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PERIODIC ROTATING WAVES IN AN UNDULATING ANNULUS AND THEIR HOMOGENIZATION LIMIT*

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Abstract. We study a mean curvature flow equation in an annulus with periodically undulating boundaries and consider the homogenization limit problem as the period of the boundary undulation tends to zero. We first establish a necessary and sufficient condition for the existence of periodic rotating waves. Then we study how the average rotating speed of the periodic rotating wave depends on the geometry of the boundaries. Our results show that boundary undulation always lowers the speed of a rotating wave. We also determine the homogenization limit of the average rotating speed. Quite surprisingly, this homogenized speed depends only on the maximum opening angles of the domain boundaries.

Key words. mean curvature flow equations, periodic rotating waves, undulating annulus, homogenization limit, average rotating speed

AMS subject classifications. 35K55, 35B10

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1. Introduction.

We study a curvature-driven motion of curves in an annulus Ω_m ($m \in \mathbf{N}$), whose boundaries undulate periodically with period $\frac{2\pi}{m}$. The motion of the curves is a mean curvature flow equation,

$$V = A + \kappa, \tag{1.1}$$

where V denotes the normal velocity of plane curves, κ denotes the curvature, and $A > 0$ is a constant. Domain Ω_m is defined as follows: Let $\tilde{g}(s)$, $\tilde{h}(s)$ be 2π-periodic smooth functions satisfying

$$\tilde{h}(0) = 0, \quad \tilde{h}(s) \geq 0, \quad \max_s \tilde{h}'(s) = \tan\alpha_1, \quad \min_s \tilde{h}'(s) = -\tan\beta_1,$$
$$\tilde{g}(0) = 0, \quad \tilde{g}(s) \geq 0, \quad \max_s \tilde{g}'(s) = \tan\alpha_2, \quad \min_s \tilde{g}'(s) = -\tan\beta_2$$

for some $\alpha_i,\ \beta_i \in (0, \frac{\pi}{2})$ $(i = 1, 2)$. For $m \in \mathbf{N}$, define

$$h(s) := \frac{H}{m}\,\tilde{h}(ms), \qquad g(s) := \frac{G}{m}\,\tilde{g}(ms),$$

where $G > H > 0$ are fixed. In what follows, we use polar coordinates (r, θ) to express points in the plane. Define

$$\Omega_m := \left\{(r,\theta) \;\middle|\; H - h(\theta) < r < G + g(\theta), \quad \theta \in [0, 2\pi]\right\}.$$

Denote the outer (resp., inner) boundary of Ω_m by $\partial_2\Omega_m$ (resp., $\partial_1\Omega_m$) and call α_2 (resp., α_1) its *maximum opening angle* (see Figure 1.1).

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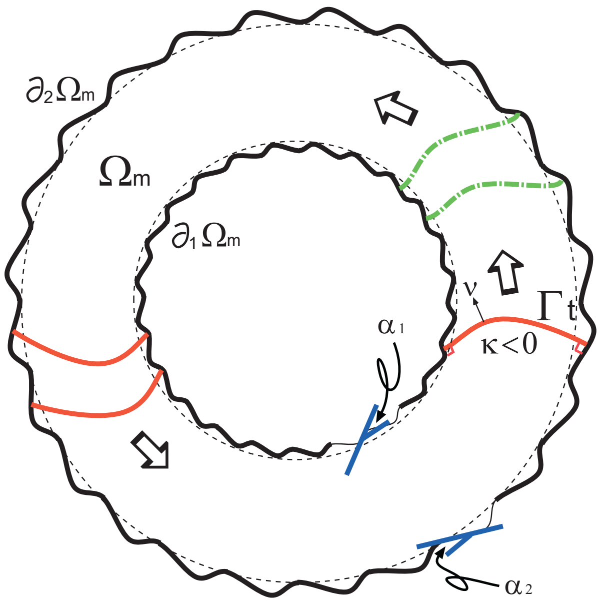

FIG. 1.1. *Undulating annulus and periodic rotating waves.*

By a solution of (1.1) we mean a time-dependent simple curve Γ_t in Ω_m which satisfies (1.1) and contacts both $\partial_1\Omega_m$ and $\partial_2\Omega_m$ perpendicularly. In this paper we are interested in those curves rotating counterclockwise along Ω_m periodically, as well as their average rotating speeds.

