.18), the numerator of the right-hand side is equal to 1.

Next we study the dependence of X and Y on $h \in \mathcal{H}_g$ if we replace Ω by Ω_h. Let (X_h, Y_h) be the inverse map of (ξ_h, η_h). Then, if a sequence $\{h_n\}$ converges to $h^* \in \mathcal{H}_g$ locally uniformly, then, by Lemma 4.1, $(\xi_{h_n}, \eta_{h_n}) \to (\xi_{h^*}, \eta_{h^*})$ as $n \to \infty$. Consequently, by (4.16),

$$X_{h_n} \to X_{h^*}, \qquad Y_{h_n} \to Y_{h^*} \quad \text{in } C^3_{loc}(\bar{D}). \tag{4.22}$$

4.3. Equation in the new coordinates

Let us now rewrite Eq. (1.2) using the new coordinates ξ, η and the new unknown $v(\xi, t)$. Differentiating the expression

$$Y\big(\xi, v(\xi,t)\big) = u\big(X\big(\xi, v(\xi,t)\big), t\big)$$

twice by ξ and once by t, and using (4.19), we obtain

$$u_x = \frac{-X_\eta + X_\xi v_\xi}{X_\xi + X_\eta v_\xi},$$
$$u_{xx} = \frac{1}{(X_\xi + X_\eta v_\xi)^3}\left\{\frac{(1+v_\xi^2)(J_\eta - J_\xi v_\xi)}{2J^2} + \frac{v_{\xi\xi}}{J}\right\},$$
$$u_t = \frac{v_t}{(X_\xi + X_\eta v_\xi)J},$$

where

$$J(\xi,\eta) := \frac{1}{X_\xi(\xi,\eta)^2 + X_\eta(\xi,\eta)^2} \quad \left(= \xi_x(X,Y)^2 + \xi_y(X,Y)^2\right).$$

Therefore, we find that (1.2) is converted into the following equation:

$$v_t = d(\xi, v, v_\xi)v_{\xi\xi} + f(\xi, v, v_\xi), \tag{4.23}$$

where

$$d(\xi,\eta,p) := \frac{J(\xi,\eta)}{1+p^2},$$
$$f(\xi,\eta,p) := -\frac{1}{2}J_\xi(\xi,\eta)p + \frac{1}{2}J_\eta(\xi,\eta) + A\sqrt{J(\xi,\eta)\left(1+p^2\right)}.$$

In deriving Eq. (4.23), we need to assume condition (4.4), the validity of which will be verified later in Lemma 4.8. Note also that, because of the orthogonality (4.17), the boundary condition (1.3) is converted to

$$v_\xi(-H,t) = v_\xi(H,t) = 0. \tag{4.24}$$

Since $X(\xi,\eta)$ is in C^3, Eq. (4.23) is a quasilinear parabolic equation whose coefficients are C^1 functions of (ξ, v, v_ξ). Furthermore, the coefficients $J(\xi,v)$, $J_\xi(\xi,v)$ and $J_\eta(\xi,v)$ are bounded thanks to (4.20).

In accordance with Definition 1.1, a classical solution of this reduced problem is defined as follows:

Definition 4.4. A function $v(\xi,t)$ defined in $[-H,H]\times[0,T)$ is said to be a classical solution of (4.23)–(4.24) in the time interval $[0,T)$ if v, v_ξ are continuous in $[-H,H]\times[0,T)$, $v_{\xi\xi}$, v_t are continuous in $(-H,H)\times(0,T)$ and if v satisfies (4.23)–(4.24) in $(-H,H)\times(0,T)$. It is called a time-global classical solution if $T=+\infty$.

Remark 4.5. In what follows, when we say v is a classical solution in the closed time-interval $[0,T]$, we mean that v is a classical solution in $[0,T)$ and that v, v_ξ, $v_{\xi\xi}$, v_t are continuous up to $t=T$.

At the end of this subsection we derive useful gradient bounds which will be needed in the next subsection to prove the existence of a time-global solution. As mentioned in the introduction, we will assume (1.5) throughout the present paper. This condition is equivalent to

$$\max\{\tan\alpha_\pm, \tan\beta_\pm\} < 1.$$

Hereafter we denote by Q_T the space–time region on which u is defined:

$$Q_T := \left\{(x,t) \,\middle|\, \zeta_-(t) < x < \zeta_+(t),\ 0<t<T\right\}.$$

Lemma 4.6 *(A priori gradient bound for u). Let u be a classical solution of (1.2)–(1.3) in the interval $[0,T]$. Then*

$$\left|u_x(x,t)\right| \leqslant \max\left\{\left\|u_0'\right\|_{L^\infty}, G\right\} \quad \textit{for } (x,t)\in \overline{Q}_T, \tag{4.25}$$

where $\|u_0'\|_{L^\infty} := \sup\{|u_0'(x)| \mid \zeta_-(0) \leqslant x \leqslant \zeta_+(0)\}$. Furthermore, if

$$\left\|u_0'\right\|_{L^\infty} < \frac{1}{G}, \tag{4.26}$$

then inequality (4.2) holds everywhere in $\overline{Q}_T$. Here the right-hand side of (4.26) is understood to be $+\infty$ if $G=0$.

Proof. The function $w := u_x(x,t)$ satisfies the following equation:

$$\begin{cases} w_t = \dfrac{w_{xx}}{1+w^2} - \dfrac{2ww_x^2}{(1+w^2)^2} + A\dfrac{ww_x}{\sqrt{1+w^2}} & \text{in } Q_T, \\ w\big(\zeta_\pm(t),t\big) = \mp g'\big(u\big(\zeta_\pm(t),t\big)\big), & \text{for } t\in(0,T). \end{cases} \tag{4.27}$$

This equation can be written in the form

$$w_t = a(x,t)w_{xx} + b(x,t)w_x,$$

with $a(x,t)>0$. By the maximum principle, the maximum of $|w|$ on $\overline{Q}_T$ is attained on the parabolic boundary of Q_T. Thus (4.25) follows from (1.5). Next, combining this gradient estimate and (4.14), we obtain

$$\xi_x + \xi_y u_x \geqslant \xi_x\left(1 - \left|\frac{\xi_y}{\xi_x}u_x\right|\right) \geqslant \xi_x(1-G\rho),$$

where $\rho = \max\{\|u_0'\|_{L^\infty}, G\}$. Since $\inf_{\overline{\Omega}} \xi_x > 0$, and since $1-G\rho>0$, the inequality (4.2) holds by setting $\delta := (1-G\rho)\inf_{\overline{\Omega}}\xi_x$. □

Definition 4.7. By an admissible function for problem (1.2)–(1.3), we mean a C^1 function $u_0(x)$ defined on some interval $\zeta_- \leqslant x \leqslant \zeta_+$ such that

(a) $(x,u_0(x))\in\Omega$ for all $\zeta_- < x < \zeta_+$;
(b) $(\zeta_\pm, u_0(\zeta_\pm))\in\partial_\pm\Omega$; and the graph of u_0 intersects $\partial_\pm\Omega$ perpendicularly;
(c) $|u_0'(x)| < \frac{1}{G}$ for all $\zeta_- \leqslant x \leqslant \zeta_+$ if $G\neq 0$.

We denote by C^1_{ad} the set of all admissible functions.

If $u_0\in C^1_{\mathrm{ad}}$, then, as we have seen in the proof of the above lemma, $\xi_x(x,u_0) + \xi_y(x,u_0)u_0'(x) > 0$. Consequently, a function $v_0(\xi)$, $-H\leqslant\xi\leqslant H$, is well-defined from the expression

$$\eta\big(x,u_0(x)\big) = v_0\big(\xi\big(x,u_0(x)\big)\big).$$

We denote by $\widetilde{C}^1_{\mathrm{ad}}$ the set of all the functions $v_0(\xi)$ obtained this way.

Lemma 4.8 *(A priori gradient bound for v). Let $v(\xi,t)$ be a classical solution of (4.23)–(4.24) in the time interval $[0,T]$ with initial data $v_0 \in \widetilde{C}^1_{\mathrm{ad}}$ which corresponds to $u_0 \in C^1_{\mathrm{ad}}$. Then inequality (4.4) holds everywhere in $[-H,H]\times[0,T]$, hence $v(\cdot,t)\in \widetilde{C}^1_{\mathrm{ad}}$ for every $t\in[0,T]$. Furthermore,*

$$\left|v_\xi(\xi,t)\right| \leqslant \frac{G+\rho}{1-G\rho} \quad \text{for } (\xi,t)\in[-H,H]\times[0,T], \tag{4.28}$$

where $\rho = \max\{\|u_0'\|_{L^\infty}, G\}$.

Proof. Let us show that (4.4) holds with the following choice of δ_1:

$$\delta_1 := \frac{1}{2\sup_{\overline{\Omega}} \xi_x}.$$

Suppose the conclusion does not hold. Then there exists $t_0 \in (0,T]$ such that

$$X_\xi + X_\eta v_\xi \begin{cases} > 0 & \text{for } 0 \leqslant t \leqslant t_0, \\ < \delta_1 & \text{for } t = t_0. \end{cases}$$

Since $X_\xi + X_\eta v_\xi > 0$ in the interval $[0,t_0]$, a function $u(x,t)$ is determined uniquely from (4.3), and it is a classical solution of (1.2)–(1.3). Therefore, by Lemma 4.6, u satisfies (4.25). Combining this and (4.14), we obtain

$$\xi_x + \xi_y u_x = \xi_x\left(1 + \frac{\xi_y}{\xi_x} u_x\right) \leqslant 2 \sup_{\overline{\Omega}} \xi_x$$

for $0 \leqslant t \leqslant t_0$. This and (4.21) yield

$$X_\xi + X_\eta v_\xi \geqslant \delta_1 \quad \text{for } t = t_0,$$

contradicting our earlier assumption. This contradiction shows that (4.4) holds for all $t\in[0,T]$. Therefore, the solution $u(x,t)$ of (1.2)–(1.3) corresponding to $v(\xi,t)$ is defined for all $t\in[0,T]$ and satisfies (4.25). This means $v(\cdot,t)\in\widetilde{C}^1_{\mathrm{ad}}$ for all $t\in[0,T]$. Furthermore,

$$|v_\xi| = \left|\frac{\eta_x + \eta_y u_x}{\xi_x + \xi_y u_x}\right| = \left|\frac{u_x - \frac{\xi_y}{\xi_x}}{1 + \frac{\xi_y}{\xi_x} u_x}\right| \leqslant \frac{\rho + G}{1 - G\rho}.$$

The lemma is proved. □

The following is an immediate consequence of the above lemma and the definition of $\widetilde{C}^1_{\mathrm{ad}}$.

Corollary 4.9. *Let v_0 be an element of $\widetilde{C}^1_{\mathrm{ad}}$ which corresponds to $u_0\in C^1_{\mathrm{ad}}$. If there exists a classical solution v for problem (4.23)–(4.24) with initial data v_0 on some time interval $[0,T)$, then there exists a classical solution u for problem (1.2)–(1.3) with initial data u_0 on the time interval $[0,T)$. Furthermore, $u(\cdot,t)$ belongs to C^1_{ad} for each $t\in[0,T)$. In particular, if v is a time-global classical solution of (4.23)–(4.24), then u is a time-global solution of (1.2)–(1.3).*

4.4. Comparison principles

Definition 4.10. A function $\hat{v} \in C^{2,1}([-H, H] \times [0, T])$ is called a *lower solution* of (4.23)–(4.24) on the interval $0 \leqslant t \leqslant T$ if

$$\begin{cases} \hat{v}_t \leqslant d(\xi, \hat{v}, \hat{v}_\xi)\hat{v}_{\xi\xi} + f(\xi, \hat{v}, \hat{v}_\xi), & (\xi, t) \in (-H, H) \times (0, T), \\ \pm\hat{v}_\xi(\pm H, t) \leqslant 0, & t \in (0, T). \end{cases}$$

A function $\hat{v} \in C^{2,1}([-H, H] \times [0, T])$ is called an *upper solution* of (4.23)–(4.24) if the reversed inequalities hold.

The following proposition follows easily from the maximum principle:

Proposition 4.11 *(Comparison principle for v). Let v^- and v^+ be a lower and an upper solution of (4.23)–(4.24) on the interval $0 \leqslant t \leqslant T$, respectively. Suppose that $v^-(\xi, 0) \leqslant v^+(\xi, 0)$ for $\xi \in [-H, H]$. Then*

$$v^-(\xi, t) \leqslant v^+(\xi, t) \quad \text{for } (\xi, t) \in [-H, H] \times [0, T].$$

Furthermore, if $v^- \not\equiv v^+$ then

$$v^-(\xi, t) < v^+(\xi, t) \quad \text{for } (\xi, t) \in [-H, H] \times (0, T].$$

Corollary 4.12 *(Growth bound on v). There exists a constant $K_f > 0$, dependent only on A, ν, H, $\|g_\pm\|_{C^{3+\nu}(\mathbb{R})}$ and δ_g such that for any classical solution v of (4.23)–(4.24) with initial data $v_0 \in \widetilde{C}^1_{\mathrm{ad}}$,*

$$\big\|v(\cdot, t)\big\|_{L^\infty} \leqslant K_f t + \|v_0\|_{L^\infty}. \tag{4.29}$$

Proof. Since J and J_η are bounded by constants depending on ν, H, $\|g_\pm\|_{C^{3+\nu}(\mathbb{R})}$ and δ_g, we have

$$K_f := \sup_{(\xi,\eta)\in[-H,H]\times\mathbb{R}} \big|f(\xi, \eta, 0)\big| < +\infty.$$

Consequently, $v^+(t) = K_f t + \|v_0\|_{L^\infty}$ is an upper solution of (4.23)–(4.24), while $v^-(t) = -K_f t - \|v_0\|_{L^\infty}$ is a lower solution of (4.23)–(4.24). Furthermore $v^-(0) \leqslant v_0(\xi) \leqslant v^+(0)$ for $\xi \in [-H, H]$. Hence, by Proposition 4.11, we have

$$v^-(t) \leqslant v(\xi, t) \leqslant v^+(t) \quad \text{for } (\xi, t) \in [-H, H] \times [0, T].$$

This proves the corollary. $\square$

Next we state the comparison principle for solution curves of (1.1).

Definition 4.13. Let $\hat{u}$ be a $C^{2,1}$ function defined for $\hat{\zeta}_-(t) \leqslant x \leqslant \hat{\zeta}_+(t)$, $0 \leqslant t \leqslant T$ such that $(\hat{\zeta}_\pm(t), \hat{u}(\hat{\zeta}_\pm(t), t)) \in \partial_\pm\Omega$, respectively. Then $\hat{u}$ is called a *lower solution* of (1.2)–(1.3) on the interval $0 \leqslant t \leqslant T$ if

$$\begin{cases} \hat{u}_t \leqslant \dfrac{\hat{u}_{xx}}{1+\hat{u}_x^2} + A\sqrt{1+\hat{u}_x^2}, & \hat{\zeta}_-(t) < x < \hat{\zeta}_+(t),\ 0 < t < T, \\ \hat{u}_x(\hat{\zeta}_+(t), t) \leqslant -g'(\hat{u}(\hat{\zeta}_+(t), t)), & 0 < t < T, \\ \hat{u}_x(\hat{\zeta}_-(t), t) \geqslant g'(\hat{u}(\hat{\zeta}_-(t), t)), & 0 < t < T. \end{cases}$$

A function $\hat{u}$ is called an *upper solution* of (1.2)–(1.3) if the reversed inequalities hold.

We easily see that $\hat{u}$ is a lower solution of (1.2)–(1.3) if and only if $\hat{v}$, the expression of $\hat{u}$ in the coordinates (ξ, η, t), is a lower solution of (4.23)–(4.24), provided that $\hat{v}$ is well-defined by (4.1).

Notation. Let $\mathcal{C}$ denote the set of all simple (non-self-intersecting) C^1-curves γ in $\overline{\Omega}$ such that:

(a) The two endpoints of γ lie on $\partial_-\Omega$ and on $\partial_+\Omega$, respectively;
(b) Every point of γ except the endpoints lies in Ω.

Each $\gamma \in \mathcal{C}$ divides Ω into two open sets. We denote the one located above γ by $\mathcal{U}(\gamma)$. We then define an order relation in $\mathcal{C}$ by

$$\gamma \preccurlyeq \tilde{\gamma} \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \mathcal{U}(\gamma) \supset \mathcal{U}(\tilde{\gamma}).$$

We also write $\gamma \ll \tilde{\gamma}$ if $\gamma \preccurlyeq \tilde{\gamma}$ and $\gamma \cap \tilde{\gamma} = \emptyset$.

