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On a reaction–diffusion equation with Robin and free boundary conditions ☆

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Abstract

This paper studies the following problem

$$\begin{cases} u_t = u_{xx} + f(u), & 0 < x < h(t),\ t > 0, \\ u(0,t) = b u_x(0,t), & t > 0, \\ u(h(t),t) = 0, \quad h'(t) = -u_x(h(t),t), & t > 0, \\ h(0) = h_0, \quad u(x,0) = \sigma\phi(x), & 0 \leqslant x \leqslant h_0 \end{cases}$$

where f is an unbalanced bistable nonlinearity, $b \in [0,\infty)$, $\sigma \geqslant 0$ and ϕ is a compactly supported C^2 function. We prove that, there exists $\sigma^* > 0$ such that, vanishing happens when $\sigma < \sigma^*$ (i.e., $h(t) < M$ for some $M > 0$ and $u(\cdot,t)$ converges as $t \to \infty$ to 0 uniformly in $[0, h(t)]$); spreading happens when $\sigma > \sigma^*$ (i.e., $h(t) - c^*t$ tends to a constant for some $c^* > 0$, $u(\cdot,t)$ converges to a positive stationary solution locally uniformly in $[0,\infty)$ and to a traveling semi-wave with speed c^* near $x = h(t)$); in the transition case when $\sigma = \sigma^*$, $\|u(\cdot,t) - V(\cdot - \xi(t))\|_{H^2([0,h(t)])}$ tends to 0 as $t \to \infty$, where $\xi(t)$ is a maximum point of $u(\cdot,t)$ and V is the unique even positive solution of $V'' + f(V) = 0$ subject to $V(\infty) = 0$. Moreover, with respect to b and f, $\xi(t) = P \ln t + Q + o(1)$ for some $P > 0$ and $Q \in \mathbb{R}$, or, $\xi(t) \to z$ for some root z of $V(-z) = bV'(-z)$.

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1. Introduction

Consider the following free boundary problem

$$\begin{cases} u_t = u_{xx} + f(u), & 0 < x < h(t),\ t > 0, \\ u(0,t) = bu_x(0,t), & t > 0, \\ u(h(t),t) = 0, \quad h'(t) = -u_x(h(t),t), & t > 0, \\ h(0) = h_0, \quad u(x,0) = u_0(x), & 0 \leqslant x \leqslant h_0 \end{cases} \tag{1}$$

where $b \in [0,\infty)$, $x = h(t)$ is a moving boundary to be determined together with $u(x,t)$ and f is an **unbalanced bistable** nonlinearity satisfying

$$\begin{cases} f \in C^1([0,\infty)),\ f(0) = 0 > f'(0),\ \ f(\cdot) < 0 \text{ in } (0,\alpha), \\ f(\cdot) > 0 \text{ in } (\alpha,1),\ f(\cdot) < 0 \text{ in } (1,\infty),\ \inf_{s>0} f(s)/s > -\infty, \\ \text{for } F(u) := -2\int_0^u f(s)ds,\ F(\theta) = 0 \text{ for some } \theta \in (\alpha,1). \end{cases} \tag{\textbf{F}}$$

The initial function u_0 is chosen from $\mathscr{X}(h_0)$, where, for some $h_0 > 0$,

$$\mathscr{X}(h_0) := \left\{ \phi \;\middle|\; \begin{array}{l} \phi \in C^2([0,h_0]),\ \phi(0) = b\phi'(0), \\ \phi \geqslant, \not\equiv 0 \text{ in } [0,h_0],\ \phi \equiv 0 \text{ in } [h_0,\infty) \end{array} \right\}.$$

Problem (1) may be used to describe the spreading of a new or invasive biological/chemical species, with the free boundary $h(t)$ representing the spreading front of the species whose density is denoted by $u(x,t)$. The (Stefan) free boundary condition indicates that the front invades at a rate that is proportional to the magnitude of the spatial population gradient there. Such a boundary condition was used for population models in [6,12,13] etc., for tumor, protocell and wound healing models in [10,22–24], etc. It can be derived from Fick's law of diffusion by considering the "population loss" based on Allee effect at the front (cf. [6] for details).

Problem (1) with Neumann boundary condition $u_x(0,t) = 0$ was recently studied by Du and Lin [12] for logistic nonlinearity $f(u) = u(1-u)$, and by Du and Lou [13] for general f. Among others they obtained a dichotomy result for the solutions of (1) with a monostable f: for any positive solution, either spreading happens (i.e., $\lim_{t\to\infty} h(t) = \infty$ and $\lim_{t\to\infty} u(\cdot,t) = 1$); or vanishing happens (i.e., $h(t) < M$ for some $M > 0$ and $\lim_{t\to\infty} u(\cdot,t) = 0$). The vanishing phenomenon is a remarkable result which reveals the difference between the free boundary problem and the Cauchy problem (in the latter case, it is known that the problem with a monostable f has the so-called *hair trigger effect*, which implies that any positive solution will spread, cf. [4,5]). In case f is a bistable nonlinearity satisfying **(F)**, [13] studied (1) with Neumann boundary condition $u_x(0,t) = 0$. They gave a rather complete description on the long time behavior of the solution $u(\cdot,t;\sigma\phi)$ of (1) with initial datum $u_0 = \sigma\phi$. They proved that there exists $\sigma^* > 0$ such that, vanishing happens for $u(\cdot,t;\sigma\phi)$ when $\sigma < \sigma^*$; spreading happens when $\sigma > \sigma^*$; and in the transition case when $\sigma = \sigma^*$, $u(\cdot,t;\sigma^*\phi)$ converges as $t \to \infty$ to $V(\cdot)$ locally uniformly in $x \in \mathbb{R}$, where V is the unique even positive solution of $V'' + f(V) = 0$ subject to $V(\infty) = 0$ (i.e. the so-called *ground state*). The approach of the last result relies on the fact that $u_x(x,t) < 0$ for

$x > h_0$ and $t > 0$. (This monotonicity, however, will no longer be true if the Neumann boundary condition is replaced by a Robin one.)

When $b = 0$ (i.e., the Dirichlet boundary condition $u(0,t) = 0$), Kaneko and Yamada [26] studied problem (1) for monostable and bistable types of f. They gave some sufficient conditions for spreading and for vanishing.

Problem (1) has been studied by the authors in [28], but only for monostable and combustion types of f. We proved that for any solution u, either spreading or vanishing happens. Because of the absence of transition cases, these situations are simpler than the case where f is bistable (see the main results below). In this paper we continue to study the equation with a bistable type of nonlinearity, supplemented with a general Robin boundary condition at $x = 0$ and a Stefan free boundary condition at $x = h(t)$. The Robin condition $u(0,t) = bu_x(0,t)$ indicates that, the flux of the species invading the habitat $[0, h(t)]$ from the boundary $x = 0$ (which is expressed by $bu_x(0,t)$ by Fick's law) depends on the density $u(0,t)$. The constant $1/b$ ($b > 0$) is also called per capita entering rate.

The main purpose of this paper is to study the influence of the Robin and free boundary conditions on the long time behavior of the solutions. We will prove a vanishing–transition–spreading trichotomy result, similar as that in [13] for problems with a Neumann boundary condition, but still with significant difference in the transition case. More precisely, in the transition case, we will prove that the solution u satisfies $\|u(\cdot,t) - V(\cdot - \xi(t))\|_{H^2([0,h(t)])} \to 0$ as $t \to \infty$, and using the maximum principle we will show that, with respect to b and f, either $\xi(t) \to z$ for some root z of $V(-z) = bV'(-z)$, or $\xi(t) \to \infty$. The latter is a very interesting phenomenon which does not occur in the problems with the homogeneous Neumann boundary condition (cf. [13]). Our proof for $u \to V(\cdot - \xi(t))$ is based on some energy estimates and the concentrated compactness argument, which is completely different from the approach in [13]. Furthermore, when $\xi(t) \to \infty$, we can derive by the center manifold argument that $\xi(t) = P \ln t + Q + o(1)$ for some constants P, Q ($P > 0$) depending only on b and f.

Without the free boundary (i.e., $h(t) \equiv \infty$), (1) reduces to a problem on the half line. Under the homogeneous Neumann boundary condition, the problem can be extended by reflection to be a Cauchy one. The long time behavior of the solutions and traveling wave solutions for such problems have been extensively studied (cf. [4,5,15,20,21,27,31]). For example, in [15], motivated by the break-through results obtained in [31], Du and Matano proved a vanishing-transition-spreading trichotomy result. Fašangová and Feireisl [17] also studied the problem on the half line with homogeneous Dirichlet or Neumann boundary condition (i.e., $u(0,t) = 0$ or $u_x(0,t) = 0$). Under the Dirichlet boundary condition they proved that, for any nonnegative function $\phi \in W_0^{1,2}([0,\infty))$ which has only one local maximum point, there exist $\sigma_* := \sigma_*(\phi)$ and $\sigma^* := \sigma^*(\phi)$ with $0 < \sigma_* \leqslant \sigma^*$ such that vanishing happens for the solution $u(\cdot,t;\sigma\phi)$ when $\sigma < \sigma_*$, spreading happens when $\sigma > \sigma^*$, and in the transition case when $\sigma \in [\sigma_*, \sigma^*]$, $\|u(\cdot,t;\sigma\phi) - V(\cdot - \xi(t))\|_{H^1([0,\infty))} \to 0$ as $t \to \infty$ with $\lim_{t\to\infty} \xi(t) = \infty$. In other words, they found the interesting phenomenon: in the transition case, u converges to the infinitely shifting ground state. But they still left some important questions:

1. Does the transition case consist of a sharp threshold value? i.e., does $\sigma_* = \sigma^*$?
2. How fast does $\xi(t) \to \infty$ as $t \to \infty$?
3. What happens for the solutions with multiple local maximum points?

Recently, [11] studied the problem on the half line with Robin boundary condition, and gave answers to these questions: 1. $\sigma_* = \sigma^*$; 2. $\xi(t) = P \ln t + O(1)$ for some P depending on f and b;

3. the results remain valid for the solutions with multiple local maximum points. Our present paper also gives similar answers to these questions but for free boundary problems. Furthermore, we study the asymptotic behavior of the free boundary $h(t)$ and show that $h(t) < M$ for some $M > 0$ when vanishing happens; $h(t) - c^*t - H_\infty \to 0$ for some $c^* > 0$ and $H_\infty \in \mathbb{R}$ when spreading happens; and $h(t) - \xi(t) \to \infty$ with $\xi(t)$ representing the unique maximum point of $u(\cdot, t)$ for large t.

This paper is organized as follows. In Section 2 we state our main results. In Section 3 we study the stationary solutions and the number of local maximum points of the solution $u(\cdot, t)$. In Section 4 we prove a spreading–transition–vanishing trichotomy result on the long time behavior of solutions. In Section 5 we give a uniform convergence for transition solutions and prove the first main result. In Section 6 we study the asymptotic speed and asymptotic profile for spreading solutions.

2. Main results

For any $h_0 > 0$ and $u_0 \in \mathscr{X}(h_0)$, we will show in the next section that the solution u of (1) is bounded if f satisfies **(F)**. Then we get the time-global existence of the solution by similar argument as in [12,13,26]. Moreover, by the Hopf lemma we have $u_x(h(t), t) < 0$, and so $h'(t) > 0$ and $h_\infty := \lim_{t\to\infty} h(t) \in (h_0, \infty]$ exists. In this paper, we will show that, as $t \to \infty$, each solution of (1) converges to a bounded, nonnegative stationary one, which means a solution of

$$v'' + f(v) = 0,\ v(0) = bv'(0),\ v \geqslant 0 \text{ in } (0, \infty) \text{ and } v \in L^\infty([0, \infty)). \tag{2}$$

By phase plane analysis it is not difficult to give all kinds of solutions of (2) (see Subsection 3.1 for details). Among them the following types are of special importance, since they can be the ω-limits of the solutions of (1):

(1) **Trivial Solution** $v \equiv 0$;
(2) **Active States** v_*: solutions of (2) satisfying $v'(x) > 0$ in $(0, \infty)$ and $v(\infty) = 1$;
(3) **Ground States**: $v(\cdot) = V(\cdot - z)$ where V is the unique even positive solution of $V'' + f(V) = 0$ on $\mathbb{R}$ subject to $V(\infty) = 0$, and $z \in \mathbf{Z} := \{z \mid V(-z) = bV'(-z)\}$.

We have the following results on the long time behavior of the solutions of (1).

Theorem 2.1. *Assume* **(F)**. *Let $u(x, t; \sigma\phi)$ be the solution of (1) with initial datum $u_0 = \sigma\phi$ for some $\phi \in \mathscr{X}(h_0)$. Then there exists $\sigma^* = \sigma^*(\phi) \in (0, \infty)$ such that the following trichotomy holds:*

(i) *if $\sigma > \sigma^*$,* **spreading** *happens in the following sense:*

$$h_\infty = \infty \quad \text{and} \quad \lim_{t\to\infty} \|u(\cdot, t; \sigma\phi) - v_*(\cdot)\|_{C^2([0,M])} = 0 \text{ for any } M > 0,$$

for some Active State v_*;

(ii) *if $0 \leqslant \sigma < \sigma^*$,* **vanishing** *happens in the following sense:*

$$h_\infty < \infty \quad \text{and} \quad \lim_{t\to\infty} \|u(\cdot, t; \sigma\phi)\|_{H^2([0,h(t)])} = 0;$$

(iii) *in the* **transition** *case when* $\sigma = \sigma^*$,

$$h_\infty = \infty \quad \textit{and} \quad \lim_{t\to\infty} \|u(\cdot, t; \sigma\phi) - V(\cdot - \xi(t))\|_{H^2([0,h(t)])} = 0, \tag{3}$$

where $\xi(t) \in (0, h(t))$ *satisfies* $\lim_{t\to\infty}[h(t) - \xi(t)] = \infty$. *Moreover,*

(1) *if* $b\sqrt{F(s)} < s$ *for all* $s \in (0, \theta)$, *then* $\lim_{t\to\infty} \xi(t) = \infty$;

(2) *if* $b\sqrt{F(s_n)} \geqslant s_n$ *for a sequence* $s_n \searrow 0$, *then* $\lim_{t\to\infty} \xi(t) = z \in \mathbf{Z}$;

(3) *if none of the above two conditions hold, then* $\lim_{t\to\infty} \xi(t) = \infty$ *for some initial data* $\sigma^*\phi$, *and* $\lim_{t\to\infty} \xi(t) = z \in \mathbf{Z}$ *for other initial data.*

We will see in the next section that the set of the solutions of (2) may include Trivial Solution, Active States, Ground States and Positive Periodic Solutions. Our theorem indicates that only the former three types can be selected as the ω-limit of u.