In the following we state the motivation and background of (i) the study of mean curvature flow equations; (ii) the study of propagation in annuli; and (iii) the study of propagation in a domain with undulating boundaries.

(i) In 1951 [3] proposed that, in the growth of crystals, the evolution of a crystal surface is governed by a curvature flow equation like (1.1). In 1956 [17] also proposed that the motion of idealized grain boundaries is governed by its curvature. Later *mean curvature flow equations* were used in the study of propagation of wave fronts in an excitable medium, in flame front propagation, in crystal growth, and in many other fields. From a mathematical point of view, it is found that sharp internal layers (or interfaces) appear in singular limit problems of reaction diffusion equations, and the motion of such layers is mean curvature flow equations. For example, a successful model used in the study of Belousov–Zhabotinsky (BZ) reactions are so-called FitzHugh–Nagumo (FHN) equations; the propagation of a pulse solution of FHN equations depends on the mean curvature of the pulse front (e.g., [11], [12], [22]), that is, the motion is a mean curvature flow equation like (1.1).

Mean curvature flow equations are also interesting in their own right from a geometrical point of view. Recently, many geometricians have studied the asymptotic behavior of mean curvature flows ([1], [2], [5], [8], [10], to name only a few). However, as far as we know, very little is known about (periodic) traveling/rotating waves of mean curvature flow equations, though periodic traveling wave solutions of reaction diffusion equations have been studied a lot (cf. [16], [23] and the references therein).

(ii) In the past twenty years many scientists have been interested in BZ reactions in annular gel (for example, [6], [18]). As we said above, one can use FHN equations to model BZ reactions, and FHN equations can be reduced to a mean curvature flow equation. So the study of mean curvature flow equations in annuli is very helpful to the study of propagation of pulse solutions in BZ reactions in annuli. Some other motivation and studies of rotating waves in annuli can also be found in [9], [20], [22].

(iii) In some cases the domains are not necessarily equipped with flat boundaries. When we consider a propagation in a space with reticulated structure (cf. [4]), or in a media mixed with impurity which blocks the propagation, we may get a domain with undulating boundaries. Such a domain may be an undulating annulus like Ω_m provided the reticulated structure or the masses of impurity are arranged in circles. The propagation of pulses in such an annulus reduces to our problem.

In this paper, we study periodic rotating waves of mean curvature flow (1.1) in annulus Ω_m. We believe that the results in this paper can be extended to more general mean curvature flows.

To avoid sign confusion, the normal vector ν to Γ_t will always be chosen to be counterclockwise; the sign of the normal velocity V and the curvature κ will be understood in accordance with this normal direction (see details below).

We will consider the case that each curve Γ_t is expressed as a graph of a $C^{2,1}$ function $\theta = u(r,t)$ with $(r, u(r,t)) \in \Omega_m$. Let $\eta_1(t)$, $\eta_2(t)$ be the r-coordinates of the end points of Γ_t lying on $\partial_1\Omega_m$, $\partial_2\Omega_m$, respectively. In other words, $\eta_1(t) = H - h(u(\eta_1(t),t))$, $\eta_2(t) = G + g(u(\eta_2(t),t))$. Write the orthogonal coordinates as $(r\cos\theta, r\sin\theta)_\perp$ and denote the unit tangent vector of Γ_t by $\mathbf{T}$ (in the positive direction of r); then $\mathbf{T} = (\cos u - r\sin u \cdot u_r,\ \sin u + r\cos u(r,t)\cdot u_r)_\perp/\sqrt{1+r^2u_r^2}$, $\nu = (-\sin u - r\cos u \cdot u_r,\ \cos u - r\sin u(r,t)\cdot u_r)_\perp/\sqrt{1+r^2u_r^2}$ and

$$V = (-r\sin u\cdot u_t,\ r\cos u\cdot u_t)_\perp\cdot\nu = \frac{ru_t}{\sqrt{1+r^2u_r^2}}, \qquad \kappa = \frac{ru_{rr} + 2u_r + r^2u_r^3}{(1+r^2u_r^2)^{3/2}}.$$