Let u be a C^1 function defined on some interval $\zeta_- \leqslant x \leqslant \zeta_+$ such that $(x, u(x)) \in \Omega$ for $\zeta_- < x < \zeta_+$ and that $(\zeta_\pm, u(\zeta_\pm)) \in \partial_\pm\Omega$. Then the graph of u, say $\mathcal{G}(u)$, belongs to $\mathcal{C}$. For such functions u_1 and u_2, we define an order relation between them by

$$u_1 \preccurlyeq u_2 \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \mathcal{G}(u_1) \preccurlyeq \mathcal{G}(u_2),$$
$$u_1 \ll u_2 \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \mathcal{G}(u_1) \ll \mathcal{G}(u_2).$$

Proposition 4.14 *(Comparison principle for u). Let u_1 and u_2 be a lower solution and an upper solution of (1.2)–(1.3) on the interval $0 \leqslant t \leqslant T$, respectively. Suppose that $u_1(\cdot, 0) \preccurlyeq u_2(\cdot, 0)$. Then*

$$u_1(\cdot, t) \preccurlyeq u_2(\cdot, t) \quad \textit{for } 0 \leqslant t \leqslant T.$$

Furthermore, if $u_1 \not\equiv u_2$ then

$$u_1(\cdot, t) \ll u_2(\cdot, t) \quad \textit{for } 0 < t \leqslant T.$$

Proof. If u_1 and u_2 satisfy the condition (4.25), then they have expressions v_1 and v_2 in the coordinates (ξ, η, t), respectively. In this case, it is clear that $u_1(\cdot, t) \preccurlyeq u_2(\cdot, t)$ (resp. $u_1(\cdot, t) \ll u_2(\cdot, t)$) if and only if $v_1(\xi, t) \leqslant v_2(\xi, t)$ (resp. $v_1(\xi, t) < v_2(\xi, t)$) for $\xi \in [-H, H]$. Therefore the conclusion follows from Proposition 4.11. The general case is basically the same: the conclusion follows from the strong maximum principle except that the Hopf boundary lemma has to be applied after an appropriate local change of coordinates near the boundary. We omit the details. □

The above proposition remains true even if the solution curve γ_t is not necessarily the graph of a function u. More precisely, we have:

Proposition 4.15 *(Comparison principle for (1.1)). Let $\{\gamma_t\}_{t\in[0,T]}$ and $\{\widetilde{\gamma}_t\}_{t\in[0,T]}$ be solutions of (1.1) which contact $\partial_\pm\Omega$ perpendicularly for $0 \leqslant t \leqslant T$ with initial data γ_0 and $\widetilde{\gamma}_0 \in \mathcal{C}$, respectively. Suppose that $\gamma_0 \preccurlyeq \widetilde{\gamma}_0$. Then*

$$\gamma_t \preccurlyeq \widetilde{\gamma}_t \quad \text{for } 0 \leqslant t \leqslant T.$$

Furthermore, if $\gamma_0 \neq \widetilde{\gamma}_0$ then

$$\gamma_t \ll \widetilde{\gamma}_t \quad \text{for } 0 < t \leqslant T.$$

The proof of this proposition is similar to that of Proposition 4.11. In fact, by using local coordinates, one can express (1.1) locally as a quasilinear parabolic equation; one can then apply the maximum principle. The details are omitted.

4.5. Uniform Hölder estimates and existence of time-global solutions

The goal of this subsection is to prove the following fundamental existence theorem:

Proposition 4.16 *(Global existence). Let $0 < \lambda < 1$. For any $u_0 \in C^1_{\mathrm{ad}} \cap C^{1+\lambda}$, there exists a time-global classical solution $u \in C^{2+\mu,1+\mu/2}_{loc}$ of (1.2)–(1.3) with initial data u_0, where the constant $\mu \in (0,1)$ depends on G, ν, $\|g_\pm\|_{C^{3+\nu}(\mathbb{R})}$ and δ_g. Furthermore, if $u_0 \in C^1_{\mathrm{ad}} \cap C^2$ then u_{xx} and u_t are continuous up to $t = 0$.*

The assumption $u_0 \in C^1_{\mathrm{ad}} \cap C^{1+\lambda}$ in the above theorem does not restrict the class of solutions significantly since any time-global solution eventually falls into this class as shown in the following proposition:

Proposition 4.17 *(Asymptotic gradient bound). Let u be a time-global classical solution of (1.2)–(1.3). Then $\max\{\|u_x(\cdot,t)\|_{L^\infty}, G\}$ is monotonically non-increasing in t and there exist positive constants C, T and λ, independent of the choice of u, such that*

$$\left\|u_x(\cdot,t)\right\|_{L^\infty} \leqslant G + Ce^{-\lambda t} \quad \text{for } t \geqslant T. \tag{4.30}$$

In particular,

$$\limsup_{t\to+\infty}\left\|u_x(\cdot,t)\right\|_{L^\infty} \leqslant G, \tag{4.31}$$

and there exists $t_* \geqslant 0$, independent of the choice of u, such that

$$u(\cdot,t) \in C^1_{\mathrm{ad}} \cap C^2 \quad \text{for any } t \geqslant t_*. \tag{4.32}$$

In the special case where $G = 0$ (that is, when Ω is a flat cylinder), the above proposition implies that any global classical solution is asymptotically planar. We postpone the proof of the above proposition until the end of this subsection.

In order to prove Proposition 4.16, we need a uniform Hölder estimate as stated in the following lemma:

Lemma 4.18. *There exists a constant $\mu \in (0,1)$ depending on G, ν, H, $\|g_\pm\|_{C^{3+\nu}(\mathbb{R})}$ and δ_g such that if $v(\xi,t)$ is a classical solution of (4.23)–(4.24) on the time interval $[0,T]$ with initial data $v_0 \in \widetilde{C}^1_{\mathrm{ad}}$ satisfying $v \in C^{2+\mu,1+\mu/2}([-H,H]\times(0,T])$, then for any $\tau \in (0,T)$ we have*

$$\|v_\xi, v_{\xi\xi}, v_t\|_{C^{\mu,\mu/2}([-H,H]\times[\tau,T])} \leqslant C_\tau, \tag{4.33}$$

$$\|v\|_{C^{\mu,\mu/2}([-H,H]\times[\tau,T])} \leqslant C_\tau + K_f T + \|v_0\|_{L^\infty}, \tag{4.34}$$

where K_f is the constant in Corollary 4.12 and C_τ is a constant independent of v and T but dependent on τ, A, G, ν, H, $\|g_\pm\|_{C^{3+\nu}(\mathbb{R})}$ and δ_g.

The above lemma can be shown by applying standard Hölder estimates for quasilinear parabolic equations [14] and the a priori estimates for linear parabolic equations [3] along with the uniform gradient bound (4.28). We omit the proof of this lemma since it is identical to those of [9, Lemmas 3.15 and 3.17].

Combining the above lemma and Corollary 4.9, we obtain:

Corollary 4.19. *Let $u(x,t)$ be a classical solution of (1.2)–(1.3) in the time interval $[0,T]$ with initial data $u_0 \in C^1_{\rm ad}$ where μ is the constant in Lemma 4.18. If, in addition, $u \in C^{2+\mu,1+\mu/2}$, then for any $\tau \in (0,T)$ we have*

$$\|u_x, u_{xx}, u_t\|_{C^{\mu,\mu/2}(\overline{Q}_{\tau,T})} \leqslant C_\tau, \tag{4.35}$$

$$\|u\|_{C^{\mu,\mu/2}(\overline{Q}_{\tau,T})} \leqslant C_\tau\big(1 + T + \|u_0\|_{L^\infty}\big), \tag{4.36}$$

where $Q_{\tau,T}$ denotes the space–time region on which u is defined for $t \in (\tau,T)$ and the constant C_τ is independent of u and T but dependent on τ, A, G, ν, H, $\|g_\pm\|_{C^{3+\nu}(\mathbb{R})}$ and δ_g.

Now we are ready to present a global existence result for v:

Lemma 4.20. *Let $0 < \lambda < 1$. For any $v_0 \in \widetilde{C}^1_{\rm ad} \cap C^{1+\lambda}([-H,H])$, there exists a time-global classical solution $v(\xi,t)$ of (4.23)–(4.24) with initial data v_0. Furthermore, $v \in C^{2+\mu,1+\mu/2}([-H,H]\times[\tau,T])$ for any $0 < \tau < T < +\infty$. If, in addition, $v_0 \in \widetilde{C}^1_{\rm ad} \cap C^2([-H,H])$, then $v_{\xi\xi}$, v_t are continuous up to $t = 0$.*

Proof. Using the uniform derivative bounds (4.28) and the estimates (4.29) and applying Lemma 4.18, one can easily show the existence of global solutions of (4.23)–(4.24). Further regularity of v is derived from standard regularity results for linear parabolic equations.

The continuity at $t = 0$ follows from the local existence theorem in [5]. We omit the details of the argument since it is basically identical to the proof of [9, Theorem 3.18]. □

Proof of Proposition 4.16. The theorem follows immediately from Lemma 4.20 and Corollary 4.9. □

Now we prove Proposition 4.17. We need the following lemma.

Lemma 4.21. *There exist constants $M > 0$ and $T > 0$, depending only on H, $g_\pm$, A, such that, for any global solution u of (1.2)–(1.3),*

$$\big\|u_x(\cdot,t)\big\|_{L^\infty} \leqslant M \quad \textit{for all } t \geqslant T. \tag{4.37}$$

Proof. Let $\widetilde{H} = H + \max\{\|g_+\|_{L^\infty(\mathbb{R})}, \|g_-\|_{L^\infty(\mathbb{R})}\}$ and $\sigma = \max\{\sqrt{2}A\widetilde{H}, 2\}$. Define

$$\psi(x) := \sqrt{\frac{b}{x}} \quad \text{for } x > 0,$$

where $b = 3\sigma/(\sqrt{2}A) > 0$. Then we easily see that

$$\psi' < 0, \qquad \psi'' > 0, \qquad \psi'' = -\frac{3}{2b}\psi^2\psi' \quad \text{for } x > 0.$$

Let $u(x,t)$ be any global solution of (1.2)–(1.3) and let $I(t) := [\zeta_-(t), \zeta_+(t)]$ be the x-interval on which the solution u is defined at $t \geqslant 0$. Then $w := u_x$ satisfies (4.27) for all $T > 0$, hence

$$\begin{cases} w_t = \dfrac{w_{xx}}{1+w^2} - \dfrac{2ww_x^2}{(1+w^2)^2} + A\dfrac{ww_x}{\sqrt{1+w^2}} & \text{in } Q_\infty, \\ w(\zeta_\pm(t),t) = \mp g'(u(\zeta_\pm(t),t)), & \text{for } t > 0, \end{cases} \tag{4.38}$$

where

$$Q_\infty := \{(x,t) \mid \zeta_-(t) < x < \zeta_+(t),\ 0 < t < +\infty\} \subset (-\widetilde{H}, \widetilde{H}) \times (0, +\infty).$$

Now we define a function $\widetilde{w}$ by

$$\widetilde{w}(x,t) := \psi(x + ct - \widetilde{H})$$

on the region $x \in J(t) := (\widetilde{H} - ct, 4\widetilde{H} - ct]$, where $c = (1-\sigma^{-1})A/\sqrt{2} > 0$. Then, for $x \in J(t)$ and $t \geqslant 0$, we have

$$\widetilde{w} \geqslant 1, \qquad \widetilde{w}_t = c\widetilde{w}_x, \qquad \frac{\widetilde{w}_{xx}}{1+\widetilde{w}^2} \leqslant \frac{\widetilde{w}_{xx}}{\widetilde{w}^2} = -\frac{3}{2b}\widetilde{w}_x, \qquad \frac{\widetilde{w}}{\sqrt{1+\widetilde{w}^2}} \geqslant \frac{1}{\sqrt{2}}.$$

Consequently

$$\widetilde{w}_t \geqslant \frac{\widetilde{w}_{xx}}{1+\widetilde{w}^2} - 2\frac{\widetilde{w}\widetilde{w}_x^2}{(1+\widetilde{w}^2)^2} + A\frac{\widetilde{w}\widetilde{w}_x}{\sqrt{1+\widetilde{w}^2}} \quad \text{for } x \in J(t),\ t > 0.$$

Furthermore, since $w(\zeta_\pm(t),t) \leqslant G < 1$ for $t \geqslant 0$, we have

$$w(x,t) \leqslant \widetilde{w}(x,t) \quad \text{for } x \in \partial(I(t) \cap J(t)),$$

so long as $4\widetilde{H} - ct \geqslant \widetilde{H}$. Thus, by the comparison principle,

$$w(x,t) \leqslant \widetilde{w}(x,t) \quad \text{for } x \in I(t) \cap J(t),\ 0 \leqslant t \leqslant T := \frac{3\widetilde{H}}{c}.$$

In particular,

$$w(x,T) \leqslant \widetilde{w}(x,T) = \psi(x + 2\widetilde{H}) \quad \text{for } x \in I(T) \cap J(T) = I(T).$$

Since $\psi' < 0$, we have $\psi(x + 2\widetilde{H}) \leqslant \psi(\widetilde{H})$ in $I(T)$, hence

$$w(x,T) \leqslant \psi(\widetilde{H}) \quad \text{for } x \in I(T).$$

Similarly, in view of the fact that $-\widetilde{w}(-x,t)$ is a subsolution of (4.38), we have $w(x,T) \geqslant -\psi(\widetilde{H})$. Therefore, by setting

$$M := \psi(\widetilde{H}) = \sqrt{\frac{b}{\widetilde{H}}},$$

we see that

$$\left|u_x(x,T)\right| \leqslant M \quad \text{for } x \in I(T).$$

Furthermore, since $M > G$, the maximum principle yields

$$\left\|u_x(\cdot,t)\right\|_{L^\infty} \leqslant M \quad \text{for } t \geqslant T.$$

The lemma is proved. □

Lemma 4.22. *Let $T > 0$ be as in Lemma 4.21. Then there exist constants $C > 0$ and $\lambda > 0$, depending only on H, $g_\pm$ and A such that, for any global solution u of (1.2)–(1.3),*

$$\left\|u_x(\cdot,t)\right\|_{L^\infty} \leqslant G + Ce^{-\lambda t} \quad \textit{for all } t \geqslant T. \tag{4.39}$$

Proof. Let $M, T > 0$ be as in Lemma 4.21 and define

$$\hat{w}(x,t) = G + \sqrt{2}Me^{-\lambda(t-T)}\cos\bigl(a(x+\widetilde{H})\bigr) \quad \text{for } x \in [-\widetilde{H}, \widetilde{H}],\ t \geqslant T,$$

where $a = \pi/(8\widetilde{H})$ and $\lambda > 0$ is a constant to be specified later. Then

$$\hat{w}_t = \frac{\lambda}{a^2}\hat{w}_{xx} < 0, \qquad \hat{w}_x < 0, \qquad G \leqslant \hat{w} \leqslant G + \sqrt{2}M \quad \text{for } x \in [-\widetilde{H}, \widetilde{H}],\ t \geqslant T.$$

Now set

$$\lambda = \frac{a^2}{1 + (\sqrt{2}M + G)^2}.$$

Then, since $\hat{w}_{xx} < 0$, $\hat{w}_x < 0$,

$$\hat{w}_t = \frac{\hat{w}_{xx}}{1 + (\sqrt{2}M + G)^2} \geqslant \frac{\hat{w}_{xx}}{1 + \hat{w}^2} - \frac{2\hat{w}\hat{w}_x^2}{(1 + \hat{w}^2)^2} + \frac{A\hat{w}\hat{w}_x}{\sqrt{1 + \hat{w}^2}}.$$

Furthermore, $\hat{w}(x,T) \geqslant M$ for $x \in [-\widetilde{H}, \widetilde{H}]$.