In the transition case, besides the limit of $h(t) - \xi(t)$ and the dichotomy result on the limit of $\xi(t)$, the propagation speeds of $\xi(t)$ and $h(t)$ when they move to infinity are also interesting problems. We guess that both $\xi(t)$ and $h(t)$ can be estimated in the form $P \ln t + Q + o(1)$ as $t \to \infty$. We present such a result for $\xi(t)$ in Section 5, and will study the speed of $h(t)$ in the future.

We also remark that the technical condition $\inf_{s>0} f(s)/s > -\infty$ in **(F)** is *only* used to ensure that $\sigma^* < \infty$, without this condition σ^* can be infinity (see Remark 4.4 below). Finally we point out that the conclusions in Theorem 2.1 remain valid for more general monotone families of initial data, rather than just the rays $\{\sigma\phi \mid \sigma \geqslant 0\}$ (see details in Remark 4.5).

When spreading happens, that is, u converges to an Active State v_*, it is an interesting problem to study the asymptotic spreading speed: $\lim_{t\to\infty} \frac{h(t)}{t}$ and the asymptotic profile of u near $x = h(t)$: $u(h(t) + z, t)$. This problem is closely related to the following problem:

$$\begin{cases} q'' - cq' + f(q) = 0 \text{ in } (0, \infty), \\ q(0) = 0,\ q'(0) = c,\ q(\infty) = 1, q(x) > 0 \text{ in } (0, \infty). \end{cases} \tag{4}$$

Du and Lin [12] and Du and Lou [13] proved that problem (4) has a unique solution pair (c^*, q_{c^*}), and $c^* = \lim_{t\to\infty} \frac{h(t)}{t} > 0$ is the asymptotic spreading speed for the case $b = \infty$. Recently, Du, Matsuzawa and Zhou [16] proved that if spreading happens for a solution of (1) with $b = \infty$, then

$$h(t) - c^*t \to H_\infty \in \mathbb{R},\ h'(t) \to c^*,\ \|u(\cdot, t) - q_{c^*}(h(t) - x)\|_{L^\infty([0,h(t)])} \to 0$$

as $t \to \infty$. We prove similar conclusions for problem (1) with Robin boundary condition.

Theorem 2.2. *Assume* **(F)** *and spreading happens in the sense:*

$$h_\infty = \infty \quad \textit{and} \quad \lim_{t\to\infty} \|u(\cdot, t) - v_*(\cdot)\|_{C^2([0,M])} = 0 \textit{ for any } M > 0,$$

where v_* *represents an Active State. Let* (c^*, q_{c^*}) *be the unique solution of* (4). *Then*

$$\lim_{t\to\infty} [h(t) - c^*t] = H_\infty \textit{ for some } H_\infty \in \mathbb{R}, \qquad \lim_{t\to\infty} h'(t) = c^*, \tag{5}$$

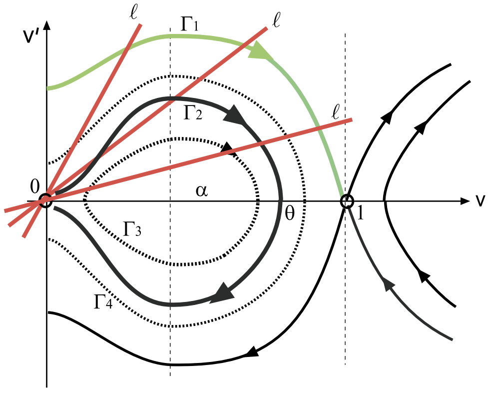

Fig. 1. Intersection points between the trajectory Γ_i and the line $\ell : v = bv'$.

and

$$\lim_{t\to\infty} \|u(x,t) - v_*(x) \cdot q_{c^*}(c^*t + H_\infty - x)\|_{L^\infty([0,h(t)])} = 0. \tag{6}$$

This theorem will be proved in Section 6 by using the moving coordinate $z = x - c^*t$. A key step is to prove the convergence of the new boundary $z = H(t) := h(t) - c^*t$ which is not necessarily monotone. We do this by zero number argument, avoiding the construction of complex upper and lower solutions as in [16].

3. Preliminaries

3.1. Stationary solutions

In this subsection we study the solutions of problem (2). On the v–v' phase plane, each solution v of $v'' + f(v) = 0$ corresponds to a trajectory $v'^2 = F(v) - q$, where q is a constant and $F(v) := -2\int_0^v f(s)ds$. The Robin boundary condition $v(0) = bv'(0)$ corresponds to intersection points of the line $\ell :\ v = bv'$ with the trajectory. By phase plane analysis it is not difficult to give all kinds of solutions of (2) (cf. Fig. 1).

(1) Trivial Solution 0.

(2) Active States v_*. Consider the trajectory Γ_1 given by $v'^2 = F(v) - F(1)$ for $v \in [0, 1)$, which connects $(0, \sqrt{F(0) - F(1)})$ to $(1, 0)$ in the half plane $\{v' > 0\}$. Assume Γ_1 intersects the line ℓ at some points, say, $(v_1, \sqrt{F(v_1) - F(1)})$. Then $\Gamma_1|_{\{v_1 \leqslant v<1\}}$ gives a solution v_* of (2), which satisfies

$$v_*'' + f(v_*) = 0 < v_*' \ \text{ in } \ [0,\infty), \quad v_*(0) = bv_*'(0), \quad v_*(\infty) = 1.$$

In this paper we call such a v_* as an Active State. If ℓ intersects Γ_1 in the region $\{\alpha < v < 1\}$, then the intersection point is unique (so is the Active State) since $\frac{dv'}{dv} = \frac{-f(v)}{v'} < 0$ on $\Gamma_1|_{\{\alpha<v<1\}}$ and $\frac{dv'}{dv} \geqslant 0$ on ℓ. On the other hand, if ℓ intersects Γ_1 in the region $\{0 < v < \alpha\}$, then the intersection points maybe more than 1 (so are the Active States) since $\frac{dv'}{dv} > 0$ on both ℓ and $\Gamma_1|_{0<v<\alpha}$.

(3) Ground States $V(\cdot - z)$. Consider the trajectory Γ_2 given by $v'^2 = F(v)$, which is a homoclinic orbit connecting $(0,0)$ to itself through $(\theta, 0)$. Denote the corresponding function by $V(x)$ with $V(0) = \theta$, then $V'' + f(V) = 0$ on $\mathbb{R}$ and $V(\pm\infty) = 0$.

If $b\sqrt{F(s_0)} = s_0$ for some $s_0 \in (0, \theta]$, then Γ_2 intersects the line ℓ at point $(s_0, \sqrt{F(s_0)})$. So $\Gamma_2|_{\{s_0 \leqslant v \leqslant \theta,\ v' \geqslant 0\} \cup \{0 < v < \theta,\ v' < 0\}}$ gives a solution v of (2), which satisfies

$$v'' + f(v) = 0 < v \ \text{ in } \ (0, \infty), \quad v(0) = bv'(0), \quad v'(0) \geqslant 0, \quad v(\infty) = 0. \tag{7}$$

In this paper we call such a solution as a Ground State (may be more than 1), which is expressed as $v(\cdot) = V(\cdot - z)$ for some positive $z \in \mathbf{Z} := \{z \mid V(-z) = bV'(-z)\}$.

(4) Positive Periodic Solutions. For any $m \in (\alpha, \theta)$, consider the trajectory Γ_3 given by $v'^2 = F(v) - F(m)$ for $v \leqslant m$, which is a closed curve connecting $(m, 0)$ to $(m_1, 0)$ $(m_1 \in (0, \alpha))$. If Γ_3 intersects the line ℓ at some points, say, $(v_2, \sqrt{F(v_2) - F(m)})$. Then Γ_3 gives a solution P of (2), which satisfies

$$P'' + f(P) = 0 < P \ \text{ in } \ (0, \infty), \quad P(0) = bP'(0), \quad P(x) = P(x + L)\ \forall x \geqslant 0,$$

for some $L > 0$. Such solutions (called Positive Periodic Solutions in this paper) are generally more than 1.

(5) Compactly Supported Solutions. For each $m \in (\theta, 1)$, consider the trajectory Γ_4 given by $v'^2 = F(v) - F(m)$ for $v \leqslant m$, which connects $(0, \sqrt{F(0) - F(m)})$ to $(0, -\sqrt{F(0) - F(m)})$ passing through $(m, 0)$. It gives a function v_m satisfying

$$v_m'' + f(v_m) = 0 < v_m \leqslant m \ \text{ in } \ (0, 2L_m), \quad v_m(0) = v_m(2L_m) = 0, \tag{8}$$

where

$$L_m := \int_0^m \frac{ds}{\sqrt{F(s) - F(m)}}, \quad m \in (\theta, 1). \tag{9}$$

Clearly, when Γ_4 intersects the line ℓ at some points, then a suitable shift of each v_m gives a solution of

$$v'' + f(v) = 0 < v \text{ in } (0, 2L_m - r) \text{ for some } r \geqslant 0,\ v(0) = bv'(0),\ v(2L_m - r) = 0.$$

In what follows we write

$$L_* := \inf_{m \in (\theta, 1)} L_m. \tag{10}$$

By the above phase plane analysis we have the following results.

Proposition 3.1. *Denote by $\mathcal{S}$ the set of solutions of (2). Then $\mathcal{S}$ contains 0 and Active States for any $b \geqslant 0$. Furthermore, it only contains these two types of solutions if $b\sqrt{F(s)} < s$ for $s \in (0, \theta)$; it also contains Ground States if $b\sqrt{F(s_0)} = s_0$ for some $s_0 \in (0, \theta)$; it also contains Ground States and Positive Periodic Solutions if $b\sqrt{F(s_1)} > s_1$ for some $s_1 \in (0, \theta)$.*

3.2. Zeros of u_x

The approach in this paper relies on some extended zero number properties, which are stated now.

Lemma 3.2. *Let $t_1 < t_2$ and $I := [0, l]$ for some $l > 0$. Assume that $w(x, t)$ is a classical solution of the following equation*

$$w_t = w_{xx} + c(x,t)w, \quad x \in (0,l),\ t \in (t_1, t_2), \tag{11}$$

where c is a bounded function in $I \times (t_1, t_2)$. Assume further that $w(l, t) \neq 0$ for $t \in (t_1, t_2)$, and $w(0, t) \neq 0$ for $t \in (t_1, t_2)$ (or $w(0, t) = 0$ for $t \in (t_1, t_2)$; or $w_x(0, t) = 0$ for $t \in (t_1, t_2)$; or $w_x(0, t) = \bar{a} w(0, t)$ for $t \in (t_1, t_2)$ and some $\bar{a} > 0$). Then the following conclusions hold:

(1) *$\mathcal{Z}_I(w(\cdot, t)) < \infty$ for any $t \in (t_1, t_2)$, where $\mathcal{Z}_I(w(\cdot, t))$ denotes the number of zeros of $w(\cdot, t)$ on I (including the possible zero $x = 0$);*
(2) *$\mathcal{Z}_I(w(\cdot, t))$ is nonincreasing in $t \in (t_1, t_2)$. Moreover, if $w(x^*, t^*) = w_x(x^*, t^*) = 0$ for some $t^* \in (t_1, t_2)$, $x^* \in [0, l)$, then*

$$\mathcal{Z}_I(w(\cdot,t)) > \mathcal{Z}_I(w(\cdot,s)) \qquad \text{for all } t \in (t_1, t^*),\ s \in (t^*, t_2).$$

Proof. When $w(0, t) \neq 0$, or $w(0, t) = 0$, or $w_x(0, t) = 0$ for all $t \in (t_1, t_2)$, the conclusions follow from [3, Theorems C and D] directly. In what follows we only consider the case:

$$w_x(0,t) = \bar{a} w(0,t) \quad \text{for } t \in (t_1, t_2) \tag{12}$$

and always assume that $S := \{t \in (t_1, t_2) : w(0, t) = 0\} \neq \emptyset$ (since the boundary condition reduces again to the case $w(0, t) \neq 0$ when $S = \emptyset$).

Using the idea in [29] one can easily show that $w(0, t)$ changes sign at most finitely many times in (τ, t_2) for any given $\tau \in (t_1, t_2)$. We now prove a more precise result, that is, $S \cap (\tau, t_2)$ contains at most finitely many elements.

First we show that S is a nowhere dense set. Indeed, if S contains an interval $\Delta \subset (t_1, t_2)$, then we can extend w to $(-l, 0)$ as an odd function, and so (12) implies that $x = 0$ is a degenerate zero of the extended function for all $t \in \Delta$. This contradicts [3, Theorem D].

Consequently, there exists a sequence $\{\tau_n\} \subset (t_1, t_2)\backslash S$ decreasing to t_1 such that

$$w(0,t) \neq 0 \quad \text{for } t \in J_n := (\tau_n - \beta_n, \tau_n + \beta_n)$$

provided $\beta_n > 0$ is sufficiently small. Hence, $\mathcal{Z}_I(w(\cdot, t)) < \infty$ for $t \in J_n$ by [3, Theorem D]. We now increase β_n to some β_n^* such that

$$w(0,t) \neq 0 \ \text{ for } t \in [\tau_n, \tau_n + \beta_n^*) \qquad \text{and} \qquad w(0, \tau_n + \beta_n^*) = 0.$$

(Without loss of generality, we assume $\tau_n + \beta_n^* < t_2$, for otherwise, 0 is not a zero of $w(x, t)$ for any $t \in [\tau_n, t_2)$, and so the conclusions in (2) hold for $t \in [\tau_n, t_2)$.) In what follows, we write $\hat{t} := \tau_n$ and $t^* := \tau_n + \beta_n^*$ for simplicity. Then

$$w(0,t)\neq 0 \ \text{ for } t\in[\hat{t},t^*) \qquad \text{and} \qquad 0=\bar{a}w(0,t^*)=w_x(0,t^*). \tag{13}$$

We will prove conclusion (2) at $t=t^*$. More precisely, we will prove

$$\begin{cases} w(0,t)\neq 0 \text{ for } t\in(t^*,t^*+\epsilon] \ \text{ for some small } \epsilon>0, \\ \mathcal{Z}_I(w(\cdot,t))\geqslant \mathcal{Z}_I(w(\cdot,t^*))>\mathcal{Z}_I(w(\cdot,s)) \text{ for all } t\in(\hat{t},t^*) \text{ and } s\in(t^*,t^*+\epsilon). \end{cases} \tag{14}$$

Once this is proved, we see that S consists of isolated elements, $\mathcal{Z}_I(w(\cdot,t))$ decreases strictly when t gets across $t^*\in S$. So $w(\cdot,t)$ can have degenerate zero in $[0,l)$ only finitely many times in $[\hat{t},t_2)$. In particular, $S\cap(\tau,t_2)$ contains at most finitely many elements. Thus the lemma can be proved.