Hence (1.1) is equivalent to

$$u_t = \frac{u_{rr}}{1+r^2u_r^2} + \frac{2u_r + r^2u_r^3}{r(1+r^2u_r^2)} + A\frac{\sqrt{1+r^2u_r^2}}{r}, \qquad \eta_1(t) < r < \eta_2(t),\ t>0. \tag{1.2}$$

Denote the *clockwise* unit tangent vector of $r = H$ (resp., $r = G$, $\partial_1\Omega_m$, $\partial_2\Omega_m$) by $\mathbf{T}_1^0$ (resp., $\mathbf{T}_2^0$, $\mathbf{T}_1$, $\mathbf{T}_2$). In what follows, we say that the curve Γ_t contacts $r = H$ (resp., $r = G$, $\partial_1\Omega_m$, $\partial_2\Omega_m$) with angle γ in the sense that $\mathbf{T}\cdot\mathbf{T}_1^0 = \cos\gamma$ (resp., $\mathbf{T}\cdot\mathbf{T}_2^0 = -\cos\gamma$, $\mathbf{T}\cdot\mathbf{T}_1 = \cos\gamma$, $\mathbf{T}\cdot\mathbf{T}_2 = -\cos\gamma$). Our boundary conditions are $\mathbf{T}\cdot\mathbf{T}_i = 0$ on $\partial_i\Omega_m$ $(i = 1,2)$, which are expressed as

$$u_r(\eta_1(t),t) = \frac{h'(u)}{(\eta_1(t))^2}, \qquad u_r(\eta_2(t),t) = \frac{-g'(u)}{(\eta_2(t))^2}. \tag{1.3}$$

Let $\Omega_0 = \{(r,\theta) \mid H < r < G,\ \theta\in[0,2\pi]\}$ be the trivial annulus which is *formally* a limit of Ω_m as $m\to\infty$. Problem (1.2) in Ω_0 with boundary conditions

$$u_r(H,t) = u_r(G,t) = 0 \tag{1.4}$$

is quite simple. In fact, as is shown in subsection 2.1.3, when $A > 0$, there exists a unique $\omega_0 > 0$ such that (1.2), (1.4) has a unique *rotating wave* $u(r,t) = \varphi(r) + \omega_0 t$, which has a certain nonplanar profile and rotating speed ω_0. Relevant study in trivial annulus can also be found in [9], [20].

On the other hand, in Ω_m, a rotating wave with a certain profile does not exist. In fact, as Γ_t propagates, its shape and speed fluctuate along with the undulation of the domain Ω_m. In such a situation, we adopt a generalized notion of rotating waves. A solution $U_m(r,t)$ of (1.2)–(1.3) is called a *periodic rotating wave* if it satisfies

$$U_m(r, t+T_m) = U_m(r,t) + \frac{2\pi}{m}$$

for some $T_m > 0$. Clearly, a periodic rotating wave changes its profile periodically in time (see Figure 1.1). The *average rotating speed* of a periodic rotating wave is

$$\omega_m = \frac{2\pi}{mT_m}.$$

In what follows we concentrate our attention on periodic rotating waves with average rotating speed $\omega_m = O(1)$ as $m \to \infty$.