Let $u(x,t)$ be any global solution of (1.2)–(1.3) and let $I(t) := [\zeta_-(t), \zeta_+(t)]$ be the x-interval on which the solution u is defined at $t \geqslant 0$. Then, by Lemma 4.21 and the comparison theorem, we have

$$u_x(x,t) \leqslant \hat{w}(x,t) \quad \text{for } x \in I(t),\ t \geqslant T.$$

Similarly $u_x(x,t) \geqslant -\hat{w}(-x,t)$, hence

$$\left|u_x(x,t)\right| \leqslant \hat{w}(x,t) \leqslant G + \sqrt{2}Me^{-\lambda(t-T)} \quad \text{for } x \in I(t),\ t \geqslant T.$$

The lemma is proved. □

Proof of Proposition 4.17. Let $I(t) := [\zeta_-(t), \zeta_+(t)]$ be the x-interval on which the solution u is defined at $t \geqslant 0$. We note that $w := u_x(x,t)$ satisfies (4.38). Since $|g'(u(\zeta_\pm(t),t))| \leqslant G$, we see by the maximum principle that

$$\bigl|w(x,t)\bigr| \leqslant \max\Bigl\{\max_{x\in I(0)}\bigl|u_0'(x)\bigr|, G\Bigr\}$$

for all $x \in I(t)$ and $t \geqslant 0$. The same argument shows that

$$\rho(t) := \max\Bigl\{\max_{x\in I(t)}\bigl|w(x,t)\bigr|, G\Bigr\}$$

is monotonically non-increasing in t. Furthermore, Lemma 4.22 implies

$$G \leqslant \rho(t) \leqslant G + Ce^{-\lambda t} \quad \text{for } t \geqslant T,$$

where the positive constants T, C and λ are independent of u. Thus we obtain

$$\lim_{t\to+\infty} \rho(t) = G. \tag{4.40}$$

The conclusion of the proposition immediately follows from (4.40) and the definition of C^1_{ad}. □

4.6. Sequence of solution-triples

In this subsection we consider Eq. (1.2) on Ω_h for each $h \in \mathcal{H}_g$. More precisely, we consider solutions of

$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2}, \quad \zeta_-(t) < x < \zeta_+(t),\ t > 0, \tag{4.41$_h$}$$

$$\begin{cases} u_x\bigl(\zeta_-(t),t\bigr) = h_-'\bigl(u\bigl(\zeta_-(t),t\bigr)\bigr), \quad u_x\bigl(\zeta_+(t),t\bigr) = -h_+'\bigl(u\bigl(\zeta_+(t),t\bigr)\bigr), \\ \bigl(\zeta_\pm(t), u(\zeta_\pm(t),t)\bigr) \in \partial_\pm\Omega_h. \end{cases} \tag{4.42$_h$}$$

and prove the continuous dependence of the solution triple (u_h, Ω_h, I) on h.

We first note that all the existence results and the a priori estimates in the previous sections remain valid for solutions on Ω_h, provided that the symbols such as $C^{2+\mu,1+\mu/2}$ or $C^{\mu,\mu/2}$ are understood to denote the corresponding function spaces on Ω_h. The class of admissible functions C^1_{ad} has also to be replaced by $C^1_{\mathrm{ad}}(h)$, which is defined as in Definition 4.7 by using Ω_h instead of Ω. (The constant G remains the same for any $h \in \mathcal{H}_g$.)

For each $h \in \mathcal{H}_g$, we have introduced the coordinate transformation $(x,y) \mapsto (\xi_h, \eta_h)$ in Section 4.2, where ξ_h is defined as the solution of

$$\begin{cases} \Delta\xi_h = 0, & \text{in } \Omega_h, \\ \xi_h = -H, & \text{on } \partial_-\Omega_h, \\ \xi_h = H, & \text{on } \partial_+\Omega_h, \end{cases} \tag{4.43}$$

and η_h is the conjugate harmonic function of ξ_h satisfying the normalization condition $\eta_h(0,0) = 0$. This coordinate transformation converts Ω_h into a flat cylinder $\Omega_0 := (-H,H) \times \mathbb{R}$, and curves in Ω_h into those in Ω_0. If a curve in Ω_h is expressed as the graph of a function $u(x)$, then the corresponding curve in Ω_0 is expressed as the graph of the function $\eta_h = v(\xi_h)$, where the function v is given by

$$\eta_h\big(x, u(x)\big) = v\big(\xi_h\big(x, u(x)\big)\big).$$

We denote this function v by $V_h[u]$. In view of (4.10), we see that

$$V_{\sigma_a h}[\sigma_a u] = V_h[u] - \eta_h(0, a). \tag{4.44}$$

In what follows convergence of solutions u_n on Ω_{h_n} will be understood in terms of the convergence of $V_{h_n}[u_n]$ on Ω_0.

Lemma 4.23. *Let $\{h_n\}$ be a sequence in $\mathcal{H}_g$ converging to $h^* \in \mathcal{H}_g$ in $L^\infty_{loc}(\mathbb{R}; \mathbb{R}^2)$, and let $w_n \in C^1_{\mathrm{ad}}(h_n)$ be a sequence of functions such that*

$$\lim_{n\to\infty} V_{h_n}[w_n] = v^* \quad \text{in } C^1\big([-H, H]\big) \tag{4.45}$$

for some C^1 function v^* and that

$$\sup_{n\in\mathbb{N}} \|w_n'\|_{L^\infty} < \frac{1}{G}. \tag{4.46}$$

Then there exists a (unique) function $w^* \in C^1_{\mathrm{ad}}(h^*)$ such that $v^* = V_{h^*}[w^*]$.

Proof. Since $w_n \in C^1_{\mathrm{ad}}(h_n)$, the graph of w_n meets $\partial_-\Omega_{h_n}$ and $\partial_+\Omega_{h_n}$ perpendicularly at its endpoints. The same is true of the graph of the converted function $V_{h_n}[w_n]$ and boundaries $\partial_\pm\Omega_0$ since the coordinate transformation $(x, y) \mapsto (\xi_{h_n}, \eta_{h_n})$ is conformal up to the boundary.

Next we denote by $(\xi, \eta) \mapsto (X_{h_n}, Y_{h_n})$ the inverse transformation of $(x, y) \mapsto (\xi_{h_n}, \eta_{h_n})$. Then, since $w_n \in C^1_{\mathrm{ad}}(h_n)$, we see from (4.4) that

$$(X_{h_n})_\xi + (X_{h_n})_\eta\big(V_{h_n}[w_n]\big)' \geqslant \delta_1 > 0,$$

where the constant δ_1 is independent of n, since it is determined by $\sup_n \|(\xi_{h_n})_x\|_{L^\infty}$, which is equal to $\|(\xi_g)_x\|_{L^\infty}$. Letting $n \to \infty$ in the above inequality and using (4.22), we obtain

$$(X_{h^*})_\xi + (X_{h^*})_\eta\big(v^*\big)' \geqslant \delta_1.$$

This implies that there exists a function w^* such that $v^* = V_{h^*}[w^*]$. Since the graph of v^* is perpendicular to $\partial_\pm\Omega_0$, the same is true of the graph of w^* and $\partial_\pm\Omega_{h^*}$. Furthermore, (4.46) yields $\|(w^*)_x\|_{L^\infty} < 1/G$. Hence $w^* \in C^1_{\mathrm{ad}}(h^*)$. The proof of the lemma is complete. □

Lemma 4.24. *Let $u \in C^{2+\mu,1+\mu/2}_{loc}$ be a global classical solution of (4.41$_h$)–(4.42$_h$) and let a_n and $t_n \to +\infty$ be sequences in $\mathbb{R}$ such that*

(i) *$\sigma_{a_n} h \to h^*$ in $L^\infty_{loc}(\mathbb{R}; \mathbb{R}^2)$ as $n \to \infty$;*
(ii) *for any $T > 0$ there exists a constant $C_T > 0$ independent of n satisfying*

$$\sup_{t\in[-T,T]} \big\|\sigma_{a_n} u(\cdot, t + t_n)\big\|_{L^\infty} \leqslant C_T$$

for all n with $t_n > T$.

Then the sequence of solution triples (u_n, Ω_{h_n}, I_n) *defined by*

$$u_n(x,t) := \sigma_{a_n} u(x, t + t_n), \qquad h_n := \sigma_{a_n} h, \qquad I_n := (-t_n, +\infty)$$

has a subsequence which converges to some solution triple $(u^*, \Omega_{h^*}, \mathbb{R})$ *with* $u^* \in C^{2+\mu, 1+\mu/2}_{loc}$ *in the sense of Definition 2.13. Furthermore the convergence* $u_n \to u^*$ *takes place in the* $C^{2,1}_{loc}$ *topology in the sense that*

$$V_{h_n}[u_n] \to V_{h^*}\big[u^*\big] \quad \text{in } C^{2,1}_{loc}\big([-H, H] \times \mathbb{R}\big),$$

where V_h *is as in Lemma 4.23.*

Proof. By Proposition 4.17, there exists $t_* \geqslant 0$ satisfying $u(\cdot, t) \in C^1_{\mathrm{ad}}$ for $t \geqslant t_*$. Then $u_n(\cdot, t) \in C^1_{\mathrm{ad}}(h_n)$ for $t \geqslant t_* - t_n$. Therefore $v_n := V_{h_n}[u_n]$ can be defined for $t \geqslant t_* - t_n$ and solves the following problem:

$$\begin{aligned} &v_t = d_{h_n}(\xi, v, v_\xi) v_{\xi\xi} + f_{h_n}(\xi, v, v_\xi), \quad \xi \in (-H, H),\ t > t_* - t_n, \\ &v_\xi(-H, t) = v_\xi(H, t) = 0, \quad t > t_* - t_n. \end{aligned} \tag{4.47}$$

Here for $h \in \mathcal{H}_g$ we define

$$\begin{aligned} J(\xi, \eta; h) &:= \frac{1}{(X_h)^2_\xi(\xi, \eta) + (X_h)^2_\eta(\xi, \eta)}, \\ d_h(\xi, \eta, p) &:= \frac{J(\xi, \eta; h)}{1 + p^2}, \\ f_h(\xi, \eta, p) &:= -\frac{1}{2} J_\xi(\xi, \eta; h) p + \frac{1}{2} J_\eta(\xi, \eta; h) + A\sqrt{J(\xi, \eta; h)\big(1 + p^2\big)}, \end{aligned} \tag{4.48}$$

where (X_h, Y_h) denotes the inverse map of (ξ_h, η_h). By Corollary 2.9, we see that the uniform estimates (Lemma 4.18) hold for solutions of (4.23)–(4.24) with d, f replaced by d_h, f_h, respectively.

We fix $T > 0$ and take $n_0 = n_0(T) \in \mathbb{N}$ satisfying $t_n > t_* + T$ for all $n > n_0$. By the assumption (ii), there exists some constant $\widetilde{C}_T > 0$ independent of n such that $\|v_n(\cdot, t)\|_{L^\infty([-H,H])} \leqslant \widetilde{C}_T$ for all $n > n_0$ and $t \in [-T, T]$. Hence, applying the uniform estimates (Lemma 4.18) to v_n, we obtain

$$\|v_n\|_{C^{2+\mu, 1+\mu/2}([-H,H]\times[-T,T])} \leqslant M_T$$

for some constant $M_T > 0$ that depends on T but is independent of n. Therefore, by Cantor's diagonal argument, one can find a subsequence of $\{v_n\}_n$ (again denoted by $\{v_n\}_n$) and a function $v^* \in C^{2+\mu, 1+\mu/2}_{loc}([-H, H] \times \mathbb{R})$ such that

$$\lim_{n\to\infty} \|v_n - v^*\|_{C^{2,1}_{loc}([-H,H]\times\mathbb{R})} = 0. \tag{4.49}$$

This, together with (4.47) and (4.22), implies that

$$\begin{aligned} v^*_t = d_{h^*}\big(\xi, v^*, v^*_\xi\big) v^*_{\xi\xi} + f_{h^*}\big(\xi, v^*, v^*_\xi\big), \quad (x, t) \in (-H, H) \times \mathbb{R}, \\ v^*_\xi(-H, t) = v^*_\xi(H, t) = 0, \quad t \in \mathbb{R}. \end{aligned} \tag{4.50}$$

For each $t \in \mathbb{R}$, we have

$$\left\|(u_n)_x(\cdot,t)\right\|_{L^\infty} = \left\|u_x(\cdot,t+t_n)\right\|_{L^\infty} \leqslant \max\left\{G, \left\|u_x(\cdot,t_*)\right\|_{L^\infty}\right\} < \frac{1}{G}$$

for all n with $t+t_n \geqslant t_*$. Hence, by Lemma 4.23, there exists $u^*(\cdot,t) \in C^1_{\rm ad}(h^*)$ for each $t \in \mathbb{R}$. Then (4.50) yields that $(u^*, \Omega_{h^*}, \mathbb{R})$ is a solution triple.

Next we show that

$$d_{\mathcal{H}}\bigl(\gamma\bigl(u_n(\cdot,t)\bigr), \gamma\bigl(u^*(\cdot,t)\bigr)\bigr) \to 0 \quad \text{as } n \to \infty$$

locally uniformly in $t \in \mathbb{R}$. In what follows we write $\gamma_t^n = \gamma(u_n(\cdot,t))$ and $\gamma_t^* = \gamma(u^*(\cdot,t))$ for brevity. We fix $T > 0$ and $t \in [-T,T]$. For any point $(x_0,y_0) \in \gamma_t^n$, there exists a unique $\xi_0 \in [-H,H]$ such that

$$x_0 = X_{h_n}\bigl(\xi_0, v_n(\xi_0,t)\bigr), \qquad y_0 = Y_{h_n}\bigl(\xi_0, v_n(\xi_0,t)\bigr).$$

Using this ξ_0, we define a point $(x^*,y^*) \in \gamma_t^*$ by

$$x^* = X_{h^*}\bigl(\xi_0, v^*(\xi_0,t)\bigr), \qquad y^* = Y_{h^*}\bigl(\xi_0, v^*(\xi_0,t)\bigr).$$

Then we have

$$\begin{aligned} d\bigl((x_0,y_0),\bigl(x^*,y^*\bigr)\bigr) \leqslant{}& \bigl|X_{h_n}\bigl(\xi_0, v_n(\xi_0,t)\bigr) - X_{h^*}\bigl(\xi_0, v^*(\xi_0,t)\bigr)\bigr| \\ &+ \bigl|Y_{h_n}\bigl(\xi_0, v_n(\xi_0,t)\bigr) - Y_{h^*}\bigl(\xi_0, v^*(\xi_0,t)\bigr)\bigr|, \end{aligned}$$

where d denotes the Euclidean metric in $\mathbb{R}^2$. This implies

$$\begin{aligned} d\bigl(\gamma_t^n, \gamma_t^*\bigr) &\leqslant \sup_{(x_0,y_0)\in\gamma_t^n} d\bigl((x_0,y_0),\bigl(x^*,y^*\bigr)\bigr) \\ &\leqslant \bigl\|X_{h_n}\bigl(\cdot, v_n(\cdot,t)\bigr) - X_{h^*}\bigl(\cdot, v^*(\cdot,t)\bigr)\bigr\| + \bigl\|Y_{h_n}\bigl(\cdot, v_n(\cdot,t)\bigr) - Y_{h^*}\bigl(\cdot, v^*(\cdot,t)\bigr)\bigr\|. \end{aligned}$$

Here we define

$$d(A,B) := \max_{(x_1,y_1)\in A} \min_{(x_2,y_2)\in B} d\bigl((x_1,y_1),(x_2,y_2)\bigr)$$

for two compact sets A, B in $\mathbb{R}^2$. In a similar way, we obtain the same inequality for $d(\gamma_t^*, \gamma_t^n)$; hence we obtain

$$\begin{aligned} d_{\mathcal{H}}\bigl(\gamma_t^n, \gamma_t^*\bigr) &= \max\bigl\{d\bigl(\gamma_t^n, \gamma_t^*\bigr), d\bigl(\gamma_t^*, \gamma_t^n\bigr)\bigr\} \\ &\leqslant \bigl\|X_{h_n}\bigl(\cdot, v_n(\cdot,t)\bigr) - X_{h^*}\bigl(\cdot, v^*(\cdot,t)\bigr)\bigr\| + \bigl\|Y_{h_n}\bigl(\cdot, v_n(\cdot,t)\bigr) - Y_{h^*}\bigl(\cdot, v^*(\cdot,t)\bigr)\bigr\|. \end{aligned}$$

By virtue of (4.22), (4.49) and the fact that X_{h^*}, Y_{h^*} are uniformly continuous, we see that $d_{\mathcal{H}}(\gamma_t^{n_j}, \gamma_t^*) \to 0$ as $j \to 0$ uniformly in $t \in [-T,T]$. Furthermore, (4.49) implies that the above convergence takes place in the $C^{2,1}$ sense. □

5. Existence and uniqueness of the entire solution

The aim of this section is to prove the existence and uniqueness of the entire solution (namely the solution defined for all $t \in \mathbb{R}$) of (1.2)–(1.3), or more generally, that of (4.41$_h$)–(4.42$_h$). As we have mentioned before in Section 1, our strategy is first to construct a monotone increasing solution $\bar{u}$ that is defined for all $t \geqslant 0$, then to construct an entire solution U by considering the ω-limit point of $\bar{u}$ in a certain sense that will be specified later. The assumption that Ω (or Ω_h) is recurrent plays a key role in the proof of the existence and uniqueness.