Under our assumption (13) we have $\mathcal{Z}_I(w(\cdot,t))\leqslant \mathcal{Z}_I(w(\cdot,\hat{t}))<\infty$ for $t\in[\hat{t},t^*)$ by [3, Theorem D]. Choose $\tilde{t}\in(\hat{t},t^*)$ such that $w(\cdot,t)$ has only nondegenerate zeros in $(0,l)$ for $t\in[\tilde{t},t^*)$. Due to the nondegeneracy, these zeros can be expressed as smooth curves:

$$x=\gamma_1(t),\dots,x=\gamma_m(t), \ \text{ with } 0<\gamma_i(t)<\gamma_{i+1}(t)<l \text{ for } i=1,\dots,m-1.$$

For each $i\in\{1,\dots,m\}$, we now examine the limit of $\gamma_i(t)$ as $t\to t^*$. Clearly

$$x_j:=\liminf_{t\to t^*}\gamma_i(t)\geqslant 0 \text{ and } x_j^*:=\limsup_{t\to t^*}\gamma_i(t)<l.$$

If $x_j<x_j^*$, then $w(x,t^*)\equiv 0$ for $x\in[x_j,x_j^*]$. We may now apply [19, Theorem 2] to w over the region $[0,l]\times[t^*-\epsilon,t^*]$, with $\epsilon>0$ sufficiently small, to conclude that $w(x,t^*)\equiv 0$ for $x\in[0,l]$. This contradicts the fact that $w(l,t^*)\neq 0$. Therefore $x_j:=\lim_{t\to t^*}\gamma_j(t)$ exists for every $j\in\{1,\dots,m\}$.

Next we prove $x_1=0$ by a contradiction. Assume $x_1>0$, then in the region $A:=\{(x,t): 0<x<\gamma_1(t),\ \tilde{t}<t\leqslant t^*\}$, we have $w(x,t)\neq 0$. Assume $w(x,t)>0$ for definiteness. Since $w(0,t^*)=0$, we can apply the Hopf boundary lemma to deduce that $w_x(0,t^*)>0$. This contradicts the boundary condition (12). Moreover, using the strong maximum principle in the region $A_i:=\{(x,t):\gamma_i(t)<x<\gamma_{i+1}(t),\ \tilde{t}\leqslant t\leqslant t^*\}$ we see that $w(x,t^*)\neq 0$ for $x\in(x_i,x_{i+1})$ if $x_i<x_{i+1}$. Combining these results we immediately obtain $\mathcal{Z}_I(w(\cdot,t^*))\leqslant m=\mathcal{Z}_I(w(\cdot,t))$ for $t\in[\tilde{t},t^*)$. Let $0=z_1<z_2<\dots<z_n<l$ denote all the zeros of $w(\cdot,t^*)$ in I (with $n\leqslant m$).

Denote $\bar{z}=\frac{z_2}{2}$. Then $w(\bar{z},t^*)\neq 0$. For definiteness, we assume $w(\bar{z},t^*)>0$ and so $w(\bar{z},t)>0$ for $t\in[t^*,t^*+\epsilon]$ with $\epsilon>0$ small. Using the strong maximum principle on $[0,\bar{z}]\times[t^*,t^*+\epsilon]$ we see that

$$w(x,t)\neq 0 \text{ for } x\in[0,\bar{z}],\ t\in(t^*,t^*+\epsilon]. \tag{15}$$

This allows us to apply [3, Theorem D] to conclude that $w(\cdot,t)$ has no degenerate zeros in I when $t\in(t^*,s_1]$ for some $s_1\in(t_1,t_1+\epsilon]$. Let $\tilde{\gamma}_1(t)<\tilde{\gamma}_2(t)<\dots<\tilde{\gamma}_p(t)$ be the nondegenerate zeros of $w(\cdot,t)$ in I, with $t\in(t^*,s_1]$. Then $x=\tilde{\gamma}_i(t)$ $(i=1,\dots,p)$ are smooth curves, and $\tilde{z}_i:=\lim_{t\to t^*}\tilde{\gamma}_i(t)$ exists for each $i\in\{1,\dots,p\}$, for otherwise $w(\cdot,t^*)$ would be identically zero over some interval of x, contradicting to what is known about $w(\cdot,t^*)$. Furthermore, $\tilde{z}_i<\tilde{z}_{i+1}$ for $i\in\{1,\dots,p-1\}$, since otherwise, we may apply the maximum principle over the region $\tilde{A}_i:=\{(x,t):\tilde{\gamma}_i(t)<x<\tilde{\gamma}_{i+1}(t),\ t^*\leqslant t\leqslant s_1\}$ to deduce that $w\equiv 0$ in $\tilde{A}_i$. Finally from (15) we know that none of these curves can connect to the point $(0,t^*)$. Thus $\tilde{z}_1<\tilde{z}_2<\dots<\tilde{z}_p$

are different zeros of $w(\cdot, t^*)$ in $I \setminus \{0\}$. It follows that $n - 1 \geqslant p$, that is, $\mathcal{Z}_I(w(\cdot, t^*)) - 1 \geqslant \mathcal{Z}_I(w(\cdot, t))$ for $t \in (t^*, s_1]$. This proves (14), and the lemma is proved. □

Remark 3.3. This lemma can be easily extended to a solution defined in a variable domain. More precisely, if we replace l in Lemma 3.2 by a continuous and positive function $\xi(t)$, then the conclusions remain valid for $\mathcal{Z}_{I(t)}(w(\cdot, t))$: the number of zeros of $w(\cdot, t)$ in $I(t) := [0, \xi(t)]$. In fact, for any given $\tau \in (t_1, t_2)$, we can find $\epsilon > 0$ and $\delta > 0$ small such that $w(x, t) \neq 0$ for $t \in J_\tau := [\tau - \delta, \tau + \delta] \subset (t_1, t_2)$ and $x \in [\xi(\tau) - \epsilon, \xi(t)]$. Therefore $\mathcal{Z}_{I(t)}(w(\cdot, t)) = \mathcal{Z}_{[0, \xi(\tau)-\epsilon]}(w(\cdot, t))$ for $t \in J_\tau$. Since the conclusions of Lemma 3.2 hold for $(x, t) \in [0, \xi(\tau) - \epsilon] \times J_\tau$, they also hold for $(x, t) \in [0, \xi(t)] \times J_\tau$. Finally, since any compact subinterval of (t_1, t_2) can be covered by finitely many such J_τ, we see that $\mathcal{Z}_{I(t)}[w(\cdot, t)]$ has the required properties in (t_1, t_2).

Using the above lemma and remark we can prove the following result.

Lemma 3.4. *Let (u, h) be a time-global solution of (1) with $u_0 \in \mathscr{X}(h_0)$. Then $\mathcal{Z}_{[0,h(t)]}(u_x(\cdot, t)) < \infty$ for all $t > 0$. Moreover, there exists $T > 0$ such that, for some integer $N > 0$, $u(\cdot, t)$ has exactly N local maximum points $\xi_i(t)$ and $N - 1$ local minimum points $\tilde{\xi}_i(t)$ in $(0, h(t))$ with*

$$0 < \xi_1(t) < \tilde{\xi}_2(t) < \xi_2(t) < \cdots < \xi_N(t) < h(t),$$

$u_{xx}(\xi_i(t), t) < 0$ $(i = 1, 2, \dots, N)$, $u_{xx}(\tilde{\xi}_i(t), t) > 0$ $(i = 12, \dots, N - 1)$, and $\xi_i(t)$, $\tilde{\xi}_i(t) \in C^1$ for each i.

Proof. By the maximum principle we have $u(x, t) > 0$ for all $t > 0$ and $x \in (0, h(t))$. Then by the Robin boundary condition at $x = 0$ and Hopf lemma we have $u_x(0, t) > 0$ and $u_x(h(t), t) < 0$ for all $t > 0$. Now applying Lemma 3.2 and Remark 3.3 to the function u_x we see that when t is large, for example, $t > T$ for some large $T > 0$, $u(\cdot, t)$ has exactly N local maximum points $\{\xi_i(t)\}_{i=1}^N$ and $N - 1$ local minimum points $\{\tilde{\xi}_i(t)\}_{i=1}^{N-1}$ in $(0, h(t))$, where N is a positive integer. In addition, these points are nondegenerate zeros of $u_x(\cdot, t)$ when $t > T$. This implies that $u_{xx}(\xi_i(t), t) < 0 < u_{xx}(\tilde{\xi}_i(t), t)$. Finally, the C^1 regularity of $\xi_i(\cdot)$ and $\tilde{\xi}_i(\cdot)$ follows from implicit function theorem for the algebraic equations $u_x(\xi_i(t), t) = 0$ and $u_x(\tilde{\xi}_i(t), t) = 0$. □

4. A trichotomy result on the long time behavior of solutions

4.1. The $C^2_{\mathrm{loc}}([0, h_\infty))$ convergence

We first give a locally uniform convergence result.

Lemma 4.1. *Assume* **(F)** *and $u_0 \in \mathscr{X}(h_0)$ for some $h_0 > 0$. Let (u, h) be the solution of (1). Then u converges to a solution v of (2) in $C^2_{\mathrm{loc}}([0, h_\infty))$:*

$$\lim_{t\to\infty} \|u(\cdot, t) - v(\cdot)\|_{C^2([0,M])} = 0 \quad \forall\, 0 < M < h_\infty, \tag{16}$$

where v is one of the following types: Trivial Solution 0, Active States and Ground States. Moreover, $h_\infty < \infty$ only happens when $v \equiv 0$.

Proof. Denote by $\omega(u)$ the ω-limit set of $u(\cdot,t)$ in the topology of $L^\infty_{loc}([0,h_\infty))$. By local parabolic estimates, the definition of $\omega(u)$ remains unchanged if the topology of $L^\infty_{loc}([0,h_\infty))$ is replaced by that of $C^2_{loc}([0,h_\infty))$. $\omega(u)$ is not empty since u is bounded. Following the idea of Du and Matano [15] and Du and Lou [13], one can prove that $\omega(u)$ consists of exactly one of the solutions of $v''+f(v)=0$ in $[0,h_\infty)$.

When $h_\infty<\infty$ we claim that $v\equiv 0$. In fact, in a similar way as in [12,13] one can show $\|h'\|_{C^{\nu/2}((1,\infty))}\leqslant C$ for some $\nu\in(0,1)$ and some C independent of t, and $\|u(\cdot,t)-v(\cdot)\|_{C^{1+\nu}([0,h(t)])}\to 0$ as $t\to\infty$. Therefore $v(h_\infty)=0$, and so $v'(h_\infty)<0$ by the Hopf lemma if v is a positive stationary solution with compact support $[0,h_\infty]$. In this case we have $h'(t)=-u_x(h(t),t)>-\frac{1}{2}v_x(h_\infty)>0$ for large t. This contradicts the assumption $h(t)\to h_\infty<\infty$. Therefore, $h_\infty<\infty$ might happen only in the case when $v\equiv 0$.

When $h_\infty=\infty$, by Lemma 3.4 we know that $\mathcal{Z}_{[0,h(t)]}(u_x(\cdot,t))$ is finite and nonincreasing in t, so v cannot be a Positive Periodic Solution. The lemma is then proved since we have excluded the Positive Periodic Solutions and positive solutions with compact supports. □

In the formula (16), if v is an Active State, then we say **spreading** happens for u; if $v=0$ we say **vanishing** happens for u.

4.2. *Sufficient conditions for spreading*

In order to prove the trichotomy result in Theorem 2.1, we need some sufficient conditions for spreading. The first one is that the initial data is not small in a sufficiently wide interval.

Lemma 4.2. *Let u be the solution of (1). Then spreading happens if u_0 satisfies one of the following conditions:*

(1) *for some $m\in(\theta,1)$, v_m in (8), and some $r\geqslant 0$, $u_0(\cdot)\geqslant v_m(\cdot-r)$ on $[r,r+2L_m]$, where L_m is defined by (9);*
(2) *for some $m\in(\alpha,1]$ and $r\geqslant 0$, $u_0(\cdot)\geqslant m$ on $[r,r+2L(m)]$, where $L(m)$ is a certain positive function of $m\in(\alpha,1]$.*

Proof. (1). When $u_0(\cdot)\geqslant v_m(\cdot-r)$, the comparison principle implies that $u(\cdot,t)\geqslant v_m(\cdot-r)$ on $[r,r+2L_m]$ for each $t>0$. Since only Active States are possible solutions of (2) bigger than $v_m(\cdot-r)$, the conclusion follows from Lemma 4.1 immediately.

(2a). First we consider the case $m\in(\theta,1]$. Choose $L(m)=L_m$ defined in (9). If $u_0\geqslant m$ on $[r,r+2L(m)]$, then $u_0(\cdot)\geqslant v_m(\cdot-r)$, and so spreading happens by (1).

(2b). Next consider $m\in(\alpha,\theta]$. Let $\eta(t)$ be the solution of

$$\eta_t=f(\eta)\quad\text{on }\ [0,\infty),\qquad \eta(0)=m.$$

Since $f(\cdot)>0$ in $(\alpha,1)$, with $\varepsilon=\frac{1-\theta}{3}$ and $T=\int_m^{\theta+2\varepsilon}\frac{ds}{f(s)}$ we have $\eta(T)=\theta+2\varepsilon$.