Before stating the main results, we give some assumptions on the boundaries:

$$\alpha_1 + \beta_1 < \frac{\pi}{2}, \qquad \alpha_2 + \beta_2 < \frac{\pi}{2}, \tag{1.5}$$

$$|h'| \cdot \overline{M} < 1, \qquad |g'| \cdot \overline{M} < 1, \tag{1.6}$$

where $\overline{M} = \max\{M_1, M_2\} + 1$ with

$$M_1 := \max\left\{\frac{\tan\alpha_1}{H}, \frac{\tan\beta_1}{H}, \frac{\tan\alpha_2}{G}, \frac{\tan\beta_2}{G}\right\}, \tag{1.7}$$

and $M_2 > 0$ is such that

$$w(2 + 5H^2w^2 + H^4w^4) > A(1+H^2w^2)^{3/2} \quad \text{for } \ w > M_2. \tag{1.8}$$

Roughly speaking these conditions require that the undulation of the boundaries be gradual. Assumption (1.5) excludes the possible singularity that the curve touches $\partial\Omega_m$ at some points besides the two end points. Assumption (1.6) guarantees the boundedness of $|u_r|$ on the boundaries (see Appendix A), and it also ensures that we can convert $u(r,t)$ into another unknown $v(z,t)$ defined on $z \in [0,1]$; we can then simply carry out a rigorous proof (see (A.9) in Appendix A). It should be pointed out that these conditions are not necessary ones and can be weakened in special cases.

About the existence we have the following.

THEOREM 1.1. *Assume* (1.5) *and* (1.6) *hold. Then for large* m, (1.2)–(1.3) *has a periodic rotating wave* $U_m(r,t)$ *if and only if*

$$A > \frac{2H\sin\alpha_1 + 2G\sin\alpha_2}{G^2 - H^2}. \tag{1.9}$$

Moreover, a periodic rotating wave is unique up to a time-shift when it exists.

In fact, (1.9) is a necessary and sufficient condition for the existence of rotating *lower* solutions (see subsection 2.1.3).

A more important aim in this paper is to study how the periodic rotating wave and its average speed depend on the shape of the boundaries. In chemical, physical, or biological experiments, traveling/rotating waves can be observed directly. In this sense, people concern themselves with the traveling/rotating speed rather than the existence, though the latter is important from a mathematical point of view. Generally, traveling/rotating waves in a trivial domain with flat boundaries can be studied by converting the problem to an ODE. However, the propagation in a domain with undulating boundaries (a domain with reticulated structure) cannot be converted to an ODE. We have to deal with a PDE directly. Thus the study for the speed of traveling/rotating waves in undulating domains is important and difficult. Very little is known so far.

We estimate the average rotating speed ω_m and determine its homogenization limit as well as the homogenization limit of $U_m(r,t)$ as $m \to \infty$.

THEOREM 1.2. *Assume* (1.5), (1.6), *and* (1.9) *hold. Then for large* m,

(i) *there exists* $C = C(h', g', H, G, A) > 0$ *such that* ω_m *satisfies*

$$\omega^* - \frac{C}{m} \ < \ \omega_m \ < \ \omega^* + \frac{C}{\sqrt{m}} < \omega_0, \tag{1.10}$$

where $\omega^* = \omega^*(\alpha_1, \alpha_2, H, G, A)$ *is given by the unique solution* $(\omega^*, \varphi^*(r;\omega^*))$ *of*

$$\begin{cases} \omega = \dfrac{\varphi''}{1+r^2\varphi'^2} + \dfrac{2\varphi' + r^2\varphi'^3}{r(1+r^2\varphi'^2)} + A\dfrac{\sqrt{1+r^2\varphi'^2}}{r}, & H < r < G, \\ \varphi'(H) = \dfrac{\tan\alpha_1}{H}, \qquad \varphi'(G) = \dfrac{-\tan\alpha_2}{G}, \end{cases} \tag{1.11}$$

where ω_0 *is given by the unique rotating wave solution of* (1.2), (1.4) *in* Ω_0*;*

(ii) *as* $m \to \infty$, $U_m(r,t) \to \varphi^*(r;\omega^*) + \omega^* t + C$ *in* $C^{2,1}([H,G] \times [-T,T])$ *for any* $T > 0$, *where* C *is a constant independent of* T.