Our standing hypothesis in this section is the following:

$$\text{There exists no stationary solution in } \Omega. \tag{5.1}$$

For some of the results we need a stronger version of (5.1):

$$\text{There exists no stationary solution in } \Omega_h \text{ for any } h \in \mathcal{H}_g. \tag{5.2}$$

A sufficient condition for (5.2) is (1.9), namely $A > 0$ and $2AH \geqslant \sin\alpha_+ + \sin\alpha_-$, as we will see in Remark 5.2 below.

5.1. A geometrical condition

Here we give a sufficient condition that guarantees non-existence of a stationary solution.

Lemma 5.1. *If $A > 0$ and $2AH \geqslant \sin\alpha_+ + \sin\alpha_-$, then problem (1.2)–(1.3) has no stationary solution.*

Proof. Suppose that there exists a stationary solution $v(x)$ of (1.2)–(1.3). Then the graph of v is a circular arc of constant curvature $-A$ whose endpoints meet the boundaries $\partial_\pm\Omega$ perpendicularly. Naturally the radius of this arc is $1/A$. Let $(x_\pm, v(x_\pm))$ be the endpoints on $\partial_\pm\Omega$, respectively. Since $A > 0$, the tangential direction of the arc is not horizontal at least at one of the two endpoints. This means that at least one of the two endpoints of the arc lies outside of the region $\{-H \leqslant x \leqslant H\}$. Consequently, we have $x_+ - x_- > 2H$.

On the other hand, since

$$v'(x_-) \leqslant \sup g'_-(y) = \tan\alpha_-, \qquad v'(x_+) \geqslant -\sup g'_+(y) = -\tan\alpha_+,$$

a simple geometric observation shows that

$$x_+ - x_- \leqslant \frac{\sin\alpha_+ + \sin\alpha_-}{A} \leqslant 2H.$$

This contradiction proves the lemma. □

Remark 5.2. Take any $h \in \mathcal{H}_g$, and consider problem (1.1) in Ω_h instead of Ω under the same Neumann boundary conditions. Then the conclusion of Lemma 5.1 holds for Ω_h, since $h_\pm$ satisfy

$$\sup h'_+ \leqslant \tan\alpha_+, \qquad \sup h'_- \leqslant \tan\alpha_-.$$

5.2. Existence of entire solutions

In this subsection we prove the following proposition on the existence of an entire solution under the assumption (5.1).

Proposition 5.3. *Assume* (5.1)*. Then* (1.2)–(1.3) *has an entire solution* $U(x,t)$ *with the following properties*:

(i) $U \in C^{2+\mu,1+\mu/2}_{loc}(\overline{Q})$, *where*

$$Q := \{(x,t) \mid (x, U(x,t)) \in \Omega,\ t \in \mathbb{R}\}; \tag{5.3}$$

(ii) $|U_x(x,t)| \leqslant G$ *and* $0 < U_t(x,t) \leqslant C$ *for all* $(x,t) \in \overline{Q}$ *for some constant* $C > 0$;
(iii) $\max_x U(x,t) \to -\infty$ *as* $t \to -\infty$, $\min_x U(x,t) \to +\infty$ *as* $t \to +\infty$;
(iv) *if, in addition,* (5.2) *holds, then there exists a constant* $\rho > 0$ *such that* $U_t(x,t) \geqslant \rho$ *for all* $(x,t) \in \overline{Q}$.

The above proposition follows immediately by combining the lemmas below. In fact, the assertions (i), (ii) follow from Lemma 5.6, and the assertions (iii), (iv) from Lemma 5.7.

We start with constructing a solution $\bar{u}(x,t)$ that is defined for all $t \geqslant 0$ and monotone increasing in t. We can take a linear function as the initial value $\bar{u}(x,0)$ (that is, the graph of $\bar{u}(x,0)$ is a line segment) due to the following lemma:

Lemma 5.4. *Assume* (G1)–(G4)*. Then there exists a relatively dense family of line segments in* $\overline{\Omega}$ *whose end-points contact* $\partial_\pm\Omega$ *perpendicularly.*

Proof. For $y \in \mathbb{R}$ we write

$$p_\pm(y) = \big(\pm\big(H + g_\pm(y)\big), y\big) \in \partial_\pm\Omega$$

and

$$d_\pm(y) = d\big(p_\pm(y), \partial_\mp\Omega\big) := \inf_{z\in\mathbb{R}} d\big(p_\pm(y), p_\mp(z)\big),$$

where d denotes the Euclidean metric in $\mathbb{R}^2$. Since

$$2H \leqslant d_\pm(y) \leqslant 2H + \|g_+ + g_-\|_{L^\infty(\mathbb{R})}$$

for $y \in \mathbb{R}$, there exists a positive constant m_0 independent of y such that

$$d_\pm(y) = \inf_{|z-y|\leqslant m_0} d\big(p_\pm(y), p_\mp(z)\big). \tag{5.4}$$

Let

$$d_0 := \inf_{y,z\in\mathbb{R}} d\big(p_+(y), p_-(z)\big). \tag{5.5}$$

We may assume that there exists some $y_0 \in \mathbb{R}$ satisfying $d_+(y_0) > d_0$ and $d_-(y_0) > d_0$; otherwise the assertion of the lemma is obvious. Then

$$d_1 := \min\{d_+(y_0), d_-(y_0)\} - d_0 > 0.$$

By (5.5), there exist $y_1, z_1 \in \mathbb{R}$ satisfying

$$d\big(p_+(y_1), p_-(z_1)\big) \leqslant d_0 + \frac{d_1}{4}.$$

Let $Y_0 > 0$ be such that $y_1, z_1 \in (-Y_0, Y_0)$ and $[y_0 - m_0, y_0 + m_0] \subset (-Y_0, Y_0)$, where m_0 is the constant in (5.4). Then we can choose $\varepsilon_0 > 0$ sufficiently small such that for any $a \in A_{\varepsilon_0, Y_0}$ we have

$$d\big(p_+(y_1 + a), p_-(z_1 + a)\big) \leqslant d_0 + \frac{d_1}{2}, \qquad d_\pm(y_0 + a) \geqslant d_0 + \frac{3}{4} d_1. \tag{5.6}$$

Here we used the notation in Definition 2.3, namely

$$A_{\varepsilon_0, Y_0} := \big\{a \in \mathbb{R} \;\big|\; \|\sigma_a g - g\|_{L^\infty((-Y_0, Y_0); \mathbb{R}^2)} < \varepsilon_0\big\}.$$

Since the functions $d_\pm(y)$ are continuous, the above inequalities (5.6) imply that for any $a_-, a, a_+ \in A_{\varepsilon_0, Y_0}$ with $y_1 + a, z_1 + a \in (y_0 + a_-, y_0 + a_+)$ the function $d(p_+(y), p_-(z))$ attains its local minimum over the interval $[y_0 + a_-, y_0 + a_+] \times [y_0 + a_-, y_0 + a_+]$ at an interior point (y^*, z^*). Therefore, the line segment connecting $p_+(y^*)$ and $p_-(z^*)$ intersects $\partial_\pm \Omega$ perpendicularly.

By the recurrence of g, the set A_{ε_0, Y_0} is relatively dense in $\mathbb{R}$. Thus the assertion of the lemma is proved. □

Let l be one of the line segment in the previous lemma and let $\bar{u}_0(x)$ be a linear function such that l is the graph of $\bar{u}_0$. Then we have $\bar{u}_0 \in C^1_{\mathrm{ad}} \cap C^2$ and

$$\big|\bar{u}_{0x}(x)\big| \leqslant G, \qquad \frac{\bar{u}_{0xx}}{(1 + \bar{u}_{0x}^2)^{3/2}} = 0 > -A. \tag{5.7}$$

Let $\bar{u}(x, t)$ be the global solution of (1.2)–(1.3) with initial data $\bar{u}_0(x)$. Then Lemma 4.6 implies that

$$\big|\bar{u}_x(x, t)\big| \leqslant G \quad \text{for all } x \in \big[\bar{\zeta}_-(t), \bar{\zeta}_+(t)\big],\ t \geqslant 0, \tag{5.8}$$

where $[\bar{\zeta}_-(t), \bar{\zeta}_+(t)]$ denotes the horizontal span of the solution curve $\bar{u}(x, t)$ at each time $t \geqslant 0$.

Lemma 5.5. *Assume (5.1) and let $\bar{u}$ be as above. Then:*

(i) *There exists $C > 0$ such that $0 \leqslant \bar{u}_t(x, t) \leqslant C$ for all $x \in [\bar{\zeta}_-(t), \bar{\zeta}_+(t)]$, $t \geqslant 0$.*
(ii) *$\bar{u}(0, t)$ diverges to $+\infty$ as $t \to +\infty$.*

Proof. (i) Since $\bar{u}_t(x, 0) > 0$, the maximum principle yields that

$$\bar{u}_t(x, t) \geqslant 0 \quad \text{for all } x \in \big[\bar{\zeta}_-(t), \bar{\zeta}_+(t)\big],\ t \geqslant 0. \tag{5.9}$$

The upper bound on $\bar{u}_t$ is derived from Corollary 4.19 and the continuity of $\bar{u}_t$ up to $t = 0$.

(ii) Suppose that $\bar{u}(0, t)$ is bounded. Then by (5.8), $\|\bar{u}(\cdot, t)\|_{L^\infty}$ is also bounded in t. This and Corollary 4.19 imply that for any $\delta > 0$, $\|\bar{u}(\cdot, t)\|_{C^{2+\mu}}$ is bounded for $t \geqslant \delta$. Therefore, $\bar{u}$ converges to some C^2 function as $t \to +\infty$, which is necessarily a stationary solution of (1.2)–(1.3). This contradicts (5.1). □

Lemma 5.6. *Problem (1.2)–(1.3) has an entire solution $U(x,t)$ with the following properties*:

(i) $U \in C^{2+\mu,1+\mu/2}_{loc}(\overline{Q})$, *where* $Q := \{(x,t) \mid (x, U(x,t)) \in \Omega,\ t \in \mathbb{R}\}$;
(ii) $|U_x(x,t)| \leqslant G$ *and* $0 \leqslant U_t(x,t) \leqslant C$ *for all* $(x,t) \in \overline{Q}$, *where* C *is the same constant as in Lemma 5.5. If, in addition,* U *is not a stationary solution, then* $U(x,t) > 0$ *for all* $(x,t) \in \overline{Q}$.

Proof. If (1.2)–(1.3) has a stationary solution $V(x)$, then $U(x,t) := V(x)$ is obviously an entire solution of (1.2)–(1.3) with $U_t \equiv 0$. Furthermore, Propositions 4.16 and 4.17 imply $V \in C^{2+\mu,1+\mu/2}_{loc}$ and $\|V\|_{L^\infty} \leqslant G$.

Next we assume (5.1). We construct an entire solution U by using a standard renormalization argument. Choose a sequence $a_1 < a_2 < a_3 < \cdots \to +\infty$ such that

$$\sigma_{a_n} g \to g \quad \text{in } L^\infty_{loc}\big(\mathbb{R}; \mathbb{R}^2\big) \text{ as } n \to \infty.$$

Without loss of generality, we may assume that $a_1 > \|\bar{u}_0\|_{L^\infty}$. By Lemma 5.5(ii), we can take a sequence $t_1 < t_2 < t_3 < \cdots \to +\infty$ satisfying $\bar{u}(0, t_n) = a_n$. Then for each $n \in \mathbb{N}$, (u_n, Ω_{g_n}, I_n) with

$$u_n := \sigma_{a_n} \bar{u}(x, t + t_n), \qquad g_n := \sigma_{a_n} g, \qquad I_n := [-t_n, \infty)$$

is a solution triple satisfying $u_n(0,0) = 0$.

We fix $T > 0$. By (5.8) and Lemma 5.5(i), we have

$$\big\|u_n(\cdot,t)\big\|_{L^\infty} \leqslant M_T$$

for all $t \in [-T, T]$ and $n \in \mathbb{N}$ with $t_n > T$, where the constant M_T depends on T but is independent of n. Hence, applying Lemma 4.24 to $\{(u_n, \Omega_{g_n}, I_n)\}_n$, we can find a subsequence (also denoted by $\{(u_n, \Omega_{g_n}, I_n)\}_n$) and a solution triple $(U, \Omega, \mathbb{R})$ with $U \in C^{2+\mu,1+\mu/2}_{loc}$ such that

$$(u_n, \Omega_{g_n}, I_n) \xrightarrow{n\to\infty} (U, \Omega, \mathbb{R}).$$

Then the function U is an entire solution of (1.2)–(1.3) with $U(0,0) = 0$.

Furthermore, since $|(u_n)_x(x,t)| \leqslant G$, $0 \leqslant (u_n)_t(x,t) \leqslant C$ for all $t \geqslant -t_n$ and since the convergence u_n to U takes place in the $C^{2,1}$ topology, we obtain the inequalities $\|U_x\|_{L^\infty} \leqslant G$ and $0 \leqslant U_t \leqslant C$. The strict inequality $U_t > 0$ follows from the strong maximum principle. □

Lemma 5.7. *Assume (5.1) and let* $U(x,t)$ *be the entire solution of (1.2)–(1.3) in Lemma 5.6. Then*

(i) $\max_x U(x,t) \to -\infty$ *as* $t \to -\infty$, $\min_x U(x,t) \to +\infty$ *as* $t \to +\infty$;
(ii) *if, in addition, (5.2) holds, then there exists a constant* $\rho > 0$ *such that* $U_t(x,t) \geqslant \rho$ *for all* $(x,t) \in \overline{Q}$.

Proof. (i) Since $U(x,t)$ is monotone increasing in t, the limit

$$p_\infty := \lim_{t\to+\infty} U(0,t) \in \mathbb{R} \cup \{+\infty\}$$

exists. If $p_\infty < +\infty$, then by the boundedness of U_x, $U(x,t)$ remains bounded, hence it converges to some function $V(x)$ in the $C^{2,1}$ sense as $t \to +\infty$. In this case, the limit function V is a stationary solution, but this is impossible by the assumption (5.1). Thus we have $p_\infty = +\infty$. This and the boundedness of U_x implies $\min_x U(x,t) \to +\infty$ as $t \to \infty$. The other assertion $U(x,t) \to -\infty$ (as $t \to -\infty$) can be shown in the same way.

(ii) Suppose that there exists a sequence $(x_n, t_n) \in \overline{Q}$ with $U_t(x_n, t_n) \to 0$ as $n \to \infty$. We may assume that $x_n \to x^*$ for some x^* as $n \to \infty$. Set

$$b_n := U(0, t_n), \qquad U_n(x, t) := \sigma_{b_n} U(x, t + t_n), \qquad h_n := \sigma_{b_n} g.$$

Then $(U_n, \Omega_{h_n}, \mathbb{R})$ $(n \in \mathbb{N})$ are solution triples with $U_n(0, 0) = 0$.