We fix $R=L_{\theta+\varepsilon}$. Let $L\gg R$ be a constant to be determined and w_0 be a function satisfying

$$w_0(x)=m\text{ when }|x|<L-1,\ w_0(\pm L)=0,\ xw_0'(x)\leqslant 0\text{ when }L-1\leqslant|x|\leqslant L.$$

Let $w(x,t)$ be the solution of the problem

$$\begin{cases} w_t = w_{xx} + f(w) & \forall x \in [-L, L], t > 0, \\ w(\pm L, t) = 0 & \forall t > 0, \\ w(x, 0) = w_0(x) & \forall x \in [-L, L]. \end{cases}$$

Let $\rho(x) = (1 + x^2)^{-1}$. Then $\zeta(x,t) := \rho(x)[w(x,t) - \eta(t)]$ satisfies $\zeta_t = \zeta_{xx} + 4x\rho\zeta_x + [2\rho + f']\zeta$. Hence, with $Q := 2 + \max_{0 \leqslant s \leqslant 1} f'(s)$ we can derive that

$$\max_{|x| \leqslant L} \{\rho(x)|w(x,t) - \eta(t)|\} \leqslant e^{Qt} \cdot \max_{|x| \leqslant L} \{\rho(x)|w_0(x) - m|\} \leqslant \frac{e^{Qt}}{1 + (L-1)^2}.$$

Taking $L = L(m) = 1 + \sqrt{(1 + R^2)e^{QT}/\varepsilon - 1}$ we have, when $|x| \leqslant R$,

$$|w(x,T) - \eta(T)| \leqslant \frac{1}{\rho(x)} \frac{e^{QT}}{1 + (L-1)^2} \leqslant \frac{(1 + R^2)e^{QT}}{1 + (L-1)^2} = \varepsilon.$$

Thus, $w(\cdot, T) \geqslant \eta(T) - \varepsilon = \theta + \varepsilon$ on $[-R, R]$.

Now if $u_0 \geqslant m$ on some interval $[r, r + 2L]$ for $r \geqslant 0$, we have $u_0(x + r + L) \geqslant w_0(x)$ for all $x \in [-L, L]$, so by comparison,

$$u(x + r + L, t) \geqslant w(x, t), \quad x \in [-L, L], \ t > 0.$$

In particular, for $t = T$ and $x \in [-R, R]$ we have $u(x + r + L, T) > \theta + \varepsilon$. The assertion of the lemma then follows from the earlier case (2a) with $m = \theta + \varepsilon$. □

We remark that the sufficient condition (2) originates from Fife and McLeod [20]. The introduction of the function ρ is due to Feireisl and Polàčik [18].

Our second sufficient condition ensuring spreading is that the initial datum is sufficiently large on any given interval.

Lemma 4.3. *Assume* **(F)** *and $\phi \in \mathscr{X}(h_0)$ for some $h_0 > 0$. Let u be the solution of* (1) *with initial datum $u_0 = \sigma\phi$. Then spreading happens when σ is sufficiently large.*

Proof. Since $\phi \geqslant, \not\equiv 0$, there exist a small $\delta > 0$ and an interval $I \subset [0, h_0]$ such that $\phi(x) > \delta$ in I. Without loss of generality, we assume $I := [0, h_1]$ for some $h_1 < h_0$. We will construct a suitable lower solution to prove the lemma. Consider the eigenvalue problem

$$(e^{\frac{x^2}{4}}\varphi')' + \lambda e^{\frac{x^2}{4}}\varphi = 0 \text{ for } x \in (0, 1), \qquad \varphi(0) = \varphi(1) = 0. \tag{17}$$

By the Sturm–Liouville theory, this problem has the first eigenvalue λ_1 with the first eigenfunction $\varphi_1(x) > 0$ $(x \in (0, 1))$. For any fixed $L_m > L_*$, where L_* is the constant defined by (10), problem (8) has at least one solution v_m defined on $[0, 2L_m]$. Choose positive constants ε, T, Λ as follows:

$$0 < \varepsilon < \min\{1, h_1^2\}, \quad T := 4L_m^2, \quad \Lambda := |\lambda_1| + K(T + 1),$$

where $K := -\inf_{s>0} f(s)/s < \infty$ by **(F)**, and choose $\rho > 0$ large such that

$$\frac{\rho}{(T+\varepsilon)^{\Lambda}}\,\varphi_1\left(\frac{x}{\sqrt{T+\varepsilon}}\right) > v_m(x) \text{ in } (0, 2L_m), \quad 1+\frac{2\rho}{(T+1)^{\Lambda}}\varphi_1'(1) < 0. \tag{18}$$

Define

$$w(x,t) := \frac{\rho}{(t+\varepsilon)^{\Lambda}}\,\varphi_1\left(\frac{x}{\sqrt{t+\varepsilon}}\right), \quad k(t) := \sqrt{t+\varepsilon}.$$

We claim that $(w(x,t), k(t))$ is a lower solution of (1) over $x \in (0, k(t))$ and $t \in [0, T]$. In fact,

$$w_t - w_{xx} - f(w) \leqslant \frac{-\rho}{(t+\varepsilon)^{\Lambda+1}}\left[\varphi_1'' + \frac{x}{2\sqrt{t+\varepsilon}}\varphi_1' + (\Lambda - K(t+\varepsilon))\varphi_1\right] \leqslant 0.$$

By (18), we have

$$k'(t) + w_x(k(t), t) \leqslant \frac{1}{2\sqrt{t+\varepsilon}}\left(1+\frac{2\rho}{(T+1)^{\Lambda}}\varphi_1'(1)\right) < 0.$$

Finally, we can choose $\sigma > 0$ sufficiently large such that

$$w(x,0) = \frac{\rho}{\varepsilon^{\Lambda}}\varphi_1\left(\frac{x}{\sqrt{\varepsilon}}\right) < \sigma\phi(x) \quad \text{for } x \in [0, \sqrt{\varepsilon}] \subset [0, h_1]. \tag{19}$$

Hence $(w(x,t), k(t))$ is a lower solution of (1) over the time interval $[0, T]$ and so we have

$$u(x, T; \sigma\phi) \geqslant w(x, T) > v_m(x) \text{ for } x \in (0, 2L_m).$$

Therefore spreading happens for $u(x,t;\sigma\phi)$ by the previous lemma. □

Remark 4.4. We remark that the condition $\inf_{s>0} f(s)/s > -\infty$ in **(F)** is only used in the above proof to ensure that spreading happens for large initial data. Without this condition, for any $\phi \in \mathscr{X}(h_0)$ with small support $[0, h_0]$, vanishing may happen for all $u(x,t;\sigma\phi)$, no matter how large σ is (cf. [13, Proposition 5.8]).

Remark 4.5. In [15] and some other references, the authors used more general monotone families of initial data, rather than just the rays $\{\sigma\phi \mid \sigma \geq 0\}$. We remark that our conclusions in the above lemma and in Theorem 2.1 remain valid for such initial data, for example, for the family of initial data $\{\phi_\sigma\}_{\sigma\geqslant 0}$ satisfying

(1) $\phi_\sigma \in \mathscr{X}(h_\sigma)$ for some $h_\sigma > 0$, the map $\sigma \mapsto \phi_\sigma$ is continuous from $[0, \infty)$ to $C([0, \infty))$;
(2) if $0 \leqslant \sigma_1 < \sigma_2$, then $h_{\sigma_1} \leqslant h_{\sigma_2}$, $\phi_{\sigma_1} \leqslant \phi_{\sigma_2}$ and $\phi_{\sigma_1} \not\equiv \phi_{\sigma_2}$;
(3) $\phi_\sigma(x) \to 0$ as $\sigma \to 0$, uniformly in $x \in [0, \infty)$;
(4) there exist $a \geqslant 0$, $\epsilon > 0$ such that, $[a, a+\epsilon] \subset [0, h_\sigma]$ for all large σ and $\phi_\sigma(x) \to \infty$ as $\sigma \to \infty$, uniformly in $x \in [a, a+\epsilon]$.

The last condition (4) is just used to ensure the spreading phenomena for the solutions starting from ϕ_σ for large σ (see (19) in the previous proof).

4.3. Sufficient conditions for vanishing

This part gives a sufficient condition for vanishing.

Lemma 4.6. *Let $u_0 \in \mathscr{X}(h_0)$ for some $h_0 > 0$. If $\|u_0\|_{C([0,h_0])} < \alpha$, then $h_\infty < \infty$ and $\lim_{t\to\infty} \|u(\cdot,t)\|_{C^2([0,h(t)])} = 0$.*

Proof. Note that $\|u_0\|_{C([0,h_0])} \leqslant \alpha - 2\varepsilon$ for some $\varepsilon > 0$. There exists a constant $\beta > 0$ such that $f(s) \leqslant -\beta s$ for $0 \leqslant s \leqslant \alpha - 2\varepsilon$. By comparison we have

$$u(x,t) \leqslant (\alpha - 2\varepsilon)e^{-\beta t}, \quad t \geqslant 0.$$

Choose $\rho := \max\left\{2h_0, \sqrt{\frac{\alpha\pi}{2\beta}}, \frac{\pi h_0}{2}\left(\arccos\frac{\alpha-2\varepsilon}{\alpha-\varepsilon}\right)^{-1}\right\}$ and define

$$k(t) := \rho(2 - e^{-\beta t}) \quad \text{and} \quad w(x,t) := (\alpha - \varepsilon)e^{-\beta t}\cos\frac{\pi x}{2k(t)},\ 0 \leqslant x \leqslant k(t),\ t \geqslant 0.$$

It is easy to verify that

$$\begin{cases} w_t - w_{xx} - f(w) \geqslant 0, & 0 < x < k(t),\ t > 0, \\ w(0,t) \geqslant u(0,t),\ w(k(t),t) = 0, & t > 0, \\ k'(t) \geqslant -w_x(k(t),t), & t > 0, \\ k(0) > h_0,\ w(x,0) \geqslant \alpha - 2\varepsilon \geqslant u(x,0) \text{ on } [0,h_0]. \end{cases}$$

Hence (w,k) is an upper solution of (1) (see the definition in [13] for instance), and so

$$h_\infty = \lim_{t\to\infty} h(t) \leqslant \lim_{t\to\infty} k(t) = 2\rho < \infty,$$

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{C([0,h(t)])} \leqslant \lim_{t\to\infty} \|w(\cdot,t)\|_{C([0,h(t)])} = 0.$$

Finally, by standard parabolic theory we have $\lim_{t\to\infty} \|u(\cdot,t)\|_{C^2([0,h(t)])} = 0$ (cf. [12, 13]). □

4.4. A trichotomy result

We now give a vanishing–transition–spreading trichotomy result on the long time behavior of solutions.

Theorem 4.7. *Assume* **(F)** *and $\phi \in \mathscr{X}(h_0)$ for some $h_0 > 0$. Let u be the solution of (1) with $u_0 = \sigma\phi$. Then there exist $\sigma_*, \sigma^* \in (0,\infty)$ with $\sigma_* \leqslant \sigma^*$ such that the following trichotomy holds:*

(i) *if $\sigma > \sigma^*$,* **spreading** *happens:*

$$h_\infty = \infty \quad \textit{and} \quad \lim_{t\to\infty} \|u(\cdot,t) - v_*\|_{C^2([0,M])} = 0$$

for each $M > 0$, and for some Active State v_*;

(ii) *if* $\sigma \in [0, \sigma_*)$, **vanishing** *happens:*

$$h_\infty < \infty \quad \text{and} \quad \lim_{t\to\infty} \|u(\cdot,t)\|_{C^2([0,h(t)])} = 0;$$

(iii) *in the* **transition** *case when* $\sigma \in [\sigma_*, \sigma^*]$, $h_\infty = \infty$, *and either*

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{C^2([0,M])} = 0 \text{ for each } M > 0,$$
$$\|u(\cdot,t)\|_{L^\infty([0,h(t)])} > \alpha \text{ for any } t > 0, \tag{20}$$

or, for some $z \in \mathbf{Z}$,

$$\lim_{t\to\infty} \|u(\cdot,t) - V(\cdot - z)\|_{C^2([0,M])} = 0 \text{ for each } M > 0. \tag{21}$$

Proof. Denote the solution of (1) by $u(x,t;u_0)$ and define

$$\Sigma_0 := \left\{\sigma \geqslant 0 \,\Big|\, \lim_{t\to\infty} \|u(\cdot,t;\sigma\phi)\|_{L^\infty([0,h(t)])} = 0\right\},$$
$$\Sigma_1 := \left\{\sigma \geqslant 0 \,\Big|\, \lim_{t\to\infty} \|u(\cdot,t;\sigma\phi) - v_*\|_{L^\infty([0,M])} = 0,\ \forall M > 0\right\},$$
$$\sigma_* := \sup\{\sigma \mid \sigma \in \Sigma_0\}, \quad \sigma^* := \inf\{\sigma \mid \sigma \in \Sigma_1\}, \tag{22}$$

where v_* is some Active State. We remark that by parabolic estimates, the $L^\infty([0,h(t)])$ norm in the definition of Σ_0 can be replaced by the $C^2([0,h(t)])$ norm, and $L^\infty([0,M])$ in the definition of Σ_1 can be replaced by $C^2([0,M])$.

If $\sigma \in \Sigma_0$, then $[0,\sigma] \subset \Sigma_0$ by comparison principle, and $\|u(\cdot,T;\sigma\phi)\|_{L^\infty} < \alpha$ for some large $T > 0$. By continuity, $\|u(\cdot,T;\hat\sigma\phi)\|_{L^\infty} < \alpha$ for every $\hat\sigma > \sigma$ with $\hat\sigma - \sigma \ll 1$. This implies $\hat\sigma \in \Sigma_0$ by Lemma 4.6. Hence, Σ_0 is open. As $0 \in \Sigma_0$, we see that $\Sigma_0 = [0,\sigma_*)$ for some $\sigma_* > 0$. On the other hand, Σ_1 is not empty by Lemma 4.3. For any $\sigma \in \Sigma_1$, since $u(\cdot,t;\sigma\phi)$ converges to an Active state v_* with $v_*(\infty) = 1$, there exists $r > 0$ large such that $u(\cdot,T_1;\sigma\phi) > v_m(\cdot - r)$ in $[r, r+2L_m]$ for some large time T_1, where v_m is the compactly supported solution of (8). By continuity, $u(\cdot,T_1;(\sigma-\varepsilon)\phi) > v_m(\cdot - r)$ in $[r, r+2L_m]$ provided $\varepsilon > 0$ is sufficiently small. Hence Σ_1 is an open set. By comparison we have $\Sigma_1 = (\sigma^*, \infty)$ for some $\sigma^* \geqslant \sigma_*$. Therefore, $[\sigma_*, \sigma^*] = [0,\infty)\backslash(\Sigma_0 \cup \Sigma_1)$ is not empty.