By an argument similar to the one in the proof of (i), taking a subsequence if necessary, we see that

$$(U_n, \Omega_{h_n}, \mathbb{R}) \xrightarrow{n\to\infty} \big(U^*, \Omega_{h^*}, \mathbb{R}\big) \tag{5.10}$$

for some $h^* \in \mathcal{H}_g$ and some entire solution U^* of (1.2)–(1.3) in Ω_{h^*}. Since $(x_n, t_n) \in \overline{Q}$, we have

$$\big(x_n, U_n(x_n, 0)\big) \in \overline{\Omega}_{h_n}, \quad n \in \mathbb{N}. \tag{5.11}$$

Furthermore, the facts that $U_n(0, 0) = 0$ and that $|(U_n)_x| \leqslant G$ imply

$$\big|U_n(x_n, 0)\big| \leqslant G\widetilde{H},$$

where $\widetilde{H} := H + \max\{\|g_+\|_{L^\infty(\mathbb{R})}, \|g_-\|_{L^\infty(\mathbb{R})}\}$. Combining this with (5.10) and (5.11), we see that $(x^*, U^*(x^*, 0)) \in \overline{\Omega}_{h^*}$, in other words, $(x^*, 0) \in \overline{Q^*}$, where Q^* denotes the space–time region on which U^* is defined. Since $U_t \geqslant 0$ in $\overline{Q}$, we have $U^*_t \geqslant 0$ in $\overline{Q^*}$. Furthermore,

$$U^*_t\big(x^*, 0\big) = \lim_{n\to\infty} (U_n)_t(x_n, 0) = \lim_{n\to\infty} U_t(x_n, t_n) = 0.$$

Suppose $(x^*, 0) \in Q^*$. Then the strong maximum principle yields $U^*_t \equiv 0$, in other words, U^* is a stationary solution of (4.41_h)–(4.42_h) with $h = h^*$, but this is impossible by the assumption (5.2). Next we assume $(x^*, 0) \in \partial Q^*$. Then differentiating (4.42_h) with $h = h^*$ and $u = U^*$ by t, we see that $(U^*_t)_x(x^*, 0) = 0$. This is again impossible by (5.2) and the Hopf boundary lemma. □

5.3. Uniqueness of entire solution

In this subsection, we prove that the entire solution $U(x, t)$ of (1.2)–(1.3) is unique up to time shift. Let $U(x, t)$ be an entire solution having the properties of Proposition 5.3(i), (ii), (iii), and let $W(x, t)$ be any entire solution of (1.2)–(1.3). Then the following function $\Lambda_{U,W}$ is well-defined:

$$\Lambda_{U,W}(t) := \inf\left\{\Lambda > 0 \;\middle|\; \begin{array}{l}\text{there exists } \tau \in \mathbb{R} \text{ such that}\\ U(\cdot, t + \tau) \preccurlyeq W(\cdot, t) \preccurlyeq U(\cdot, t + \tau + \Lambda)\end{array}\right\}, \tag{5.12}$$

where the symbol $\preccurlyeq$ is defined in Section 4.4. The function $\Lambda_{U,W}(t)$ has the following properties:

Lemma 5.8. *Assume* (5.1)*.*

(i) *The function $\Lambda_{U,W}(t)$ is monotone decreasing and is bounded in $t \in \mathbb{R}$.*
(ii) *If $\Lambda_{U,W}(t_0) = 0$ for some t_0, then there exists $\tau \in \mathbb{R}$ such that $U(\cdot, t + \tau) \equiv W(\cdot, t)$ for $t \geqslant t_0$. If $\Lambda_{U,W}(t_0) > 0$ for some t_0, then $\Lambda_{U,W}(t)$ is positive and is strictly decreasing in $t < t_0$.*

Before proving this lemma, we first show the following:

Lemma 5.9. *Assume (5.1) and let $W(x,t)$ be any entire solution of (1.2)–(1.3). Denote by $\overline{Q}_W$ the set of all (x,t) for which $W(x,t)$ is defined. Then*

(i) $|W_x(x,t)| \leqslant G$ *for all* $(x,t) \in \overline{Q}_W$;
(ii) $\max_x W(x,t) - \min_x W(x,t) \leqslant 2G\widetilde{H}$ *for* $t \in \mathbb{R}$*, where*

$$\widetilde{H} := H + \max\{\|g_+\|_{L^\infty(\mathbb{R})}, \|g_-\|_{L^\infty(\mathbb{R})}\}; \tag{5.13}$$

(iii) *there exists a constant* $C > 0$ *such that* $|W_t(x,t)| \leqslant C$ *for all* $(x,t) \in \overline{Q}_W$;
(iv) $W(0,t) \to \pm\infty$ *as* $t \to \pm\infty$.

Proof. (i) By Lemma 4.22 we have, for any $t \in \mathbb{R}$ and $\tau \in (-\infty, t-T]$,

$$|W_x(x,t)| \leqslant G + Ce^{-\lambda(t-\tau)}.$$

Letting $\tau \to -\infty$, we obtain $|W_x(x,t)| \leqslant G$ for all $(x,t) \in \overline{Q}_W$.

(ii) The assertion follows immediately from (i).

(iii) The statement (i) implies $W(\cdot,t) \in C^1_{\mathrm{ad}}$ for all $t \in \mathbb{R}$. In view of this and Corollary 4.19, we obtain the boundedness of W_t.

(iv) Let $U(x,t)$ be the entire solution in Lemma 5.7. Then we can choose $t_1 < t_2$ such that $U(\cdot,t_1) \preccurlyeq W(\cdot,0) \preccurlyeq U(\cdot,t_2)$. Therefore, the assertion follows from Lemma 5.7 and the comparison theorem. □

Now we are ready to prove Lemma 5.8.

Proof of Lemma 5.8. (i) By the definition of $\Lambda_{U,W}$, for each fixed $t \in \mathbb{R}$, there exist $\tau \in \mathbb{R}$ such that

$$U(\cdot, t+\tau) \preccurlyeq W(\cdot,t) \preccurlyeq U\big(\cdot, t+\tau+\Lambda_{U,W}(t)\big). \tag{5.14}$$

Therefore, it follows from the comparison principle (Proposition 4.14) that for any $s > 0$,

$$U(\cdot, t+s+\tau) \preccurlyeq W(\cdot,t+s) \preccurlyeq U\big(\cdot, t+s+\tau+\Lambda_{U,W}(t)\big).$$

This implies $\Lambda_{U,W}(t+s) \leqslant \Lambda_{U,W}(t)$ for $s > 0$.

Next we prove the boundedness of $\Lambda_{U,W}(t)$. What we have to show is

$$\lim_{t\to-\infty} \Lambda_{U,W}(t) < +\infty.$$

Choose a returning sequence $a_n \to -\infty$ such that $\sigma_{a_n} g \to g$ as $n \to \infty$. Then Lemma 5.7(ii) implies that there exists a sequence $t_n \to -\infty$ satisfying $U(0,t_n) = a_n$ for all $n \in \mathbb{N}$. Without loss of generality we may assume that

$$\sigma_{a_n} U(x, t+t_n) \to U^*(x,t) \quad \text{as } n \to \infty$$

in the sense of Lemma 4.24. Then $U^*(x,t)$ is an entire solution on Ω_g satisfying $U^*_t \geqslant 0$ since $U_t > 0$. By Lemma 5.9(iv), we can find $T > 0$ such that

$$U^*(0,T) - U^*(0,0) = U^*(0,T) \geqslant 4G\widetilde{H} + 1,$$

where $\widetilde{H}$ is the constant in (5.13). It follows that, for all sufficiently large n, we have

$$U(0, T + t_n) - U(0, t_n) \geqslant 4G\widetilde{H},$$

hence

$$\min_x U(x, T + t_n) - \max_x U(x, t_n) \geqslant 2G\widetilde{H}.$$

By Lemma 5.9, we can find $s_n \in \mathbb{R}$ such that

$$U(\cdot, t_n) \preccurlyeq W(\cdot, s_n) \preccurlyeq U(\cdot, T + t_n).$$

This implies that

$$\Lambda_{U,W}(s_n) \leqslant T$$

for all sufficiently large n. Since $t_n \to -\infty$ as $n \to \infty$ and since W_t is bounded, it is clear that $s_n \to -\infty$ as $n \to \infty$. In view of this and the monotonicity of $\Lambda_{U,W}(t)$, we obtain

$$\lim_{t \to -\infty} \Lambda_{U,W}(t) \leqslant T.$$

(ii) The former statement is obvious. Suppose that $\Lambda_{U,W}(t_0) > 0$ for some t_0. Then by (i), $\Lambda_{U,W}(t) > 0$ for any fixed $t < t_0$. Therefore (5.14) and the comparison principle (Proposition 4.14) yield

$$U(\cdot, t + s + \tau) \ll W(\cdot, t + s) \ll U\big(\cdot, t + s + \tau + \Lambda_{U,W}(t)\big)$$

for $s > 0$. Consequently, by the continuity of $U(x, t)$ in t, there exists $\delta = \delta(t, s) > 0$ such that

$$U(\cdot, t + s + \tau + \delta) \ll W(\cdot, t + s) \ll U\big(\cdot, t + s + \tau + \Lambda_{U,W}(t) - \delta\big).$$

From this it follows that $\Lambda_{U,W}(t + s) \leqslant \Lambda_{U,W}(t) - 2\delta$. This proves statement (ii). $\square$

The following is the main result of this subsection:

Proposition 5.10. *Assume (5.1). Let $U(x, t)$ be the entire solution of (1.2)–(1.3) in Proposition 5.3 and let $W(x, t)$ be any entire solution of (1.2)–(1.3). Then $W(x, t)$ is a time shift of $U(x, t)$.*

Proof. We only need to show that $\Lambda_{U,W}(t) = 0$ for all $t \in \mathbb{R}$. Suppose that $\Lambda_{U,W}(t_0) > 0$ for some $t_0 \in \mathbb{R}$. Then Lemma 5.8(i) implies that $\Lambda_{U,W}(t)$ converges to some $\overline{\Lambda} > 0$ as $t \to -\infty$.

Choose a returning sequence $\{a_n\}_{n\in\mathbb{N}}$ for g with $a_n \to -\infty$ as $n \to \infty$ and take a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $W(0, t_n) = a_n$. By (5.14), there exists a sequence $\{\tau_n\}$ such that

$$U(\cdot, t_n + \tau_n) \preccurlyeq W(\cdot, t_n) \preccurlyeq U\big(\cdot, t_n + \tau_n + \Lambda_{U,W}(t_n)\big)$$

for $n \in \mathbb{N}$. This implies that

$$\begin{aligned} 0 &\leqslant W(0, t_n) - U(0, t_n + \tau_n) \\ &\leqslant U\big(0, t_n + \tau_n + \Lambda_{U,W}(t_n)\big) - U(0, t_n + \tau_n) \leqslant C\overline{\Lambda}, \end{aligned}$$

where C is the constant in Proposition 5.3(ii). Set

$$U_n(x,t) := \sigma_{a_n} U(x, t + t_n + \tau_n), \qquad W_n(x,t) := \sigma_{a_n} W(x, t + t_n).$$

Then $(U_n, \sigma_{a_n}\Omega, \mathbb{R})$ and $(W_n, \sigma_{a_n}\Omega, \mathbb{R})$ are solution triples, and $W_n(0,0) = 0$, $U_n(0,0) \in [-C\overline{\Lambda}, 0]$ for each $n \in \mathbb{N}$. Furthermore, U_n and W_n satisfy all the properties of W in Lemma 5.9. Using a similar discussion as that in the proof of Lemma 5.6, we can find subsequences of $\{U_n\}$, $\{W_n\}$, again denoted by $\{U_n\}$, $\{W_n\}$, and functions $U_\infty, W_\infty \in C^{2+\mu,1+\mu/2}_{loc}$ such that

$$(U_n, \sigma_{a_n}\Omega, \mathbb{R}) \xrightarrow{n\to\infty} (U_\infty, \Omega, \mathbb{R}), \qquad (W_n, \sigma_{a_n}\Omega, \mathbb{R}) \xrightarrow{n\to\infty} (W_\infty, \Omega, \mathbb{R}).$$

Hence $(U_\infty, \Omega, \mathbb{R})$ and $(W_\infty, \Omega, \mathbb{R})$ are also solution triples.

By the definition of $\Lambda_{U_n,W_n}(t)$, for any $t \in \mathbb{R}$, there exists $\hat{\tau} \in \mathbb{R}$ such that

$$U_n(\cdot, t + \hat{\tau}) \preccurlyeq W_n(\cdot, t) \preccurlyeq U_n\big(\cdot, t + \hat{\tau} + \Lambda_{U_n,W_n}(t)\big)$$

and that the graph of $W_n(\cdot,t)$ has intersections with those of $U_n(\cdot, t+\hat{\tau})$ and $U_n(\cdot, t+\hat{\tau}+\Lambda_{U_n,W_n}(t))$. Writing the above inequalities in terms of U and W, we have

$$U(\cdot, t + \hat{\tau} + t_n + \tau_n) \preccurlyeq W(\cdot, t + t_n) \preccurlyeq U\big(\cdot, t + \hat{\tau} + t_n + \tau_n + \Lambda_{U_n,W_n}(t)\big).$$

Hence

$$\Lambda_{U_n,W_n}(t) = \Lambda_{U,W}(t + t_n) \to \overline{\Lambda} \quad \text{as } n \to \infty.$$

On the other hand, $\Lambda_{U_n,W_n}(t) \to \Lambda_{U_\infty,W_\infty}(t)$ $(n \to \infty)$ locally uniformly in t. Therefore $\Lambda_{U_\infty,W_\infty}(t) \equiv \overline{\Lambda}$ for all $t \in \mathbb{R}$. This, however, contradicts Lemma 5.8(ii) and the fact that $\overline{\Lambda} > 0$.

Thus we have $\Lambda_{U,W}(t) = 0$ for all $t \in \mathbb{R}$, and hence there exists a constant τ such that $U(\cdot, t+\tau) \equiv W(\cdot, t)$ for all $t \in \mathbb{R}$. □

5.4. *Further properties of the entire solution*

Let us begin with the following lemma:

Lemma 5.11. *Assume (5.1) and let $U(x,t)$ be the (unique) entire solution of (1.2)–(1.3), and let c_- denote its lower average speed (see (2.17)). Then the following conditions are equivalent:*

$$\text{(a)} \quad c_- > 0; \qquad \text{(b)} \quad \text{(5.2) } \textit{holds}.$$

Proof. The assertion (b) $\Rightarrow$ (a) is an immediate consequence of Proposition 5.3(iv). To prove the assertion (a) $\Rightarrow$ (b), choose $h \in \mathcal{H}_g$ arbitrarily, and let $\{a_n\} \subset \mathbb{R}$ be such that $\sigma_{a_n} g \to h$ in $L^\infty_{loc}(\mathbb{R};\mathbb{R}^2)$ as $n \to \infty$. Since $U(x,t) \to \pm\infty$ as $t \to \pm\infty$, we can take a sequence $\{t_n\} \subset \mathbb{R}$ such that $U(0,t_n) = a_n$ for all $n \in \mathbb{N}$. Replacing $\{t_n\}$ by its subsequence if necessary, we see, by Lemma 4.24, that there exists an entire solution $U^*(x,t)$ in Ω_h such that

$$\sigma_{a_n} U(x, t + t_n) = U(x, t + t_n) - a_n \to U^*(x,t) \quad \text{as } n \to \infty$$

locally uniformly in the region where U^* is defined. Since $c_- > 0$, (2.17) implies that there exists a constant $T > 0$ satisfying

$$U(0, t + T) - U(0, t) \geqslant \frac{c_-}{2} T \quad \text{for all } t \in \mathbb{R}.$$

Therefore, we obtain

$$U^*(0,t+T)-U^*(0,t)\geqslant\frac{c_-}{2}T\quad\text{for all }t\in\mathbb{R}.$$

This implies that $U^*(0,t)\to\pm\infty$ as $t\to\pm\infty$. Hence by the comparison principle, there exists no stationary solution in Ω_h. □

Now let us assume (5.2). For each $h\in\mathcal{H}_g$, we define $U(x,t;h)$ to be the entire solution of (1.2) in the domain Ω_h which satisfies the following normalization condition

$$U(0,0;h)=0.$$

The existence of such an entire solution follows from Proposition 5.3. The uniqueness result (Proposition 5.10) clearly holds if Ω is replaced by Ω_h and if $U(x,t)$ is replaced by $U(x,t;h)$. In view of this, and considering that

$$\sigma_a U(x,t+\tau;h):=U(x,t+\tau;h)-a$$

is an entire solution in $\Omega_{\sigma_a h}$ for any constant $\tau\in\mathbb{R}$, we see that the following identity holds:

$$U(x,t;\sigma_a h)=\sigma_a U\bigl(x,t+\tau(a);h\bigr),\tag{5.15}$$

where $\tau(a)$ is a constant determined by the condition

$$U\bigl(0,\tau(a);h\bigr)=a.\tag{5.16}$$

Lemma 5.12. *Assume (5.2) and let $\{h_n\}$ be a sequence in $\mathcal{H}_g$ converging to $h^*\in\mathcal{H}_g$ in $L^\infty_{loc}(\mathbb{R};\mathbb{R}^2)$. Then the following hold:*

$$U(x,t;h_n)\to U\bigl(x,t;h^*\bigr)\quad(n\to\infty),\tag{5.17}$$

$$U_t(x,t;h_n)\to U_t\bigl(x,t;h^*\bigr)\quad(n\to\infty)\tag{5.18}$$

locally uniformly in (x,t) in the sense of Lemma 4.24.

Proof. Suppose that the convergence in (5.17) does not hold. Then, by using the parabolic estimates in Section 4.5, we can find a subsequence $\{h_{n_i}\}$ of $\{h_n\}$ and a function $U^*(x,t)$ such that

$$U(x,t;h_{n_i})\to U^*(x,t)\not\equiv U\bigl(x,t;h^*\bigr)\quad(i\to\infty)$$

locally uniformly in the sense of Lemma 4.24. Clearly, $(U^*,\Omega_{h^*},\mathbb{R})$ is a solution triple and $U^*(0,0)=0$. Hence, by Proposition 5.10, $U^*(x,t)\equiv U(x,t;h^*)$, contradicting the above assumption. This contradiction proves (5.17). The assertion (5.18) follows from (5.17) and parabolic estimates. The lemma is proved. □

Corollary 5.13. *Assume (5.2). Then, for any $h\in\mathcal{H}_g$, there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that*

$$U(x,t;h)=\lim_{n\to\infty}\bigl(U\bigl(x,t+\tau(a_n);g\bigr)-a_n\bigr).\tag{5.19}$$

Proof. By the definition of $\mathcal{H}_g$, there exists a sequence $\{a_n\} \subset \mathbb{R}$ such that $\sigma_{a_n} g \to h$ as $n \to \infty$. By (5.15),

$$U(x,t;\sigma_{a_n} g) = U\big(x, t+\tau(a_n); g\big) - a_n.$$

Combining this and Lemma 5.12, we obtain (5.19). □

Lemma 5.14. *Assume (5.2). Then there exist positive constants δ, C_1 and C_2 which depend on $g_\pm$ such that*

$$\delta \leqslant U_t(0,t;h) \leqslant C_1 \quad \textit{for all } t \in \mathbb{R},\ h \in \mathcal{H}_g; \tag{5.20}$$

$$\big|U_{tt}(0,t;h)\big| \leqslant C_2 \quad \textit{for all } t \in \mathbb{R},\ h \in \mathcal{H}_g. \tag{5.21}$$

Proof. The assertion (5.20) is a direct consequence of Proposition 5.3 and Corollary 5.13. The assertion (5.21) follows from Lemma 5.15 below and Corollary 5.13. □

Lemma 5.15 *(Interior estimates for U_{tt}). Let $U(x,t)$ be the entire solution of (1.2)–(1.3) and Q be the space–time domain defined in (5.3). Then U is differentiable twice by t in Q and for any $H_0 \in (0,H)$ there exists a constant $C > 0$ such that*

$$\|U_{tt}\|_{L^\infty([-H_0,H_0]\times\mathbb{R})} \leqslant C. \tag{5.22}$$

Proof. That U is twice differentiable in t follows from standard interior parabolic estimates. Differentiating Eq. (1.2) by t, we see that $w = U_t$ satisfies

$$w_t = \frac{w_{xx}}{1+U_x^2} - \frac{2U_x U_{xx} w_x}{(1+U_x^2)^2} + A\frac{U_x w_x}{\sqrt{1+U_x^2}} \quad \text{in } Q.$$

This equation can be regarded as a linear parabolic equation

$$w_t = a(x,t)w_{xx} + b(x,t)w_x.$$

By (4.35), a, b and w are uniformly bounded and Hölder continuous in $\overline{Q}$. The desired estimate (5.22) then follows from standard interior parabolic estimates. □

6. Traveling waves

In this section we show the existence of recurrent traveling waves under the condition (5.2) and study their average speed. We also prove the asymptotic stability of the recurrent traveling waves.

As in Section 5.4, for each $h \in \mathcal{H}_g$, we use the notation $U(x,t;h)$ to denote the entire solution of (1.2) in Ω_h satisfying the normalization condition $U(0,0;h) = 0$. In what follows, we put

$$p_h(t) := U(0,t;h). \tag{6.1}$$

The function $p_h(t)$ represents the position where the curve $\gamma(U(\cdot,t;h))$ crosses the y-axis. We will call it *the current position* of the entire solution $U(x,t;h)$. For simplicity, in the case where $h = g$, we also write $U(x,t;g)$ and $p_g(t)$ as $U(x,t)$ and $p(t)$, respectively.

6.1. Existence of a regular traveling wave

Proof of Theorem 3.1. We show that the above-mentioned entire solution $U(x,t)$ is a regular traveling wave.

First, $p(t) := U(0,t)$ is a function from $\mathbb{R}$ to $\mathbb{R}$. It satisfies $p'(t) = U_t(0,t) \geqslant \delta > 0$ by (5.20).

Secondly, substituting $a = p_h(s)$ in (5.16), we obtain

$$U\bigl(0, \tau\bigl(p_h(s)\bigr); h\bigr) = p_h(s) = U(0,s;h) \quad \text{for } s \in \mathbb{R}.$$

Hence

$$\tau\bigl(p_h(s)\bigr) = s \quad \text{for any } s \in \mathbb{R}.$$

Putting $a = p_h(s)$ and $h = g$ in (5.15) yield

$$U(x,t;\sigma_{p(s)}g) = U(x,t+s;g) - p(s).$$

Setting $t = 0$, we obtain $U(x,0;\sigma_{p(s)}g) = U(x,s;g) - p(s) = U(x,s) - p(s)$. Hence

$$\gamma\bigl(\sigma_{p(s)}U(\cdot,s)\bigr) = \gamma\bigl(U(\cdot,0;\sigma_{p(s)}g)\bigr).$$

Notice that, by Lemma 5.12, $\mathcal{F} : h \mapsto \gamma(U(\cdot,0;h))$ is a continuous map from $\mathcal{H}_g$ into $\mathcal{K}$, where $\mathcal{K}$ is as in Definition 2.14. The above identity is then written as $\gamma(\sigma_{p(s)}U(\cdot,s)) = \mathcal{F}(\sigma_{p(s)}g)$. Thus, by Definition 2.14, $U(x,t)$ is a regular traveling wave. Its uniqueness is obvious from Proposition 5.10, since any traveling wave is an entire solution. □

6.2. Average speed

In this subsection we prove that a regular traveling wave has an average speed if g is strictly ergodic. Before proceeding further, we first prove Lemma 2.15, which has been stated under a more general assumption that g is recurrent.

Proof of Lemma 2.15. We put $p(t) = U(0,t)$ and define

$$\rho^+(T) := \sup_{t\in\mathbb{R}}\bigl(p(t+T) - p(t)\bigr).$$

Then by Proposition 5.3(ii), we have

$$0 \leqslant \rho^+(T) < +\infty \quad \text{for every } T > 0.$$

Furthermore, for any $T_1, T_2 > 0$,

$$\rho^+(T_1+T_2) \leqslant \rho^+(T_1) + \rho^+(T_2).$$

This means that $\rho^+(T)$ is subadditive, hence, by Fekete's lemma, the limit

$$\lim_{T\to\infty}\frac{\rho^+(T)}{T} = \lim_{T\to\infty}\sup_{t\in\mathbb{R}}\frac{p(t+T)-p(t)}{T}$$

exists. This proves the convergence in (2.16). The convergence in (2.17) can be shown similarly. Finally the last assertion of the lemma is obvious. The lemma is proved. □

Now, for any $h \in \mathcal{H}_g$, we define a function $\Phi_h : \mathbb{R} \to \mathbb{R}$ by

$$p'_h(t) = \Phi_h\big(p_h(t)\big). \tag{6.2}$$

Since $p_h(t) := U(0,t;h)$ is strictly monotone increasing in t and $p_h(t) \to \pm\infty$ as $t \to \pm\infty$, the function Φ_h is well-defined. Eq. (6.2) gives the law of motion for the current position of $U(x,t;h)$.

Lemma 6.1. *Let δ, C_1 and C_2 be the constants in Lemma 5.14. Then*

$$\delta \leqslant \Phi_h(p) \leqslant C_1 \quad \text{for } p \in \mathbb{R},\ h \in \mathcal{H}_g; \tag{6.3}$$

$$\big|\Phi'_h(p)\big| \leqslant C_2/\delta \quad \text{for } p \in \mathbb{R},\ h \in \mathcal{H}_g. \tag{6.4}$$

Furthermore,

$$\sigma_a \Phi_h = \Phi_{\sigma_a h} \quad \text{for } a \in \mathbb{R},\ h \in \mathcal{H}_g. \tag{6.5}$$

Proof. The inequalities (6.3) follow from (5.20). By (5.21) we have $|U_{tt}(0,t;h)| \leqslant C_2$. Differentiating (6.2) in t yields

$$\Phi'_h\big(p_h(t)\big) \cdot p'_h(t) = p''_h(t).$$

Hence

$$\big|\Phi'_h\big(p_h(t)\big)\big| = \left|\frac{p''_h(t)}{p'_h(t)}\right| = \left|\frac{U_{tt}(0,t;h)}{U_t(0,t;h)}\right| \leqslant \frac{C_2}{\delta}.$$

This proves (6.4).

Next we note that (5.15) implies

$$p_h\big(t+\tau(a)\big) = p_{\sigma_a h}(t) + a.$$

Differentiating this identity yields $p'_h(t+\tau(a)) = p'_{\sigma_a h}(t)$. Consequently,

$$\Phi_h\big(p_h\big(t+\tau(a)\big)\big) = \Phi_{\sigma_a h}\big(p_{\sigma_a h}(t)\big).$$

Combining this and the previous identity, we obtain

$$\sigma_a \Phi_h\big(p_{\sigma_a h}(t)\big) = \Phi_{\sigma_a h}\big(p_{\sigma_a h}(t)\big).$$

Since $p_{\sigma_a h}(t)$ varies over $\mathbb{R}$ as t varies, this implies (6.5). The lemma is proved. □

Lemma 6.2. *Let $\{h_n\}$ be a sequence in $\mathcal{H}_g$.*

(i) *If $h_n \xrightarrow{n\to\infty} h^*$ in $L^\infty_{loc}(\mathbb{R};\mathbb{R}^2)$, then $\Phi_{h_n} \xrightarrow{n\to\infty} \Phi_{h^*}$ in $L^\infty_{loc}(\mathbb{R})$;*
(ii) *If $h_n \xrightarrow{n\to\infty} h^*$ in $L^\infty(\mathbb{R};\mathbb{R}^2)$, then $\Phi_{h_n} \xrightarrow{n\to\infty} \Phi_{h^*}$ in $L^\infty(\mathbb{R})$.*

Proof. Let us first prove (i). By Lemma 5.12,

$$p_{h_n}(t) \xrightarrow{n\to\infty} p_{h^*}(t) \quad \text{locally uniformly in } t \in \mathbb{R},$$
$$\Phi_{h_n}\bigl(p_{h_n}(t)\bigr) \xrightarrow{n\to\infty} \Phi_{h^*}\bigl(p_{h^*}(t)\bigr) \quad \text{locally uniformly in } t \in \mathbb{R}.$$

By (6.4) we have

$$\bigl|\Phi_{h_n}\bigl(p_{h_n}(t)\bigr) - \Phi_{h_n}\bigl(p_{h^*}(t)\bigr)\bigr| \leqslant \frac{C_2}{\delta}\bigl|p_{h_n}(t) - p_{h^*}(t)\bigr|,$$

hence

$$\Phi_{h_n}\bigl(p_{h^*}(t)\bigr) \xrightarrow{n\to\infty} \Phi_{h^*}\bigl(p_{h^*}(t)\bigr) \quad \text{locally uniformly in } t \in \mathbb{R}.$$

Since $p_{h^*}(t) \to \pm\infty$ as $t \to \pm\infty$, the above convergence implies

$$\Phi_{h_n}(p) \xrightarrow{n\to\infty} \Phi_{h^*}(p) \quad \text{locally uniformly in } p \in \mathbb{R}.$$

This proves (i).

Next we prove (ii). Assume that $h_n \to h^*$ in $L^\infty(\mathbb{R};\mathbb{R}^2)$ but $\Phi_{h_n} \not\to \Phi_{h^*}$ in $L^\infty(\mathbb{R})$. Replacing $\{h_n\}$ by its subsequence if necessary, we may assume that

$$\|\Phi_{h_n} - \Phi_{h^*}\|_{L^\infty(\mathbb{R})} > \varepsilon_0 \quad \text{for } n = 1, 2, 3, \ldots,$$

for some $\varepsilon_0 > 0$. Hence, there exists a sequence $\{a_n\}$ satisfying

$$\bigl|\Phi_{h_n}(a_n) - \Phi_{h^*}(a_n)\bigr| = \bigl|\sigma_{a_n}\Phi_{h_n}(0) - \sigma_{a_n}\Phi_{h^*}(0)\bigr| > \varepsilon_0.$$

This means

$$\bigl|\Phi_{\sigma_{a_n}h_n}(0) - \Phi_{\sigma_{a_n}h^*}(0)\bigr| > \varepsilon_0 \tag{6.6}$$

by (6.5). Since $\mathcal{H}_g$ is compact in $L^\infty_{loc}(\mathbb{R};\mathbb{R}^2)$, we can select a subsequence of $\{a_n\}$, denoted again by $\{a_n\}$, such that

$$\sigma_{a_n}h_n \xrightarrow{n\to\infty} h_\infty, \qquad \sigma_{a_n}h^* \xrightarrow{n\to\infty} h^*_\infty \quad \text{in } L^\infty_{loc}\bigl(\mathbb{R};\mathbb{R}^2\bigr).$$

Letting $n \to \infty$ in (6.6) and using the assertion (i), we have

$$\bigl|\Phi_{h_\infty}(0) - \Phi_{h^*_\infty}(0)\bigr| \geqslant \varepsilon_0. \tag{6.7}$$

On the other hand, since $h_n \to h^*$ in L^∞,

$$\bigl\|\sigma_{a_n}h_n - \sigma_{a_n}h^*\bigr\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} = \bigl\|h_n - h^*\bigr\|_{L^\infty(\mathbb{R};\mathbb{R}^2)} \to 0 \quad \text{as } n \to \infty.$$

This implies $h_\infty = h^*_\infty$, contradicting (6.7). The assertion (ii) is proved. □

Lemma 6.3.

(i) *If g is periodic, then Φ_g is periodic*;
(ii) *If g is almost periodic, then Φ_g is almost periodic*;
(iii) *If g is recurrent, then Φ_g is recurrent*;
(iv) *If g is strictly ergodic, then Φ_g is strictly ergodic.*

Proof. (i) Assume that g has period L. Then by (6.5), $\sigma_L \Phi_g = \Phi_{\sigma_L g} = \Phi_g$, hence Φ_g has period L.

(ii) Since g is almost periodic, by Bochner's criterion, for any sequence $\{a_n\}$, there exists a subsequence of $\{a_n\}$, denoted again by $\{a_n\}$, such that $\sigma_{a_n} g \to g^*$ in $L^\infty(\mathbb{R};\mathbb{R}^2)$ for some $g^* \in \mathcal{H}_g$. By (ii) of Lemma 6.2

$$\sigma_{a_n}\Phi_g = \Phi_{\sigma_{a_n} g} \to \Phi_{g^*} \quad \text{in } L^\infty(\mathbb{R}).$$

This implies that Φ_g is almost periodic.

(iii) By Lemma 6.1, Φ_g is bounded and uniformly continuous on $\mathbb{R}$. Suppose that

$$\sigma_{a_n}\Phi_g \to \Phi^* \quad \text{in } L^\infty_{loc}(\mathbb{R}).$$

We can choose a subsequence of $\{a_n\}$, denoted again by $\{a_n\}$, such that

$$\sigma_{a_n} g \to g^* \quad \text{in } L^\infty_{loc}\big(\mathbb{R};\mathbb{R}^2\big)$$

for some $g^* \in \mathcal{H}_g$. Hence $\sigma_{a_n}\Phi_g = \Phi_{\sigma_{a_n} g} \to \Phi_{g^*}$. Consequently, $\Phi_{g^*} = \Phi^*$.