For each $\sigma \in [\sigma_*, \sigma^*]$, by Lemma 4.1, either (21) holds, or

$$\|u(\cdot,t)\|_{L^\infty([0,M])} \to 0 \text{ for each } M > 0 \text{ but } \|u(\cdot,t)\|_{L^\infty([0,h(t)])} \not\to 0, \quad \text{as } t \to \infty.$$

In the latter case we must have the second inequality in (20). Indeed, if $\|u(\cdot,t)\|_{L^\infty([0,h(t)])} \leqslant \alpha$ for some $t > 0$, then by comparison principle, $\|u(\cdot,t)\|_{L^\infty([0,h(t)])} < \alpha$ for all larger t. Then vanishing happens by Lemma 4.6, a contradiction.

This proves the theorem. □

Remark 4.8. As we have seen in Subsection 3.1, problem (2) may have more than one Active State. So it is natural to ask: which Active State is the ω-limit of a given solution $u(\cdot,t;\sigma\phi)$ when spreading happens? This question remains open now.

5. Transition solutions

In the rest of the paper we call u a **transition solution** when (20) or (21) holds. In this section we always assume that $u(x, t; u_0)$ is such a solution.

5.1. Uniqueness of the local maximum point

To give a uniform convergence for the transition solutions we need the following uniqueness of the maximum point.

Lemma 5.1. *Let $\{\xi_i(t)\}_{i=1}^{N}$ be the local maximum points in Lemma 3.4. Then $N = 1$, that is, $u(\cdot, t)$ indeed has a unique local maximum point for $t > T$.*

Proof. We prove the lemma for two cases: (i) $\xi_N(t) \leqslant M - 1 < \infty$ for some $M > 1$ and (ii) $\sup_{t>0} \xi_N(t) = \infty$.

It is clear that, in case (i), the limit (21) rather than (20) in Theorem 5.6(iii) holds. Since $V(\cdot - z)$ has a unique maximum point $x = z$ with $V''(x - z)|_{x=z} < 0$, the C^2 convergence in (21) implies that, for sufficiently large t, $u(\cdot, t)$ has a unique maximum point in $[0, M]$, and so in $[0, \infty)$. This is indeed true for $t > T$ since we have assumed in Lemma 3.4 that the number N of the local maximum points of $u(\cdot, t)$ does not change when $t > T$.

Now we consider the case (ii) and prove the conclusion by contradiction. For this purpose we assume $n \geqslant 2$ and $\xi_1(t) < \xi_N(t)$ for all $t > T$, where T is the time in Lemma 3.4. Since $u(x, T) > 0$ in $x \in (0, h(T))$ and $u(x, T)$ is strictly decreasing in $(\xi_N(T), h(T))$, there exists $L \in (\xi_N(T), h(T))$ such that

$$2L - h(T) > \xi_1(T) \quad \text{and} \quad u(L, T) < u(x, T) \quad \forall x \in [\xi_1(T), L).$$

Define

$$T_1 := \inf\{t > T \mid \xi_N(t) = L\} \in (T, \infty).$$

Then $\xi_1(t) < \xi_N(t) \leqslant \xi_N(T_1) = L$ for all $t \in [T, T_1]$. By continuity of $\xi_i(t)$ we have $\xi_1(t) < L$ and so $2L - \xi_1(t) > \xi_1(t)$ for $t \in [T, T_1 + \varepsilon]$ provided $\varepsilon > 0$ is sufficiently small.

Set $\eta(t) := \max\{\xi_1(t), 2L - h(t)\}$, $I(t) := [\eta(t), 2L - \eta(t)]$ for $t \in [T, T_1 + \varepsilon]$ and define

$$\zeta(x, t) := u(x, t) - u(2L - x, t) \ \text{ on } \ I(t) \times [T, T_1 + \varepsilon].$$

We claim that $\zeta(\eta(t), t) > 0$ for $t \in [T, T_1]$. In fact, when $\xi_1(t) \leqslant 2L - h(t)$ (this is in particular true when $0 < t - T \ll 1$), we have $\eta(t) = 2L - h(t)$ and so

$$\zeta(\eta(t), t) = u(2L - h(t), t) - u(h(t), t) = u(2L - h(t), t) > 0. \tag{23}$$

When $2L - h(t) < \xi_1(t)$, $\eta(t) = \xi_1(t)$. Since $\xi_1(t)$ is a local maximum point of $u(\cdot, t)$, we have $u(\xi_1(t), t) > u(\xi_1(t) + \delta, t)$ for sufficiently small $\delta > 0$. On the other hand, $u(2L - \eta(t) - \delta, t) > u(2L - \eta(t), t)$ since $u(\cdot, t)$ with $t \in [T, T_1]$ is strictly decreasing in $[L, h(t)]$. Therefore

$$\zeta(\eta(t), t) > \zeta(\eta(t) + \delta, t) \quad \forall t \in [T, T_1]. \tag{24}$$

Since $\zeta(\cdot, t)$ is antisymmetric around $x = L$ on $I(t)$ and

$$\zeta(x, T) > 0 = \zeta(L, T) \quad \forall x \in [\eta(T), L).$$

By the maximum principle we have $\zeta(x, t) > 0$ in $x \in [\eta(t), L)$ as long as $\zeta(\eta(t), t) > 0$. Combining (23) with (24) we have $\zeta(\eta(t), t) > 0$ for $t \in [T, T_1]$. By continuity this is true even for $t \in [T, T_1 + \varepsilon]$ provided $\varepsilon > 0$ is small. Consequently, $\mathcal{Z}_{I(t)}(\zeta(\cdot, t)) = 1$ for all $t \in [T, T_1 + \varepsilon]$. By Lemma 3.2, such a result contradicts the fact $x = L$ is a degenerate zero of $\zeta(\cdot, T_1)$:

$$\zeta(L, T_1) = \zeta_x(L, T_1) = 2u_x(L, T_1) = 2u_x(\xi_N(T_1), T_1) = 0.$$

This contradiction proves the lemma. □

5.2. *Convergence of transition solutions in $H^2([0, h(t)])$ norm*

In order to understand further properties of transition solutions, besides the $C^2_{loc}([0, \infty))$ convergence in Theorem 4.7 we study the convergence of $u - V$ in $H^2([0, h(t)])$ norm by using concentrated compactness argument as Fašangová and Feireisl [17] and Poul [30] did. Our approach is similar as that in [11] for problems on the half line, but still with some difference.

5.2.1. *Energy estimates*

For any $\psi \in H^1([0, h(t)])$, define

$$\mathbf{E}[\psi] := \int_0^{h(t)} [\psi_x^2 + F(\psi)]dx + b\psi_x^2(0),$$

where $b \in [0, \infty)$. Let $u(x, t)$ be the solution of (1). A direct calculation shows that

$$\frac{d}{dt}\mathbf{E}[u(\cdot, t)] = -2\|u_t(\cdot, t)\|^2_{L^2([0,h(t)])} - (h'(t))^3. \tag{25}$$

Lemma 5.2. *$-\mathbf{E}[u(\cdot, t)]$, $\|u(\cdot, t)\|_{H^1([0,h(t)])}$, $\|f(u)\|_{L^2([0,h(t)])} \leqslant C$ for some $C > 0$ independent of t.*

Proof. Since u is a transition solution, it has upper bound $C_1 > 0$. For any $m \in (\alpha, \theta)$, Lemma 4.2 implies that

$$J_m(t) := \{0 < x < h(t) : u(x, t) \geqslant m\}$$

is a bounded domain, that is, $\text{mes}(J_m(t)) \leqslant M$ for some $M > 0$. It then follows that $\int_0^{h(t)} F(u)\text{d}x$ is bounded from below by $-C$, and so is $\mathbf{E}[u(\cdot, t)]$.

For $x \notin J_m(t)$ we have $0 \leqslant u \leqslant m$ and so $F(u) \geqslant \varepsilon u^2$ for some $0 < \varepsilon < 1$. Hence

$$\int_0^{h(t)} F(u)\mathrm{d}x = \int_{J_m(t)} F(u)\mathrm{d}x + \int_{[0,h(t)]\setminus J_m(t)} F(u)\mathrm{d}x$$
$$\geqslant -C + \varepsilon \int_{[0,h(t)]\setminus J_m(t)} u^2\mathrm{d}x \geqslant -C + \varepsilon\|u(t,\cdot)\|^2_{L^2([0,h(t)])} - \varepsilon C_1^2 M.$$

Therefore,

$$\mathbf{E}[u(\cdot,0)] \geqslant \mathbf{E}[u(\cdot,t)] \geqslant \varepsilon\|u(\cdot,t)\|^2_{H^1([0,h(t)])} - C - \varepsilon C_1^2 M.$$

This gives the bound of $\|u(t,\cdot)\|_{H^1([0,h(t)])}$. Finally, $\|f(u)\|_{L^2} \leqslant \|K_1 u\|_{L^2} \leqslant C$, where $K_1 := \sup_{0\leqslant s\leqslant C_1} |f'(s)|$. □

Lemma 5.3. $\|u_t(\cdot,t)\|_{L^2([0,h(t)])} \to 0$ *and* $h'(t) \to 0$ *as* $t \to \infty$.

Proof. Integrating (25) on $[0,\infty)$ and using the results in Lemma 5.2 we have

$$\int_0^{\infty} \left[2\|u_t(\cdot,t)\|^2_{L^2([0,h(t)])} + (h'(t))^3\right] ds = \mathbf{E}[u(\cdot,0)] - \mathbf{E}[u(\cdot,\infty)] < \infty.$$

Then the lemma is proved by the fact that both $\|u_t(\cdot,t)\|^2_{L^2([0,h(t)])}$ and $(h'(t))^3$ are uniformly continuous for $t > 1$ (cf. [12,13,26]). □

Lemma 5.4. $\|u(\cdot,t)\|_{H^2([0,h(t)])} \leqslant C$ *for some* C *independent of* t.

Proof. From the equation $u_t = u_{xx} + f(u)$ we see that

$$\|u_{xx}\|^2_{L^2} \leqslant 2\left(\|u_t\|^2_{L^2} + \|f(u)\|^2_{L^2}\right).$$

Hence the conclusion follows from the above two lemmas. □

5.2.2. Convergence in moving coordinates

By Lemma 5.1, for any $t > T$, $u(\cdot,t)$ has a unique local maximum point $\xi(t)$. In the transition case we have $u(\xi(t),t) > \alpha$ for all $t > T$.

Suppose (20) holds, then we must have $\lim_{t\to\infty}\xi(t) = \infty$. Set

$$y := x - \xi(t) \quad \text{and} \quad w(y,t) := u(y+\xi(t),t), \quad t > T.$$

Then w solves

$$w_t - \xi'(t)w_y = w_{yy} + f(w), \quad -\xi(t) < y < h(t) - \xi(t),\ t > T.$$

For any $M > 0$, there exists $T_M > 0$ such that when $t > T_M$, $-\xi(t) < -M$ and so $w(\cdot,t)$ is well defined in $[-M,0]$. Lemma 5.4 implies that w is bounded in $H^2([-M,0])$. Since $H^2([-M,0])$

is a Hilbert space, $\{w(\cdot,t) : t > T_M\}$ is weakly relatively compact in $H^2([-M,0])$. Therefore, there exist a sequence $\{t_n\}$ with $t_n \to \infty$ and a function $w_M \in H^2([-M,0])$ such that $w(y,t_n)$ converges weakly to w_M in $H^2([-M,0])$. By Lemma 5.3 we have

$$\begin{aligned}\|w_{yy}+f(w)\|_{L^2([-\xi(t),h(t)-\xi(t)])} &= \|w_t(\cdot,t)-\xi'(t)w_y(\cdot,t)\|_{L^2([-\xi(t),h(t)-\xi(t)])}\\ &= \|u_t(\cdot,t)\|_{L^2([0,h(t)])} \to 0 \quad \text{as } t\to\infty.\end{aligned}$$

Hence $\{w_{yy}(\cdot,t_n)\}$ is a convergence sequence in $L^2([-M,0])$ (note that $f(w(y,t_n)) \to f(w_M(y))$ in $L^2([-M,0])$). Therefore, $w(y,t_n)$ converges to w_M in $H^2([-M,0])$, which means that $w_M \in H^2([-M,0])$ ($\|w_M\|_{H^2([-M,0])} \leqslant C$ by Lemma 5.4) and

$$\begin{cases} w_M''(y)+f(w_M(y))=0 \ \text{ in } L^2([-M,0]),\\ w_M'(y)\geqslant 0 \text{ in } [-M,0], \quad w_M(0)\geqslant\alpha, \quad w_M'(0)=0.\end{cases}$$

Since $H^2([-M,0])$ is embedded into $C^{1+\nu}([-M,0])$ for any $\nu \in (0,\frac{1}{2})$, we have $w_M \in C^{1+\nu}([-M,0])$, combining with the equation of w_M we indeed have $w_M \in C^{3+\nu}([-M,0])$.