Since g is recurrent, there exists a sequence $\{b_n\}$ such that

$$\sigma_{b_n} g^* \to g \quad \text{in } L^\infty_{loc}\big(\mathbb{R};\mathbb{R}^2\big).$$

Therefore

$$\sigma_{b_n}\Phi^* = \sigma_{b_n}\Phi_{g^*} = \Phi_{\sigma_{b_n} g^*} \to \Phi_g \quad \text{in } L^\infty_{loc}(\mathbb{R}).$$

By Lemma 2.4, this means that Φ_g is recurrent.

(iv) We recall that a function is strictly ergodic if it is recurrent and uniquely ergodic. Let $F : \mathcal{H}_{\Phi_g} \to \mathbb{R}$ be any continuous map. Since $\Phi : h \mapsto \Phi_h$ defines a continuous map from $\mathcal{H}_g$ onto $\mathcal{H}_{\Phi_g}$, the composite map $F \circ \Phi : \mathcal{H}_g \to \mathbb{R}$ is continuous. Therefore, the unique ergodicity of g implies that the limit

$$\lim_{L\to\infty} \frac{1}{L} \int_\alpha^{\alpha+L} (F \circ \Phi)(\sigma_a g)\, da$$

exists uniformly in $\alpha \in \mathbb{R}$. By (6.5), the above limit equals

$$\lim_{L\to\infty} \frac{1}{L} \int_\alpha^{\alpha+L} F(\sigma_a \Phi_g)\, da.$$

Hence Φ_g is uniquely ergodic. Since Φ_g is recurrent by (iii) above, it is strictly ergodic. The lemma is proved. □

Lemma 6.4. *For any $h \in \mathcal{H}_g$, the average speed $\bar{c}_h$ of $U(x,t;h)$ exists if and only if $1/\Phi_h$ has the uniform mean $\langle 1/\Phi_h \rangle$ in the sense of (2.3). Furthermore, $\bar{c}_h$ coincides with the harmonic mean of Φ_h, namely,*

$$\bar{c}_h = \left\langle \frac{1}{\Phi_h} \right\rangle^{-1}.$$

Proof. Since $p_h'(t) = \Phi_h(p_h(t))$ and since $\Phi_h(p) \geqslant \delta > 0$ we have

$$\int_{p_h(t)}^{p_h(t+T)} \frac{dp}{\Phi_h(p)} = T.$$

Therefore,

$$\frac{p_h(t+T) - p_h(t)}{T} = \frac{p_h(t+T) - p_h(t)}{\int_{p_h(t)}^{p_h(t+T)} \frac{dp}{\Phi_h(p)}}.$$

The conclusion follows from this equality easily. □

Now we are ready to present the main result of this subsection, which is a restatement of Theorem 3.5.

Proposition 6.5. *A regular traveling wave of (1.2)–(1.3) has an average speed if g is strictly ergodic.*

Proof. By Lemma 6.4, it suffices to show that $1/\Phi_g$ has an arithmetic mean. For the clarity of the argument, let us begin with the special case where g is periodic. In this case, by Lemma 6.3, Φ_g is periodic. Hence, by (6.3), $1/\Phi_g$ is also periodic, therefore it has an arithmetic mean $\langle 1/\Phi_g \rangle$.

In the general case, by Lemma 6.3, Φ_g is uniquely ergodic. Furthermore, by (6.3), every element $\Phi \in \mathcal{H}_{\Phi_g}$ satisfies $\Phi \geqslant \delta$, hence the operation $\Phi \mapsto 1/\Phi(0)$ defines a continuous mapping from $\mathcal{H}_{\Phi_g}$ to $\mathbb{R}$. Therefore the limit

$$\lim_{L\to\infty} \frac{1}{L} \int_{\alpha}^{\alpha+L} \frac{1}{\Phi_{\sigma_a g}(0)}\, da$$

exists uniformly in $\alpha \in \mathbb{R}$. Since $\Phi_{\sigma_a g}(0) = \Phi_g(a)$, this implies that $1/\Phi_g$ has an arithmetic mean. The proof of the proposition is complete. □

6.3. Asymptotic stability of the recurrent traveling wave

In this subsection we prove Theorem 3.4; namely we show that the recurrent traveling wave U is stable and that any solution of (1.2)–(1.3) with $u_0 \in C^1_{\mathrm{ad}} \cap C^{1+\lambda}$ $(\lambda > 0)$ converges to a time shift of U as $t \to +\infty$. The stability (Theorem 3.4(i)) is an easy consequence of the comparison principle, but to prove the asymptotic stability (Theorem 3.4(ii)) requires a more improved argument.

Proof of Theorem 3.4(i). Let $\mathcal{C}$ denote the set of C^1-curves in $\overline{\Omega}$ defined in Section 4.4. Since g is uniformly continuous, there exists a constant $\sigma' > 0$ such that if $\Gamma^1, \Gamma^2 \in \mathcal{C}$ satisfy $\Gamma^1 \ll \Gamma^2$ and $d_{\mathcal{H}}(\Gamma^1, \Gamma^2) < \sigma'$, then $d_{\mathcal{H}}(\gamma^1, \gamma^2) < \sigma$ for all $\gamma^1, \gamma^2 \in \mathcal{C}$ with $\Gamma^1 \preccurlyeq \gamma^j \preccurlyeq \Gamma^2$ $(j = 1, 2)$.

We may assume that $\tau = 0$ without loss of generality. By Proposition 5.3(ii), the normal velocity of Γ_t is bounded in t. Therefore, we can choose sufficiently small $s > 0$ such that

$$d_{\mathcal{H}}(\Gamma_{t+s}, \Gamma_{t-s}) < \sigma', \quad t \in \mathbb{R}. \tag{6.8}$$

Fix such $s > 0$. Then

$$\Gamma_{t-s} \ll \Gamma_t \ll \Gamma_{t+s}, \quad t \in \mathbb{R}. \tag{6.9}$$

In particular $\Gamma_{-s} \ll \Gamma_0 \ll \Gamma_s$, hence there exists a constant $\delta > 0$ such that $d_{\mathcal{H}}(\gamma_0, \Gamma_0) < \delta$ implies

$$\Gamma_{-s} \preccurlyeq \gamma_0 \preccurlyeq \Gamma_s.$$

The comparison principle (Proposition 4.14) then yields

$$\Gamma_{t-s} \preccurlyeq \gamma_t \preccurlyeq \Gamma_{t+s}, \quad t \geqslant 0.$$

Combining this with (6.8) and (6.9), we obtain

$$d_{\mathcal{H}}(\gamma_t, \Gamma_t) < \sigma, \quad t \geqslant 0.$$

The assertion (i) is proved. □

Proof of Theorem 3.4(ii). By Proposition 4.17, there exists some $t_* \geqslant 0$ such that $u(\cdot, t) \in C^1_{ad}$ for $t \geqslant t_*$. Let $\tau_0 \in \mathbb{R}$ and $\Lambda_0 > 0$ be constants such that

$$U(\cdot, t_* + \tau_0) \preccurlyeq u(\cdot, t_*) \preccurlyeq U(\cdot, t_* + \tau_0 + \Lambda_0).$$

Then the comparison principle yields

$$U(\cdot, t + \tau_0) \preccurlyeq u(\cdot, t) \preccurlyeq U(\cdot, t + \tau_0 + \Lambda_0), \quad t \geqslant t_*.$$

These inequalities imply that the function

$$\hat{\tau}(t) := \sup\bigl\{\tau \in \mathbb{R} \bigm| U(\cdot, t + \tau) \preccurlyeq u(\cdot, t)\bigr\}$$

is well-defined and satisfies $\hat{\tau}(t) \in [\tau_0, \tau_0 + \Lambda_0]$ for all $t \geqslant t_*$. Furthermore, by the comparison principle, $\hat{\tau}(t)$ is nondecreasing in t. Thus $\hat{\tau}(t)$ converges to some $\tau_* \in \mathbb{R}$ as $t \to +\infty$.

Choose a sequence $a_1 < a_2 < a_3 < \cdots \to +\infty$ with $a_1 > \|u(\cdot, t_*)\|_{L^\infty}$ such that

$$\sigma_{a_n} g \to g \quad \text{in } L^\infty_{loc}\bigl(\mathbb{R}; \mathbb{R}^2\bigr) \text{ as } n \to \infty.$$

Since $U(0, t) \to +\infty$ as $t \to +\infty$, we can take a sequence $\{t_n\}$ satisfying $u(0, t_n) = a_n$. Then for each $n \in \mathbb{N}$, (u_n, Ω_{h_n}, I_n) with

$$u_n := \sigma_{a_n} u(\cdot, t + t_n), \qquad h_n := \sigma_{a_n} g, \qquad I_n := [-t_n, +\infty)$$

is a solution triple satisfying $u_n(0, 0) = 0$. On the other hand, letting $U_n := \sigma_{a_n} U(\cdot, t + t_n + \hat{\tau}(t + t_n))$ and $W_n := \sigma_{a_n} U(\cdot, t + t_n + \tau_*)$, we see that $(U_n, \Omega_{h_n}, \mathbb{R})$ and $(W_n, \Omega_{h_n}, \mathbb{R})$ are solution triples for each $n \in \mathbb{N}$.

Arguing as in the proof of Proposition 5.10, we can find a subsequence of $\{a_n\}$, again denoted by $\{a_n\}$, and functions u^*, U^*, W^* such that

$$(u_n, \Omega_{h_n}, I_n) \xrightarrow{n\to\infty} (u^*, \Omega, \mathbb{R}), \qquad (U_n, \Omega_{h_n}, \mathbb{R}) \xrightarrow{n\to\infty} (U^*, \Omega, \mathbb{R}),$$
$$(W_n, \Omega_{h_n}, \mathbb{R}) \xrightarrow{n\to\infty} (W^*, \Omega, \mathbb{R}).$$

Furthermore, Proposition 5.10 implies that $u^* = U$ and that each of the functions U^* and W^* is a time shift of U. Since $U_t > 0$ and since the graph of $u_n(\cdot, t)$ has an intersection with that of $U_n(\cdot, t)$ for all $t \geqslant 0$, we obtain $U^* = U$. On the other hand, since $\hat{\tau}(t_n) \to \tau_*$ as $n \to \infty$ and since the normal velocity of Γ_t is bounded in t, we see that

$$d_{\mathcal{H}}\big(\gamma\big(U_n(\cdot,0)\big), \gamma\big(W_n(\cdot,0)\big)\big) = d_{\mathcal{H}}(\Gamma_{t_n+\hat{\tau}(t_n)}, \Gamma_{t_n+\tau_*}) \to 0 \quad \text{as } n \to \infty. \tag{6.10}$$

In view of this and the fact that $U_t > 0$, we have $W^* = U^* = U$. Therefore,

$$d_{\mathcal{H}}(\gamma_{t_n}, \Gamma_{t_n+\tau_*}) = d_{\mathcal{H}}\big(\gamma\big(u_n(\cdot,0)\big), \gamma\big(W_n(\cdot,0)\big)\big) \to 0 \quad \text{as } n \to \infty.$$

Combining this and Theorem 3.4(i), we obtain

$$\lim_{t\to+\infty} d_{\mathcal{H}}(\gamma_t, \Gamma_{t+\tau_*}) = 0,$$

which means that γ_t converges to the traveling wave (after an appropriate time shift) in the sense of the Hausdorff distance. The C^2 convergence follows from standard parabolic estimates similar to those used in the proof of Lemma 4.24. □

7. The pinning case

In Sections 4–6, we have considered the case $2AH \geqslant \sin\alpha_+ + \sin\alpha_-$, where there are no stationary solutions. If A is small such that $2AH < \sin\alpha_+ + \sin\alpha_-$, then propagation may be blocked. In this section, we explain that when $2AH < \sin\alpha_+ + \sin\alpha_-$ holds, and if we consider the problem (1.2)–(1.3) with $g_\pm$ replaced by $g_\pm^\varepsilon$ as in (1.11), then there is a family of stationary solutions in Ω^ε for sufficiently small $\varepsilon > 0$.

Without loss of generality we may assume that $\partial_+\Omega^\varepsilon$ and $\partial_-\Omega^\varepsilon$ are both non-flat. Indeed, if one of them, say $\partial_+\Omega^\varepsilon$, is flat and the other is non-flat, then by reflecting the solution u and Ω^ε with respect to $\partial_+\Omega^\varepsilon$, we can convert (1.2)–(1.3) into an equivalent problem on the extended domain $\widetilde{\Omega}^\varepsilon$ whose boundaries are $\partial_-\Omega^\varepsilon$ and its reflection with respect to $\partial_+\Omega^\varepsilon$, both of which are non-flat. Thus, in what follows we assume that $g_+ \not\equiv 0$, $g_- \not\equiv 0$; hence

$$\alpha_+ > 0, \qquad \alpha_- > 0.$$

Set $\delta_0 := \min\{\alpha_-, \alpha_+\}/2 > 0$.

Lemma 7.1. *For any $\delta \in (0, \delta_0)$, there exists $M_\delta > 0$ such that for any $b \in \mathbb{R}$, the interval $[b, b + M_\delta]$ contains points $y_\pm$ such that*

$$g_+'(y_+) = \tan(\alpha_+ - \delta), \qquad g_-'(y_-) = \tan(\alpha_- - \delta). \tag{7.1}$$

Proof. First we choose $y_1, y_2 > 0$ such that

$$g_+'(y_1) = \tan(\alpha_+ - \delta/2), \qquad g_+'(y_2) = \tan(\alpha_+ - 2\delta).$$

Since g_+' is recurrent,

$$A_{\delta_1,Y} := \left\{a \in \mathbb{R} \;\middle|\; \left\|\sigma_a g'_+ - g'_+\right\|_{L^\infty((-Y,Y);\mathbb{R})} < \delta_1\right\}$$

is relatively dense in $\mathbb{R}$, where $Y := y_1 + y_2$ and

$$\delta_1 := \min\left\{\tan(\alpha_+ - \delta/2) - \tan(\alpha_+ - \delta), \tan(\alpha_+ - \delta) - \tan(\alpha_+ - 2\delta)\right\}.$$

In other words, there exists $\widetilde{M}_\delta > 0$ such that any interval of the form $[b, b + \widetilde{M}_\delta] \subset \mathbb{R}$ contains a point b_0 which belongs to $A_{\delta_1,Y}$. This means that

$$\left\|g'_+ - \sigma_{b_0} g'_+\right\|_{L^\infty((-Y,Y);\mathbb{R})} < \delta_1.$$

Especially, $|g'_+(y_i) - g'_+(y_i + b_0)| < \delta_1$ for $i = 1, 2$, and hence

$$g'_+(y_1 + b_0) > \tan(\alpha_+ - \delta/2) - \delta_1 \geqslant \tan(\alpha_+ - \delta),$$
$$g'_+(y_2 + b_0) < \tan(\alpha_+ - 2\delta) + \delta_1 \leqslant \tan(\alpha_+ - \delta).$$

Therefore, letting $M_\delta^+ := \widetilde{M}_\delta + Y$, we can find $y_b^+ \in [b, b + M_\delta^+]$ satisfying $g'_+(y_b^+) = \tan(\alpha_+ - \delta)$. In a similar way, there exists $M_\delta^- > 0$ such that any interval of the form $[b, b + M_\delta^-]$ contains a point y_b^- satisfying $g'_-(y_b^-) = \tan(\alpha_- - \delta)$. Finally, setting $M_\delta := \max\{M_\delta^+, M_\delta^-\}$, we obtain the assertion of the lemma. □

Lemma 7.2. *Assume that $2AH < \sin\alpha_+ + \sin\alpha_-$. Then (1.2)–(1.3) has a relatively dense family of stationary solutions in Ω^ε for any small $\varepsilon > 0$. More precisely, if $\{v_\lambda\}_{\lambda\in\Lambda}$ denotes the set of all stationary solutions of (1.2)–(1.3) in Ω^ε, then $\{v_\lambda(0)\}_{\lambda\in\Lambda}$ is relatively dense in $\mathbb{R}$.*

See the beginning of Section 2.1 for the relatively dense set.

Proof of Lemma 7.2. As we have shown in Lemma 5.4, there exists a relatively dense family of line segments in $\overline{\Omega}^\varepsilon$. Furthermore, all solutions of (1.2)–(1.3) starting from such linear initial functions are monotone increasing in t. If these solutions are bounded, then each of them must converge to a stationary solution. Therefore, in order to prove the theorem, it suffices to show that there exists a relatively dense family of time-independent upper solutions of (1.2)–(1.3).