Using Cantor's diagonal argument, we see that $w(y,t_n)$ (taking a subsequence if necessary) converges to w_∞ in $H^2([-M,0])$ for any $M>0$, where $w_\infty(y) \in H^2((-\infty,0]) \cap C^{2+\nu}((-\infty,0])$ with $\|w_\infty\|_{H^2((-\infty,0])} \leqslant C$ satisfies

$$w_\infty''+f(w_\infty)=0 \text{ and } w_\infty'\geqslant 0 \text{ in } (-\infty,0], \quad w_\infty'(0)=0, \quad w_\infty(0)\geqslant\alpha.$$

Therefore $w_\infty \equiv V$. Finally, by the uniqueness of V we have

$$\lim_{t\to\infty}\|w(\cdot,t)-V(\cdot)\|_{H^2([-M,0])}=0 \quad \forall M>0. \tag{26}$$

Lemma 5.5. *In the transition case, $h(t)-\xi(t)\to\infty$ as $t\to\infty$, and*

$$\lim_{t\to\infty}\|w(\cdot,t)-V(\cdot)\|_{H^2([-M,M])}=0 \quad \forall M>0. \tag{27}$$

Proof. Note that $u(x,t) \leqslant C_1$ for all $t>0$ and for some $C_1>0$. Following the proof of Lemma 2.2 in [12] we can construct an upper solution of the form

$$\tilde{u}(x,t) := C_1[2M(h(t)-x)-M^2(h(t)-x)^2]$$

for some suitable $M>0$, such that $u(x,t) \leqslant \tilde{u}(x,t)$ in

$$Q_M := \{(x,t) \mid h(t)-M^{-1} < x < h(t),\ t>0\}.$$

Assume $h(t)-\xi(t) \not\to \infty$, then there exist $M_0 \in [0,\infty)$ and a sequence $\{t_n\}$ with $t_n\to\infty$ such that

$$\lim_{n\to\infty}[h(t_n)-\xi(t_n)] = \liminf_{t\to\infty}[h(t)-\xi(t)] = M_0.$$

In the same way as proving (26) we get

$$\lim_{t\to\infty} \|w(\cdot,t) - V(\cdot)\|_{H^2([-M,M_0-\varepsilon])} = 0 \quad \forall \varepsilon \in (0,\varepsilon_1),\ \forall M > \varepsilon_1,$$

where $\varepsilon_1 := V(M_0)/(8MC_1)$. It is easily seen that

$$\tilde{u}(x,t_n) \leqslant \hat{u}(x,t_n) := C_1[2M(\xi(t_n) + M_0 + \varepsilon - x) - M^2(\xi(t_n) + M_0 + \varepsilon - x)^2]$$

since $h(t_n) < \xi(t_n) + M_0 + \varepsilon$ for large t_n. Now thanks to $u(x,t) \leqslant \tilde{u}(x,t)$ in Q_M we have

$$w(M_0 - \varepsilon, t_n) \leqslant \tilde{u}(M_0 - \varepsilon + \xi(t_n), t_n) \leqslant \hat{u}(M_0 - \varepsilon + \xi(t_n), t_n) \leqslant 4MC_1\varepsilon.$$

Taking limit as $n \to \infty$ we have

$$V(M_0 - \varepsilon) = \lim_{n\to\infty} w(M_0 - \varepsilon, t_n) \leqslant 4MC_1\varepsilon \leqslant \frac{V(M_0)}{2}.$$

This is impossible as $\varepsilon \to 0$. Thus we obtain $h(t) - \xi(t) \to \infty$ $(t \to \infty)$.

Finally (27) can be proved in the same way as proving (26). □

5.2.3. Convergence in $H^2([0,h(t)])$ norm

We first consider the case where (20) holds, and so (27) holds.

Let $\varepsilon_0 > 0$ be a number such that $f' < 0$ in $[0,\varepsilon_0]$. Fix an arbitrary $\varepsilon \in (0,\varepsilon_0)$. Let $z_\varepsilon > 0$ be the point such that $V(z_\varepsilon) = \varepsilon$. Set $J_\varepsilon(t) := \{x \geqslant 0 \mid u(x,t) \geqslant \varepsilon\}$. For $t > T$, by (27) and Lemma 5.1 we have $J_\varepsilon(t) = [a(t), b(t)]$ and $\lim_{t\to\infty}[\xi(t) - a(t)] = \lim_{t\to\infty}[b(t) - \xi(t)] = z_\varepsilon$. In addition,

$$\lim_{t\to\infty} \|u(\cdot,t) - V(\cdot - \xi(t))\|_{H^2(J_\varepsilon(t))} = \lim_{t\to\infty} \|w(\cdot,t) - V(\cdot)\|_{H^2([-z_\varepsilon,z_\varepsilon])} = 0.$$

Set $J_\varepsilon^c(t) := [0,h(t)]\backslash J_\varepsilon(t) = [0,a(t)) \cup (b(t),h(t)]$. By integrating $uu_t = uu_{xx} + uf(u)$ over $J_\varepsilon^c(t)$ we can derive $\|u\|^2_{H^1(J_\varepsilon^c(t))} = O(\varepsilon)$. Sending $\varepsilon \searrow 0$ we obtain $\|u(\cdot,t) - V(\cdot - \xi(t))\|_{H^1([0,h(t)])} \to 0$ as $t \to \infty$. Finally using $u_{xx} + f(u) = u_t \to 0$ in $L^2([0,h(t)])$ we derive

$$\lim_{t\to\infty} \Big\| u(\cdot,t) - V(\cdot - \xi(t)) \Big\|_{H^2([0,h(t)])} = 0. \tag{28}$$

When (21) instead of (20) holds, by a similar argument as above we can show that

$$\lim_{t\to\infty} \Big(\|u(\cdot,t) - V(\cdot - \xi(t))\|_{H^2([0,h(t)])} + |\xi(t) - z| \Big) = 0. \tag{29}$$

We summarize the above results as follows:

Theorem 5.6. *Let u be a transition solution. Then $h(t) - \xi(t) \to \infty$ as $t \to \infty$. Moreover, when (20) holds we have (28); when (21) holds we have (29) for some $z \in \mathbf{Z}$.*

5.3. Sharp threshold

For any $\phi \in \mathscr{X}(h_0)$ with $h_0 > 0$, denote by $\sigma^*(\phi)$ and $\sigma_*(\phi)$ the constants σ^* and σ_* defined in (22), respectively. Denote

$$u^*(x,t) := u(x,t;\sigma^*(\phi)\phi) \text{ in } [0,h^*(t)], \quad u_*(x,t) := u(x,t;\sigma_*(\phi)\phi) \text{ in } [0,h_*(t)].$$

Let $\xi^*(t)$ and $\xi_*(t)$ be the unique maximum point of $u^*(\cdot,t)$ and $u_*(\cdot,t)$ respectively for large t.

Theorem 5.7. *For each $\phi \in \mathscr{X}(h_0)$ with $h_0 > 0$, $\sigma_*(\phi) = \sigma^*(\phi)$.*

Proof. Assume by contradiction $\sigma^*(\phi) > \sigma_*(\phi)$. Then $[\sigma^*(\phi) - \sigma_*(\phi)]\phi \geqslant, \not\equiv 0$, and by the strong maximum principle we have

$$u^*(x,t) > u_*(x,t) \quad \forall x \in [0,h_*(t)],\ t > 0. \tag{30}$$

In particular, at $t = 1$, there exists $\varepsilon_1 > 0$ such that

$$u^*(x+\varepsilon,1) \geqslant u_*(x,1) \quad \forall x \in [0,h_*(1)],\ \forall \varepsilon \in [0,\varepsilon_1].$$

By the Robin boundary condition we have $\xi^*(t) > 0$ and by Theorem 5.6 we have $\liminf_{t\to\infty} \xi^*(t) > 0$. Hence, $\xi^*(t) > \varepsilon_2$ ($\forall t \geqslant 0$) for some $\varepsilon_2 > 0$, that is, $u_x^*(x,t) > 0$ in $[0,\varepsilon_2] \times [0,\infty)$. Consequently,

$$u^*(\varepsilon,t) \geqslant u^*(0,t) \geqslant u_*(0,t) \quad \forall t \geqslant 0,\ \varepsilon \in [0,\varepsilon_2].$$

Now set $\varepsilon_3 := \min\{\varepsilon_1,\varepsilon_2\}$. For every $\varepsilon \in [0,\varepsilon_3]$ we can compare $u^*(\cdot+\varepsilon,\cdot)$ and $u_*(\cdot,\cdot)$ to derive that

$$u^*(x+\varepsilon,t) \geqslant u_*(x,t) \quad \forall x \in [0,h_*(t)],\ t \geqslant 1.$$

Now applying (28) or (29) we find that

$$V(x+\varepsilon-\xi^*(t)) \geqslant V(x-\xi_*(t)) + o(1) \quad \forall x \geqslant 0,\ \varepsilon \in [0,\varepsilon_3]$$

where $o(1) \to 0$ as $t \to \infty$. In particular, evaluating the above inequality at $x = \xi_*(t)$ and take $t = T$ large enough such that $|o(1)| \leqslant V(0) - V(\varepsilon_3/4)$ we derive that

$$V(\eta+\varepsilon) \geqslant V\left(\frac{\varepsilon_3}{4}\right) = V\left(-\frac{\varepsilon_3}{4}\right) \quad \forall \varepsilon \in [0,\varepsilon_3]$$

where $\eta = \xi_*(T) - \xi^*(T)$ is a number independent of ε. Clearly, this is impossible since $V' < 0$ in $(0,\infty)$ and $V' > 0$ in $(-\infty,0)$. Thus, we must have $\sigma^*(\phi) = \sigma_*(\phi)$. □

In fact, besides the sharpness of the threshold value in transition case: $\sigma_*(\phi) = \sigma^*(\phi)$, one can show that $\sigma^*(\phi)$ is continuous in $\phi \in \cup_{h_0>0}\mathscr{X}(h_0)$ under $\|\cdot\|_{L^\infty}$ topology (cf. [11]).

5.4. Proof of Theorem 2.1

The vanishing–transition–spreading trichotomy and the sharp threshold value in transition case have been proved in Theorems 4.7, 5.6 and 5.7. To finish the proof of Theorem 2.1 we only need to specify the asymptotic behavior of $\xi(t)$ in transition case.

(1) It is easily seen from Subsection 3.1 that when $b\sqrt{F(s)} < s$ for all $s \in (0,\theta)$, $\mathbf{Z} = \emptyset$ and (2) has no Ground States. The conclusion in Theorem 2.1(iii) (1) then follows immediately.

(2) We prove (iii) (2) by contradiction. Assume there exists an initial datum ϕ_1 such that $u(\cdot,t;\phi_1)$ converges to $V(\cdot-\xi(t))$ with $\xi(t)\to\infty$ $(t\to\infty)$. Take $x_n > 0$ be such that $V(-x_n) = s_n$, then for each n, $V'(-x_n) = \sqrt{F(V(-x_n))} = \sqrt{F(s_n)}$, hence $bV'(-x_n) \geqslant V(-x_n)$. Since $x_n \to \infty$ as $s_n \to 0$, we can take a large N such that

$$u(x,1;\phi_1) > V(x-x_N) \text{ in } [0,\bar{x})$$

for some $\bar{x} > 0$ and $u(x,1;\phi_1) < V(x-x_N)$ in $(\bar{x}, h(1)]$. Denote $w(x,t) := u(x,t;\phi_1) - V(x-x_N)$. Then $w(\cdot,t)$ has a unique zero $\gamma(t)$ in $(0,h(t))$ for $0 < t-1 \ll 1$ with $\gamma(1) = \bar{x}$.

By our assumption $u(0,t;\phi_1) \to 0$ as $t\to\infty$, so there exists a time $T_1 > 1$ such that $u(0,t;\phi_1) < V(-x_N)$ for all $t > T_1$. Let us study what happens in the time interval $[1,T_1]$.

In this time interval, $u(0,t;\phi_1)$ must go down across $V(-x_N)$ at some time. Denote t_1 the first of such times. Then $\gamma(t)\to 0$ as $t\to t_1$, and so $x=0$ is the unique zero of $w(x,t_1)$. If $V(\cdot - x_N)$ satisfies the Robin boundary condition $bV'(-x_n) = V(-x_n)$, then $x=0$ is a degenerate zero of $w(\cdot,t_1)$, and so $\mathcal{Z}_{[0,h(t)]}w(\cdot,t) = 0$ for $t > t_1$ by Lemma 3.2 and its proof. This implies that $u(x,t;\phi_1) < V(x-x_N)$ for all $x\in[0,h(t)]$ and $t > t_1$. If $bV'(-x_n) > V(-x_n)$, then $u(x,t_1;\phi_1) < V(x-x_N)$ for all $x > 0$. Later, $u(0,t;\phi_1)$ may go up across $V(-x_N)$ again. But each time when this happens $\mathcal{Z}_{[0,h(t)]}(w(\cdot,t))$ will become 1 again. When $u(0,t;\phi)$ finally goes down across $V(-x_N)$ at some time $T_2 < T_1$ and does not go up again, we have $u(x,t;\phi_1) < V(x-x_N)$ for all $x\in[0,h(t)]$, $t > T_2$.

Now by comparison we have $\lim_{t\to\infty} u(\cdot,t;\phi_1) \leqslant V(\cdot - x_N)$. This, however, contradicts our assumption $u(\cdot,t;\phi_1) \to V(\cdot-\xi(t))$ with $\xi(t)\to\infty$.

(3) First we construct some initial datum ϕ such that $u(x,t;\sigma^*(\phi)\phi) \to V(x-z)$ for some $z\in\mathbf{Z}$. For this purpose we choose h_0 sufficiently small and set $\phi(x)\in\mathscr{X}(h_0)$ be a function by modifying $2(h_0-x)/h_0$ a little bit at $x=0$ such that $\phi(x)$ satisfies the Robin boundary condition. Then $\phi(0) > 1$, $\int_0^{h_0}\phi(x)dx$ is small so that vanishing happens for $u(x,t;\phi)$ (cf. [13, Theorem 3.2]). So $\sigma^*(\phi) > 1$, and $\sigma^*(\phi)\phi$ intersects $V(\cdot - z)$ (for each $z\in\mathbf{Z}$) at exactly one point. If $u(\cdot,t;\sigma^*(\phi)\phi) \to V(\cdot-\xi(t))$ in $H^2([0,h(t)])$ with $\xi(t)\to\infty$, then $u(0,t;\sigma^*(\phi)\phi) < V(-z)$ for large t. By a similar argument as above (using Lemma 3.2) we have $u(x,t;\sigma^*(\phi)\phi) < V(x-z)$ for all large t, and so $V(x-\xi(t)) \leqslant V(x-z)$ with $\xi(t)\to\infty$. This is impossible.