Let $\delta \in (0, \delta_0)$ be such that

$$2AH \leqslant \sin(\alpha_+ - 2\delta) + \sin(\alpha_- - 2\delta). \tag{7.2}$$

Then, by Lemma 7.1, the sets $S_\pm = \{y \in \mathbb{R} \mid g'_\pm(y) = \tan(\alpha_\pm - \delta)\}$ are relatively dense in $\mathbb{R}$. We fix a small $\varepsilon > 0$ such that

$$\varepsilon A\left(\|g_+\|_{L^\infty(\mathbb{R})} + \|g_-\|_{L^\infty(\mathbb{R})}\right) \leqslant \sin(\alpha_+ - \delta) - \sin(\alpha_+ - 2\delta) \tag{7.3}$$

and that

$$\varepsilon A M_\delta \leqslant \cos(\alpha_- - 2\delta) - \cos(\alpha_- - \delta), \tag{7.4}$$

where $M_\delta > 0$ is the constant in Lemma 7.1.

Let $y_-/\varepsilon \in S_-$. Then $(g_-^\varepsilon)'(y_-) = \tan(\alpha_- - \delta)$. For $s \in [0, 1]$ we define $\theta_-^s := \alpha_- - (s + 1)\delta$ and

$$\chi_s(x) := y_- - \frac{\cos\theta_-^s}{A} + \frac{1}{A}\sqrt{1 - \left\{A(x + H + \vartheta) - \sin\theta_-^s\right\}^2},$$

where $\vartheta = g_-^{\varepsilon}(y_-)$. Since

$$\chi_s(-H-\vartheta) = y_-, \qquad \chi_s'(-H-\vartheta) = \tan\theta_-^s,$$

we see that the graph of χ_s is a circular arc, say C_s, with curvature $-A$ and that C_s intersects $\partial_-\Omega^{\varepsilon}$ at $P_- := (-H-\vartheta, y_-)$ with contact angle $\pi/2+s\delta \geqslant \pi/2$. By (7.2) and (7.3), C_s has a unique intersection point $P_+^s := (H+g_+^{\varepsilon}(y_+^s), y_+^s)$ with $\partial_+\Omega^{\varepsilon}$ for each $s \in [0,1]$. Obviously, y_+^s depends continuously on $s \in [0,1]$. Let $\theta_+^s > 0$ be such that

$$\chi_s'\big(H+g_+^{\varepsilon}(y_+^s)\big) = -\tan\theta_+^s.$$

Then a simple geometric observation shows that

$$\frac{1}{A}\sin\theta_-^s + \frac{1}{A}\sin\theta_+^s = 2H + g_-^{\varepsilon}(y_-) + g_+^{\varepsilon}\big(y_+^s\big),$$
$$y_+^s - y_- = -\frac{1}{A}\cos\theta_-^s + \frac{1}{A}\cos\theta_+^s.$$

Combining these equalities with (7.2) and (7.3), we obtain

$$0 < \theta_+^s \leqslant \alpha_+ - \delta \quad \text{for all } s \in [0,1], \qquad \theta_+^0 \leqslant \theta_+^1,$$

hence, by (7.4),

$$y_+^0 - y_+^1 \geqslant \frac{1}{A}\big(\cos(\alpha_- - 2\delta) - \cos(\alpha_- - \delta)\big) \geqslant \varepsilon M_\delta.$$

Therefore, in view of Lemma 7.1 and the continuity of y_+^s in s, we see that there exists $\sigma \in [0,1]$ satisfying $y_+^{\sigma}/\varepsilon \in S_+$, namely, $(g_+^{\varepsilon})'(y_+^{\sigma}) = \tan(\alpha_+ - \delta)$. Thus the circular arc C_σ intersects $\partial_+\Omega^{\varepsilon}$ at P_+^{σ} with contact angle $\pi/2 + (\alpha_+ - \delta) - \theta_+^{\sigma} \geqslant \pi/2$ and hence $\chi_\sigma(x)$ is a time-independent upper solution of (1.2)–(1.3).

Since S_- is relatively dense in $\mathbb{R}$, we can construct a relatively dense family of time-independent upper solutions of (1.2)–(1.3). The lemma is proved. □

Proof of Theorem 3.2. Let $u(x,t)$ be a time-global classical solution of (1.2)–(1.3) with initial data $u_0 \in C_{ad}^1$ and let $I(t) = [\zeta_-(t), \zeta_+(t)]$ denote the horizontal span of the solution curve $u(x,t)$ at each time $t \geqslant 0$. By Lemma 7.2, we can find stationary solutions w_1, w_2 of (1.2)–(1.3) such that $w_1 \preccurlyeq u_0 \preccurlyeq w_2$. Hence by the comparison principle,

$$w_1 \preccurlyeq u(\cdot,t) \preccurlyeq w_2. \tag{7.5}$$

In view of this and (4.35), we see that $\|u(\cdot,t)\|_{C^{2+\mu}(I(t))}$ is bounded for $t \geqslant \tau$ with some fixed $\tau > 0$.

Next we show that (1.2)–(1.3) has a Lyapunov functional. We define

$$E\big[u(\cdot,t)\big] := \int_{\zeta_-(t)}^{\zeta_+(t)} \mathcal{E}\big(u(x,t), u_x(x,t)\big)\,dx + \mathcal{B}_+\big(u\big(\zeta_+(t),t\big)\big) + \mathcal{B}_-\big(u\big(\zeta_-(t),t\big)\big), \tag{7.6}$$

where

$$\mathcal{E}(u,p) := \frac{1}{A}\sqrt{1+p^2} - u, \qquad \mathcal{B}_\pm(u) = \int_0^u s g_\pm'(s)\, ds. \tag{7.7}$$

Then

$$\frac{d}{dt} E\big[u(\cdot,t)\big] = -\frac{1}{A} \int_{\zeta_-(t)}^{\zeta_+(t)} \frac{u_t(x,t)^2}{\sqrt{1+u_x(x,t)^2}}\, dx \leqslant 0.$$

Since $E[u(\cdot,t)]$ is bounded in $t \in \mathbb{R}$, a standard dynamical systems theory shows that the ω-limit set of u is non-empty and is contained in the set of stationary solutions. The uniqueness of the ω-limit point can be shown by the same zero-number argument as in [6], or it also follows from the result in [16]. (The result in [6] is given for semilinear equations, but the proof is virtually the same for a quasilinear equation.) Consequently, $u(x,t)$ converges to a stationary solution of (1.2)–(1.3) as $t \to +\infty$ in the C^2 topology. □

8. Classification of the long-time behavior

In this section we prove the results in Section 3.3 concerning the classification of the long-time behavior of solutions (Proposition 3.6, Corollary 3.7 and Theorem 3.8).

8.1. Proof of Theorem 3.8

Since the latter half of (A1) — the convergence — follows from the former half — the existence of a regular traveling wave — by Theorem 3.4(ii) and Remark 3.3, what we have to show is that

$$(\mathrm{A1}') \Leftrightarrow (\mathrm{A2}), \qquad (\mathrm{B1}) \Leftrightarrow (\mathrm{B2}), \qquad (\mathrm{C1}) \Leftrightarrow (\mathrm{C2}),$$

where

(A1′) There exists a regular traveling wave in Ω_g.
(A2) There exists no stationary solution in Ω_h for any $h \in \mathcal{H}_g$.
(B1) There exists an entire solution $U(x,t)$ in Ω_g such that $U(x,t) \to \pm\infty$ as $t \to \pm\infty$ and such that its lower average speed c_- equals 0 (see (2.17)).
(B2) There exists no stationary solution in Ω_g, but there exists a stationary solution in Ω_h for some $h \in \mathcal{H}_g$, $h \neq g$.
(C1) There exists no entire solution $U(x,t)$ in Ω_g such that $U(x,t) \to \pm\infty$ as $t \to \pm\infty$.
(C2) There exists a stationary solution in Ω_g.

Proof of (C1) ⇔ (C2). The assertion (C2) ⇒ (C1) is obvious by the comparison theorem. To prove (C1) ⇒ (C2), it suffices to prove the contraposition; namely that non-existence of a stationary solution in Ω_g implies the existence of an entire solution $U(x,t)$ satisfying $U(x,t) \to \pm\infty$ as $t \to \pm\infty$. This is already shown in Proposition 5.3(iii). □

Proof of (A1′) ⇔ (A2). We will prove (A1′) ⇒ (A1″) ⇒ (A2) ⇒ (A1′), where

(A1″) There exists an entire solution $U(x,t)$ in Ω_g such that $U(x,t) \to \pm\infty$ as $t \to \pm\infty$ and such that its lower average speed c_- is positive.

The assertion (A1′) ⇒ (A1″) is obvious. The assertion (A2) ⇒ (A1′) follows from Proposition 5.3(iv) and Section 6.1. The assertion (A1″) ⇒ (A2) has already been proved in Lemma 5.11. □

Proof of (B1) ⇔ (B2). The statement (B1) is equivalent to:

$$\text{neither } (\mathrm{A1}'') \text{ nor } (\mathrm{C1}) \text{ holds.}$$

On the other hand, the statement (B2) is equivalent to:

$$\text{neither } (\mathrm{A2}) \text{ nor } (\mathrm{C2}) \text{ holds.}$$

Since we have (A1″) ⇔ (A2), (C1) ⇔ (C2), it follows that (B1) ⇔ (B2). □

The proof of Theorem 3.8 is complete. □

8.2. Proof of Proposition 3.6 and Corollary 3.7

If g is periodic, then the existence of a stationary solution in Ω_g is clearly equivalent to the existence of a stationary solution in Ω_h for any $h \in \mathcal{H}_g$. This means that the cases (B1), (B2) in Theorem 3.8 do not occur; hence Proposition 3.6 follows immediately from Theorem 3.8.

To prove Corollary 3.7, suppose that (a) does not hold. Then by Proposition 3.6, there exists a stationary solution V in Ω_g. Then the functions

$$\sigma_{nL} V \quad (n \in \mathbb{Z})$$

are all stationary solutions in Ω_g, where L is the period of g. The assertion (b) then follows by using the same argument as in the proof of Theorem 3.2. □

8.3. An example of virtual pinning

In this subsection we give an example of a non-periodic domain Ω_g in which virtual pinning occurs. As we discussed before, "virtual pinning" refers to the situation where case (B1) in Theorem 3.8 holds.

To make the argument simpler, we consider the case where the domain Ω is flat on the left side, namely $g_- = 0$. Thus Ω is expressed as

$$\Omega = \big\{(x, y) \in \mathbb{R}^2 \bigm| -H < x < H + g_+(y)\big\}.$$

Let us begin with the following general criterion for virtual pinning in the domain Ω.

Lemma 8.1. *Let $g_+ : \mathbb{R} \to [0, \infty)$ be a smooth recurrent function satisfying $g_+ \not\equiv 0$ and $\sup_{y\in\mathbb{R}} g_+'(y) < 1$. Then virtual pinning occurs for (1.2)–(1.3) if and only if*

$$\sup_{y\in\mathbb{R}} k(y) = A \quad \textit{and} \quad k(y) < A \quad \textit{for any } y \in \mathbb{R}, \tag{8.1}$$

where

$$k(y) := \frac{g_+'(y)}{(2H + g_+(y))\sqrt{1 + (g_+'(y))^2}}. \tag{8.2}$$

Proof. It suffices to show that the function $g = (0, g_+)$ satisfies (B2) in Theorem 3.8.

Let Γ be a circular arc in Ω_g contacting $\partial_\pm\Omega_g$ perpendicularly. Then a simple geometric observation yields that the curvature of Γ is equal to $-k(y)$, where y is the y-coordinate of the endpoint of Γ on $\partial_+\Omega_g$. In view of this and the latter part of (8.1), we see that there exists no stationary solution in Ω_g.

On the other hand, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence satisfying $k(a_n) \to A$ as $n \to \infty$. Replacing $\{a_n\}$ by its subsequence if necessary, we may assume that $\sigma_{a_n} g$ converges to some $h = (0, h_+) \in \mathcal{H}_g$ locally uniformly in the C^1 sense. The same geometric observation as above implies that the curvature of a circular arc contacting $\partial_\pm\Omega_h$ perpendicularly is $-k_h(y)$, where y is the y-coordinate of the right endpoint and

$$k_h(y) = \frac{h'_+(y)}{(2H + h_+(y))\sqrt{1 + h'_+(y)^2}}.$$

Since

$$k_h(0) = \lim_{n\to\infty} k(a_n) = A,$$

there exists a circular arc with curvature $-A$ intersecting $\partial_\pm\Omega_h$ perpendicularly. Clearly this circular arc is a stationary solution of (1.1) in Ω_h. $\square$

Now we construct an example that meets the above criterion.

Example 8.2. Let $g_0 : \mathbb{R} \to [0, \infty)$ be a smooth periodic function satisfying

$$g_0 \not\equiv 0, \qquad g_0(y+1) \equiv g_0(y), \qquad \max_{y\in\mathbb{R}} g'_0(y) < 1.$$

We define

$$k_0(y) := \frac{g'_0(y)}{(2H + g_0(y))\sqrt{1 + (g'_0(y))^2}}. \tag{8.3}$$

This is again a periodic function of period 1. We then set

$$A = \sup_{y\in\mathbb{R}} k_0(y) \quad \Bigl(= \max_{y\in[0,1)} k_0(y)\Bigr).$$

We assume that k_0 attains its maximum A at discrete values of y. This is always the case if, for example, g_0 is real analytic. Let $0 \leqslant a_1, a_2, \dots, a_n < 1$ be such that

$$k_0(y) = A \quad \Leftrightarrow \quad y \in \{a_1, a_2, \dots, a_n\} \quad \text{mod } 1.$$

Next let $\omega > 0$ be any irrational number and define

$$g_+(y) := g_0\bigl(q(y) + C\bigr), \quad \text{where } q(y) := \frac{3}{4}y + \frac{1}{4\omega\pi}\sin\omega\pi y.$$

Here C is a constant to be specified later. It is clear that g_+ is a quasi-periodic function, therefore it is recurrent. We define k as in (8.2). Then we have

$$k(y)=k_0\big(q(y)+C\big)\frac{q'(y)\sqrt{1+(g_0'(q(y)+C))^2}}{\sqrt{1+(q'(y)g_0'(q(y)+C))^2}}. \tag{8.4}$$

Since $0<q'(y)=(3+\cos\omega\pi y)/4\leqslant 1$, the above formula yields

$$\sup_{y\in\mathbb{R}} k(y)\leqslant \sup_{y\in\mathbb{R}} k_0\big(q(y)+C\big)=A.$$

Next we show that $\sup_{y\in\mathbb{R}} k(y)=A$. Substituting $y=2m/\omega$ ($m\in\mathbb{Z}$) into (8.4), we obtain

$$k\left(\frac{2m}{\omega}\right)=k_0\left(\frac{3m}{2\omega}+C\right)\quad \text{for any } m\in\mathbb{Z}.$$

Since ω is irrational, the set $\{\frac{3m}{2\omega}+C \bmod 1 \mid m\in\mathbb{Z}\}$ forms a dense subset of $\mathbb{R}/\mathbb{Z}$. Therefore we can find a sequence of integers m_j ($j=1,2,3,\ldots$) such that $\frac{3m_j}{2\omega}+C\to a_1 \bmod 1$ as $j\to\infty$. It follows that

$$k\left(\frac{2m_j}{\omega}\right)\to k_0(a_1)=A.$$

This implies $\sup_{y\in\mathbb{R}} k(y)=A$.

In order to apply the criterion (8.1), it remains to check whether there exists a value of y satisfying $k(y)=A$ or not. If $k(y^*)=A$ for some y^*, then we see from (8.4) that $q'(y^*)=1$, $k_0(q(y^*)+C)=A$. This is equivalent to:

$$y^*=\frac{2m}{\omega}\quad \text{for some } m\in\mathbb{Z},\qquad \frac{3}{4}y^*+C\in\{a_1,\ldots,a_n\}\quad \text{mod } 1.$$

Such y^* exists if and only if

$$C\in\bigcup_{j=1}^{n}\left\{a_j+l-\frac{3m}{2\omega}\;\middle|\; l,m\in\mathbb{Z}\right\}. \tag{8.5}$$

In other words, if C does not satisfy (8.5), then virtual pinning occurs for (1.2)–(1.3). Since the right-hand side of (8.5) is a countable set, such C does exist.

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