Next, we construct some initial datum ϕ such that $u(x,t;\sigma^*(\phi)\phi) \to V(x-\xi(t))$ with $\xi(t)\to\infty$. Under our assumption: $b\sqrt{F(s_0)} = s_0$ for some $s_0\in(0,\theta)$, $\mathbf{Z}$ is not empty. Denote its maximum by z_0. Taking $m\in(\theta,1)$ with $m-\theta \ll 1$, then we can find a stationary solution v_m of (8) such that $bv_m'(z_m) = v_m(z_m)$ for some $z_m > 0$. Moreover, by our assumption: $b\sqrt{F(s)} < s$ for small $s > 0$, we have

$$v_m(z_m) < V(-z_0) - \varepsilon \quad \text{for some small } \varepsilon > 0.$$

Now we choose h_0 sufficiently large and choose $\phi(x)\in\mathscr{X}(h_0)$ such that it equals δ for some small $\delta > 0$ on $[0,2L_m]$ and equals 1 on $[2L_m+1, 4L_m+1]$. Then $\sigma^*(\phi) < 1$ since spreading happens for $u(\cdot, t; \phi)$ by Lemma 4.2, $\sigma^*(\phi)\phi$ intersects $v_m(x + z_m)$ at exactly one point near $2L_m - z_m$. If $u(\cdot, t; \sigma^*(\phi)\phi) \to V(\cdot - z)$ with $z \in \mathbf{Z}$, then $u(0, t; \sigma^*(\phi)\phi) \geqslant V(-z_0) - \varepsilon > v_m(z_m)$ for large t. By a similar argument as above we have $\mathcal{Z}_{[0,h(t)]}[u(\cdot, t; \sigma^*(\phi)\phi) - v_m(\cdot + z_m)] = 0$ for large t. This implies that $u(x, t; \sigma^*(\phi)\phi) > v_m(x + z_m)$ for large t, and so spreading happens for $u(\cdot, t; \sigma^*(\phi)\phi)$ by Lemma 4.2. This is a contradiction.

This completes the proof of Theorem 2.1. □

5.5. *The propagation speed of $\xi(t)$*

From above we know that a transition solution converges to $V(\cdot - \xi(t))$ with $\xi(t) \to \infty$ $(t \to \infty)$ only if $b\lambda \leqslant 1$, where $\lambda := \sqrt{-f'(0)}$. Denote

$$A := \theta e^{\int_0^\theta [\frac{\lambda}{\sqrt{F(s)}} - \frac{1}{s}]ds}, \quad c(a) := \frac{\lambda^2(1 - b\lambda)A^2}{[1 + b\lambda]\int_0^\theta \sqrt{F(s)}ds}, \quad c_k := \frac{f^{(k)}(0)A^{k+1}}{2(k+1)!\int_0^\theta \sqrt{F(s)}ds}.$$

Proposition 5.8. *Assume (3) holds with $\xi(t) \to \infty$. Then there exists a C^1 function $y(t)$ such that $y(t) - \xi(t) \to 0$ as $t \to \infty$. In addition, $y(t)$ has the following estimates:*

(1) *if $0 \leqslant b\lambda < 1$ and if $f \in C^2([0, \infty))$, then $y(t) = \frac{1}{2\lambda}\ln[2\lambda c(a)t] + O(t^{-1/2})$ as $t \to \infty$;*
(2) *if $b\lambda = 1$ and if, for some positive integer k,*

$$f \in C^{k+1}([0, \infty)), \quad f^{(i)}(0) = 0 \ (i = 2, 3, \ldots, k-1) \quad \textit{and} \quad f^{(k)}(0) > 0,$$

then $y(t) = \frac{1}{(k+1)\lambda}\ln[(k+1)\lambda c_k t] + O(t^{-1/(k+1)})$ as $t \to \infty$.

This proposition can be proved by studying the slow motion on the (approximate) center manifold:

$$\left\{ V(\xi - x) - \frac{V(\xi) + bV'(\xi)}{1 + b\lambda} e^{-\lambda x} \;\middle|\; \xi \geqslant 0 \right\}.$$

We omit the proof since it is similar as that in [11], where the authors obtained the same estimates for problem (1) without free boundary (i.e., $h(t) \equiv \infty$), using the "center manifold" technique developed by Carr and Pego [7], Fusco and Hale [25], Alikakos, Bates and Fusco [1], Alikakos and Fusco [2], and Chen et al. [8,9].

6. Asymptotic speed and asymptotic profile for spreading solutions

We will prove Theorem 2.2 in this section. So we always assume that spreading happens for the solution u of (1) as in Theorem 2.1 (i).

First we will show the boundedness of $h(t) - c^*t$ by constructing upper and lower solutions as in [16,20]. Then we will convert the problem to be a new one in the moving coordinates $z = x - c^*t$. Since the new boundary $H(t) := h(t) - c^*t$ is not necessarily monotone, we will use zero number argument to prove its convergence. The C^2_{loc} convergence of $u(z + c^*t, t)$ then follows. Our approach in the latter part is different from that in [16] and we avoid the construction of complex upper and lower solutions.

6.1. Estimates of $h(t) - c^* t$ and $u(x,t)$

To construct upper and lower solutions we need a lot of notation. Denote

$$\mu := \frac{1}{2}\sqrt{-f'(1)} \quad \text{and} \quad \kappa := \frac{1}{4}\Big[\sqrt{(c^*)^2 - 4f'(1)} - c^*\Big] \ \ (<\mu).$$

Choose $s_0 \in (\frac{1}{2}, 1)$ such that $1 + s_0 > 2\theta$ and $-f'(s) \in (\mu^2, 16\mu^2)$ for $s_0 \leqslant s \leqslant 1$. Since v_* is a solution of $v'' + f(v) = 0$ with $v_*'(x) > 0$ and $v_*(\infty) = 1$, there exists $\tilde{X} > 0$ large such that $v_*(\tilde{X}) > \frac{1+s_0}{2}$ and

$$(v_*')^2 = F(v_*) - F(1) = -f'(\tilde{v})(v_* - 1)^2 \text{ for } x > \tilde{X},$$

where $\tilde{v} \in (s_0, 1)$ and so $-f'(\tilde{v}) \in (\mu^2, 16\mu^2)$. Hence

$$\mu(1 - v_*) \leqslant v_*' \leqslant 4\mu(1 - v_*) \text{ for } x > \tilde{X}.$$

Integrating these inequalities we find that, for some $\Theta = \Theta(\tilde{X}) > 0$, there holds

$$1 - v_*(x) \leqslant \frac{\Theta}{4} e^{-\mu x}, \quad v_*'(x) \leqslant \mu\Theta e^{-\mu x} \quad \text{for } x \geqslant \tilde{X}. \tag{31}$$

Since (c^*, q_{c^*}) is the unique solution of (4) with $q_{c^*}(x) \to 1$ as $x \to \infty$, we have

$$q_{c^*}' = [-2\kappa + o(1)](q_{c^*} - 1) \quad \text{as } x \to \infty.$$

(This can be seen by the analysis on the q–q' phase plane around the point $(1, 0)$, which is a saddle point with a stable manifold tangent to $q' = -2\kappa(q - 1)$.) Thus

$$-\kappa(q_{c^*} - 1) < q_{c^*}' < -3\kappa(q_{c^*} - 1) \quad \text{as } x \to \infty,$$

and so, for any $X > \tilde{X}$ sufficiently large, we have $e^{\mu X} > 3\Theta$ and

$$1 - q_{c^*}(x) \leqslant C(X) e^{-\kappa x}, \quad q_{c^*}'(x) \leqslant C(X) e^{-\kappa x} \quad \text{for some } C(X) > 0. \tag{32}$$

Moreover, we need the following notation:

$$\begin{cases} K := \max\limits_{0 \leqslant s \leqslant 1} |f'(s)|, \quad f_0 := -\min\limits_{0 \leqslant s \leqslant 1} f(s), \quad \gamma := \frac{1 - v_*(X)}{2} \\ X_1 := v_*^{-1}(1-\gamma), \quad \bar{x} := X_1 - X, \quad X_2 := q_{c^*}^{-1}(1-\gamma), \\ \delta := \min\left\{\dfrac{\kappa c^*}{2}, -\dfrac{(1-\gamma)f'(1)}{4}\right\}, \quad p_\gamma := \min\limits_{0 \leqslant x \leqslant X_2} q_{c^*}'(x) > 0, \\ Q := \dfrac{4KC(X) + 8\mu\Theta p_\gamma + 4\Theta K + 8\mu\Theta C(X)}{-f'(1)(1-\gamma)}, \quad H_0 := X_1 + X_2 + M \\ M := \max\left\{\dfrac{(\Theta + Q)c^*}{\delta}, \dfrac{3[2\mu\Theta p_\gamma + 2K\Theta + \delta Q + f_0 Q + KQ]}{\delta p_\gamma}\right\}. \end{cases} \tag{33}$$

Note that these constants and the above $\mu, \kappa, s_0, \tilde{X}, X$ depend essentially only on f, a and b. Choose $T_1 > 0$ large such that

$$\begin{cases} \gamma > Qe^{-\delta T_1}, \quad 1-2\gamma > 2Qe^{-\delta T_1}, \quad 1-\gamma-s_0 > (\Theta+Q)e^{-\delta T_1}, \\ 1-2\gamma-s_0 > (Q+C(X))e^{-\delta T_1}, \quad 3\Theta e^{-2\delta T_1} < 1, \\ u(X_1,t) \geqslant (1-\gamma)v_*(X_1) \text{ for } t \geqslant T_1. \end{cases} \tag{34}$$

The last inequality follows from Theorem 4.7.

Now we define

$$\underline{h}(t) := c^* t + M(e^{-\delta t} - e^{-\delta T_1}) + H_0, \quad t \geqslant T_1, \tag{35}$$

$$\underline{u}(x,t) := [v_*(x-\bar{x}) - Qe^{-\delta t}]q_{c^*}(\underline{h}(t)-x), \quad x \in [X_1, \underline{h}(t)], \ t \geqslant T_1. \tag{36}$$

We verify that $(\underline{u}(x,t), \underline{h}(t))$ is a lower solution of (1) on $[X_1, \underline{h}(t)] \times [T_1, \infty)$.

1. For $x = X_1$ and $t \geqslant T_1$, by (34) we have

$$\underline{u}(X_1,t) = [v_*(X) - Qe^{-\delta t}]q_{c^*}(\underline{h}(t) - X_1) \leqslant v_*(X) < (1-\gamma)v_*(X_1) \leqslant u(X_1,t).$$

2. For $x = \underline{h}(t)$ and $t \geqslant T_1$, we have $\underline{u}(\underline{h}(t),t) \equiv 0$, $\underline{h}(t) - \bar{x} \geqslant c^* t$, and so

$$\begin{aligned} -\underline{u}_x(\underline{h}(t),t) &= [v_*(\underline{h}(t)-\bar{x}) - Qe^{-\delta t}]q'_{c^*}(0) \geqslant c^*[1 - \Theta e^{-\mu c^* t} - Qe^{-\delta t}] \\ &\geqslant c^* - (\Theta+Q)c^* e^{-\delta t} \geqslant c^* - M\delta e^{-\delta t} = \underline{h}'(t). \end{aligned}$$

3. For $x \in [X_1, \underline{h}(t)]$ and $t \geqslant T_1$,

$$\mathcal{N}\underline{u} := \underline{u}_t - \underline{u}_{xx} - f(\underline{u}) = [2v'_*(x-\bar{x}) - \delta M e^{-\delta t}(v - Qe^{-\delta t})]q'_{c_*}(\underline{h}(t)-x) + R(x,t),$$

where

$$R(x,t) := qf(v) - f(\underline{u}) + \delta Q e^{-\delta t} q + (v - Qe^{-\delta t}) f(q).$$

Hereinafter, we also write $v_*(x-\bar{x})$ and $q_{c^*}(\underline{h}(t)-x)$ as v and q for simplicity. Denote

$$x_1(t) := X_1 + \frac{c^* t}{2}, \ I_1 := [X_1, x_1(t)], \ I_2 := [x_1(t), \underline{h}(t) - X_2], \ I_3 := [\underline{h}(t) - X_2, \underline{h}(t)].$$

On I_3, we have $-\mu(x-\bar{x}) < -\mu c^* t \leqslant -2\delta t$. So by (31) we have

$$\begin{aligned} R(x,t) &\leqslant f(v) + f(q) - f(\underline{u}) + \delta Q e^{-\delta t} q - Qe^{-\delta t} f(q) \\ &\leqslant f(v) + K[1 - v + Qe^{-\delta t}] + \delta Q e^{-\delta t} - Qe^{-\delta t} f(q) \\ &\leqslant K[1-v] + K[1 - v + Qe^{-\delta t}] + \delta Q e^{-\delta t} + Qe^{-\delta t} f_0 \\ &\leqslant 2K\Theta e^{-\mu(x-\bar{x})} + [\delta Q + f_0 Q + KQ]e^{-\delta t} \\ &\leqslant [2K\Theta e^{-\delta t} + \delta Q + f_0 Q + KQ]e^{-\delta t}. \end{aligned}$$

Since $q'_{c^*}(\underline{h}(t)-x) \geqslant p_\gamma$ on I_3, we have

$$\begin{aligned}
\mathcal{N}\underline{u} &\leqslant \Big[2v'_*(x-\bar{x}) - \frac{1}{2}\delta M v_*(x-\bar{x})e^{-\delta t}\Big] q'_{c^*}(\underline{h}(t)-x) + R(x,t) \\
&\leqslant \Big[2\mu\Theta e^{-\mu(x-\bar{x})} + \frac{1}{2}\delta M\Theta e^{-3\delta t} - \frac{1}{2}\delta M e^{-\delta t}\Big] q'_{c^*}(\underline{h}(t)-x) + R(x,t) \\
&\leqslant \Big[2\mu\Theta e^{-\delta t} - \frac{1}{3}\delta M\Big] p_\gamma e^{-\delta t} + R(x,t), \quad \text{by } 3\Theta e^{-2\delta t} < 1 \\
&\leqslant \Big[2\mu\Theta p_\gamma e^{-\delta t} - \frac{1}{3}\delta M p_\gamma + 2K\Theta e^{-\delta t} + \delta Q + f_0 Q + KQ\Big] e^{-\delta t} \leqslant 0
\end{aligned}$$

by the choice of M.

On I_1, since $\underline{h}(t) - x \geqslant \frac{c^* t}{2}$ we have

$$1 - q_{c^*}(\underline{h}(t)-x), \;\; q'_{c^*}(\underline{h}(t)-x) \leqslant C(X)e^{-\delta t}.$$

We also have

$$1 \geqslant \underline{u}(x,t) \geqslant [v_*(X) - Qe^{-\delta T_1}][1 - C(X)e^{-\delta T_1}] \geqslant s_0.$$

Hence,

$$\begin{aligned}
R(x,t) &\leqslant f(v) - f(\underline{u}) + f(q) + \delta Q e^{-\delta t} \\
&\leqslant f'(\xi_1)[v(1-q)] + f'(\xi_1)Qe^{-\delta t} q + K(1-q) + \delta Q e^{-\delta t}, \text{ for some } \xi_1 \in (s_0, 1) \\
&\leqslant [f'(\xi_1)q + \delta]Qe^{-\delta t} + K(1-q) \leqslant \frac{f'(1)(1-\gamma)}{4} Qe^{-\delta t} + KC(X)e^{-\delta t}.
\end{aligned}$$

So,

$$\begin{aligned}
\mathcal{N}\underline{u} &\leqslant 2v'_*(x-\bar{x}) q'_{c^*}(\underline{h}(t)-x) + R(x,t) \\
&\leqslant 2\mu\Theta e^{-\mu(x-\bar{x})} C(X)e^{-\delta t} + R(x,t) \\
&\leqslant \Big[2\mu\Theta C(X) + \frac{f'(1)(1-\gamma)}{4} Q + KC(X)\Big] e^{-\delta t} \leqslant 0.
\end{aligned}$$

On I_2, $\underline{h}(t) - x \geqslant X_2$ and $x - \bar{x} > \frac{c^* t}{2} + X$. Hence

$$\begin{aligned}
&q_{c^*}(\underline{h}(t)-x) \geqslant 1-\gamma, \quad f(q_{c^*}(\underline{h}(t)-x)) > 0, \quad 1 - v_*(x-\bar{x}) \leqslant \Theta e^{-\delta t}, \\
&1 \geqslant \underline{u}(x,t) \geqslant [1 - (\Theta + Q)e^{-\delta T_1}](1-\gamma) \geqslant 1 - \gamma - (\Theta + Q)e^{-\delta T_1} > s_0.
\end{aligned}$$

So,

$$\begin{aligned}
R(x,t) &\leqslant f(v) + f(q) - f(\underline{u}) + \delta Q e^{-\delta t} \\
&\leqslant f(v) + f'(\xi_2) q(1 - v + Q e^{-\delta t}) + \delta Q e^{-\delta t}, \text{ for some } \xi_2 \in (s_0, 1) \\
&\leqslant K(1-v) + f'(\xi_2) q Q e^{-\delta t} + \delta Q e^{-\delta t} \\
&\leqslant K\Theta e^{-\delta t} + \frac{f'(1)}{2}(1-\gamma) Q e^{-\delta t} + \delta Q e^{-\delta t} \\
&\leqslant \Big[K\Theta + \frac{f'(1)(1-\gamma)Q}{4} \Big] e^{-\delta t}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{N}\underline{u} &\leqslant 2v_*'(x-\bar{x}) q_{c^*}'(\underline{h}(t) - x) + R(x,t) \\
&\leqslant \Big[2\mu\Theta p_\gamma + K\Theta + \frac{f'(1)(1-\gamma)Q}{4} \Big] e^{-\delta t} \leqslant 0.
\end{aligned}$$

4. There exists $T_2 > 0$ such that

$$\underline{u}(x, T_1) \leqslant u(x, T_1 + T_2) \text{ for } x \in [0, \underline{h}(T_1)], \quad \underline{h}(T_1) \leqslant h(T_1 + T_2).$$

Consequently, $(\underline{u}, \underline{h})$ is a lower solution of (1) on $[X_1, \underline{h}(t)] \times [T_1, \infty)$ (see the definition in [13, Lemma 2.2]), and

$$\underline{u}(x,t) \leqslant u(x, T_2 + t) \text{ for } x \in [X_1, \underline{h}(t)], \quad \underline{h}(t) \leqslant h(T_2 + t) \quad \text{for } t \geqslant T_1. \tag{37}$$

Define

$$\bar{h}(t) := c^* t + M'(e^{-\delta' T'} - e^{-\delta' t}) + H' \quad \text{and} \quad \bar{u}(x,t) := (1 + M' e^{-\delta' t}) q_{c^*}(\bar{h}(t) - x).$$

In a similar way as above (see also [16, Lemma 3.2]) one can show that for some suitable $M', \delta', T' > 0$, $(\bar{u}, \bar{h})$ is an upper solution of (1) on $[0, h(t)] \times [T', \infty)$.

We remark that the construction of the above upper and lower solutions are inspired by [13, 16,20]. In summary we have the following result.

Lemma 6.1. *Assume* **(F)** *and spreading happens for the solution u of (1). Then there exist $H_1, H_2 \in \mathbb{R}$ with $H_1 < H_2$ and $\bar{x}, \bar{X}, Q, T^*, \delta > 0$ with $c^* T^* + H_1 > \bar{X}$ such that*

$$c^* t + H_1 \leqslant h(t) \leqslant c^* t + H_2, \quad t \geqslant T^*, \tag{38}$$

$$[v_*(x - \bar{x}) - Q e^{-\delta t}] q_{c^*}(c^* t + H_1 - x) \leqslant u(x,t), \; x \in [\bar{X}, c^* t + H_1], \; t \geqslant T^*, \tag{39}$$

and

$$u(x,t) \leqslant (1 + Q e^{-\delta t}) q_{c^*}(c^* t + H_2 - x), \quad x \in [\bar{X}, h(t)], \; t \geqslant T^*. \tag{40}$$

6.2. C^2_{loc} convergence in moving coordinates

Set

$$H(t) := h(t) - c^* t,\ w(z,t) := u(z + c^* t, t),\ z \in [-c^* t, H(t)],\ t > 0. \tag{41}$$

Then w is a solution of

$$\begin{cases} w_t = w_{zz} + c^* w_z + f(w), & -c^* t < z < H(t),\ t > 0, \\ w(-c^* t, t) = b w_z(-c^* t, t), & t > 0, \\ w(H(t), t) = 0, \quad H'(t) = -w_z(H(t), t) - c^*, & t > 0, \\ w(z, 0) = u_0(z), & 0 \leqslant z \leqslant H(0). \end{cases} \tag{42}$$

On the asymptotic behavior of H and w we have the following result.

Lemma 6.2. *Assume* **(F)** *and spreading happens for the solution u of (1). Let H, w be defined by (41). Then*

$$\lim_{t\to\infty} H(t) = H_\infty \text{ for some } H_\infty \in \mathbb{R}, \qquad \lim_{t\to\infty} H'(t) = 0, \tag{43}$$

and

$$\lim_{t\to\infty} \|w(\cdot,t) - q_{c^*}(H_\infty - \cdot)\|_{C^2([-\varepsilon^{-1}, H_\infty - \varepsilon])} = 0 \qquad \forall \varepsilon > 0. \tag{44}$$

Proof. For any $r \in \mathbb{R}$, $\hat{w}(z) := q_{c^*}(r - z)$ is a solution of

$$\begin{cases} \hat{w}_{zz} + c^* \hat{w}_z + f(\hat{w}) = 0, & -\infty < z < r, \\ \hat{w}(-\infty) = 1, \quad \hat{w}(r) = 0, & \hat{w}_z(r) = -c^*. \end{cases} \tag{45}$$

Now we compare $w(z,t)$ and $\hat{w}(z)$ in the interval $J(t) := [-c^* t, \min\{r, H(t)\}]$. At the left end of $J(t)$, $w(-c^* t, t) = u(0,t) \to v_*(0)$ and $\hat{w}(-c^* t) = q_{c^*}(r + c^* t) \to 1$, as $t \to \infty$. Hence $w(-c^* t, t) < \hat{w}(-c^* t)$ for large t. The right end $H(t)$ of $w(z,t)$ may get across r. Using the zero number argument as in Lemma 3.2 and in [14, Lemma 2.3], we deduce that $\mathcal{Z}_{J(t)}[w(\cdot,t) - \hat{w}(\cdot)]$ is finite, and it decreases strictly when $H(t)$ get across r. So $H(t) - r$ changes sign at most finitely many times, namely, $H(t) > r$, or $H(t) < r$, or $H(t) \equiv r$ for all large t. Since $H(t)$ is bounded by (38) and r is arbitrary, we see that $H_\infty := \lim_{t\to\infty} H(t)$ exists and it is finite.

Using the limit of $H(t)$ and the uniform Hölder estimate for $H'(t)$: $\|H'(t)\|_{C^{\nu/2}([1,\infty))} \leqslant C$ for C independent of t (cf. [12,13]), it is easy to show that $\lim_{t\to\infty} H'(t) = 0$.

Finally we consider the ω-limit set $\omega(w)$ of w in the topology of $C^2_{\mathrm{loc}}((-\infty, H_\infty))$. In the same way as proving Lemma 4.1, one can show that $\omega(w)$ is not empty, and it consists of only solutions of $\hat{w}_{zz} + c^* \hat{w}_z + f(\hat{w}) = 0,\ z \in (-\infty, H_\infty)$. For each $\hat{w} \in \omega(w)$, by (39) and the definition of w we have $\hat{w}(-\infty) = 1$, and by (43) we have $\hat{w}(H_\infty - 0) = 0$. Therefore, $\omega(w) = \{q_{c^*}(H_\infty - z)\}$. This proves (44). $\square$

6.3. Proof of Theorem 2.2

For convenience, we extend $u(x,t)$ to be 0 for $x \geqslant h(t)$, extend $q_{c^*}(x)$ to be 0 for $x < 0$, and denote the extended functions again by u and q_{c^*}. We will prove

$$D(x,t) := u(x,t) - v_*(x) \cdot q_{c^*}(c^* t + H_\infty - x) \to 0 \ \text{ in } \ L^\infty([0,h(t)]), \quad \text{as } t \to \infty.$$

We use the notation in Lemma 6.1. For any given small $\varepsilon > 0$, there exist $x_0 > \max\{\bar{x}, \bar{X}, |H_\infty|, H_\infty - H_1\}$ and $\delta_1 > 0$ such that

$$\begin{aligned} &v_*(x - \bar{x}) > 1 - \varepsilon \text{ for } x > x_0, \\ &q_{c^*}(x) > 1 - \varepsilon \text{ for } x > x_0, \quad q_{c^*}(x) < \varepsilon \text{ for } x \in [0, 2\delta_1]. \end{aligned} \tag{46}$$

Since $f(s) < 0$ for $s > 1$, there exists $T_0 > T^*$ for T^* in Lemma 6.1 with $c^* T_0 > 3x_0 + |H_\infty|$ such that

$$u(x,t) \leqslant 1 + \varepsilon \quad \text{for } x \in [0, h(t)],\ t > T_0.$$

As in [12], it is easy to show that $\hat{u}(x,t) := 4M(h(t) - x) - 2M^2(h(t) - x)^2$ for some suitable $M > 0$ is an upper solution of (1) over $\{(x,t) \mid h(t) - M^{-1} < x < h(t), t > T_0\}$. So there exists $0 < \delta < \delta_1$ such that

$$u(x,t) \leqslant \hat{u}(x,t) \leqslant \hat{u}(h(t) - 2\delta, t) = 8M\delta(1 - M\delta) \tag{47}$$

for $x \in [h(t) - 2\delta, h(t)],\ t > T_0$.

For $I_1 := [0, x_0]$, by Lemma 4.1 there exists $T_1 > T_0$ such that

$$\|u(\cdot,t) - v_*(\cdot)\|_{L^\infty(I_1)} < \varepsilon \quad \text{for } t > T_1, \tag{48}$$

and so, for $t > T_1$,

$$\|D(\cdot,t)\|_{L^\infty(I_1)} \leqslant \|u(\cdot,t) - v_*(\cdot)\|_{L^\infty(I_1)} + \|v_*(\cdot)[1 - q_{c^*}(c^* t + H_\infty - \cdot)]\|_{L^\infty(I_1)} \leqslant 2\varepsilon.$$

For the above chosen δ, there exists $T_2 > T_1$ such that $|h(t) - c^* t - H_\infty| \leqslant \delta$ for $t > T_2$. So $c^* t + H_\infty - \delta \geqslant h(t) - 2\delta$ and on $I_2 := [c^* t + H_\infty - \delta, h(t)]$ we have by (47),

$$\|D(\cdot,t)\|_{L^\infty(I_2)} \leqslant \|u(\cdot,t)\|_{L^\infty(I_2)} + \|v_*(\cdot) \cdot q_{c^*}(c^* t + H_\infty - \cdot)\|_{L^\infty(I_2)} \leqslant 2\varepsilon.$$

On $I_3 := [c^* t + H_\infty - 2x_0, c^* t + H_\infty - \delta]$, by (44) in Lemma 6.2, there exists $T_3 > T_2$ such that for any $t > T_3$ we have

$$\|u(\cdot,t) - q_{c^*}(c^* t + H_\infty - \cdot)\|_{L^\infty(I_3)} \leqslant \varepsilon, \tag{49}$$

and so

$$\|D(\cdot,t)\|_{L^\infty(I_3)} \leqslant \|u(\cdot,t) - q_{c^*}(c^*t + H_\infty - \cdot)\|_{L^\infty(I_3)} + \|q_{c^*}(c^*t + H_\infty - \cdot)[1 - v_*(\cdot)]\|_{L^\infty(I_3)} \leqslant 2\varepsilon.$$

Finally consider $I_4 := [x_0, c^*t + H_\infty - 2x_0] \subset [\bar{X}, c^*t + H_1]$. By (39) we have $u(x,t) \geqslant (1 - \varepsilon - Qe^{-\delta t})(1 - \varepsilon)$. Choose $T_4 > T_3$ such that $Qe^{-\delta T_4} < \varepsilon$. Then for $x \in I_4$, $t > T_4$ we have $u(x,t) \geqslant (1 - 2\varepsilon)(1 - \varepsilon) > 1 - 3\varepsilon$. Hence

$$\|D(\cdot,t)\|_{L^\infty(I_4)} \leqslant \|u(\cdot,t) - 1\|_{L^\infty(I_4)} + \|1 - v_*(\cdot)\|_{L^\infty(I_4)} + \|v_*(\cdot)[1 - q_{c^*}(c^*t + H_\infty - \cdot)]\|_{L^\infty(I_4)} \leqslant 5\varepsilon.$$

Combining the above results we obtain (6) since $[0, h(t)] = I_1 \cup I_2 \cup I_3 \cup I_4$. (5) follows from (43). This proves Theorem 2.2. □

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