$\omega_m < \omega_0$ in (1.10) implies that boundary undulation always lowers the speed of the rotating wave, and $\omega^* < \omega_0$ implies that the effect of spatial inhomogeneity of Ω_m is left to the homogenization limit. Moreover, the fact that homogenized speed ω^* depends only on α_1, α_2 (besides H, G, A) is a surprising result.

In [15], we studied periodic traveling waves of (1.1) in an undulating band domain, obtaining results similar to those above. The problem in that paper is different from the present one on several points. First, since the boundaries of an annulus have period 2π anyway, Theorem 1.1 remains valid even if the smallest periods of h and g are 2π, provided the undulation of h and g is gradual. (We omit the detail in this paper.) Second, the backgrounds are different. Mean curvature flows in an *unbounded* band domain are reduced from a traveling front or a traveling pulse, but in a *bounded* annulus can be reduced only from a rotating pulse. Third, the symmetries of a band domain and an annulus are different. This can be seen from the following fact. A flat band domain is quite simple, in which we have a planar traveling wave with speed A, while a trivial annulus is not symmetric in the direction of radius, in which the rotating wave of $V = \kappa + A$ is $\varphi(r) + \omega_0 t$; its profile is not planar. Especially, in case $G - H$ is large, the graph of $\varphi(r)$ may be a spiral which turns around the origin for several rounds (cf. [9], [20], [22]). Fourth, the boundaries of the band domain in [15] are symmetric and hence the boundary conditions are symmetric. But in this paper, outer and inner boundaries of Ω_m are given by different functions h, g, and the boundary conditions (1.3) are also not symmetric.

In section 2, we prove Theorem 1.1. First, we give a *global solution* of an initial-boundary value problem for appropriate initial data. Next, we use the global solution to construct an *entire solution* by using the *renormalization method*. Then we prove the uniqueness (up to a time-shift) of the entire solution; this immediately implies the existence and uniqueness of *periodic rotating wave*. The necessity of (1.9) is explained in section 2.5. Finally, we state without proof the stability of periodic rotating waves.

In section 3 we prove Theorem 1.2: we estimate the average rotating speed by constructing a precise upper solution. We point out that our construction for an upper solution is a peculiar method, since the upper solution is larger than the solution not only on one period and not for all time, but just in time-interval $[0,1]$. However, this is good enough to give the upper bound of the average rotating speed.

2. Existence of periodic rotating waves.

2.1. Global solutions of the initial-boundary value problem. The proof for the existence of global solutions is divided into several steps. In subsection 2.1.1 we state the comparison principle as a preliminary. In subsection 2.1.2 we construct appropriate initial data $u_0(r)$. In subsection 2.1.3 we study rotating waves in trivial annuli, and select two of them as the lower solution and the upper solution. Using them we give the estimate of $|u|$ in finite time-interval $(0,T]$. In subsection 2.1.4 we give the a priori estimates for $|u_r|$, $|u_r|_\mu$ and for $|u|_\mu$ by converting the problem of $u(r,t)$ to a problem of a new unknown $v(z,t)$ ($z \in (0,1)$). Also we show the global existence, smoothness, and $C^{2+\mu,1+\frac{\mu}{2}}$ bound of u. The proofs in subsection 2.1.4 are long and the idea is similar to that of [15]; we move the details to the appendices.

In what follows, we also write $\frac{1}{m} = \varepsilon$ for convenience.

2.1.1. Comparison principle.

Definition 2.1. *Let* $u_1(r,t)$, $u_2(r,t)$ $(t \geq 0)$ *be two functions satisfying* $(r, u_i(r,t)) \in \overline{\Omega}_m$ $(i = 1,2)$. *Then* u_1 *is called a lower solution of* (1.2)–(1.3) *if*

$$u_{1t} \leq \frac{u_{1rr}}{1 + r^2 u_{1r}^2} + \frac{2u_{1r} + r^2 u_{1r}^3}{r(1 + r^2 u_{1r}^2)} + A\frac{\sqrt{1 + r^2 u_{1r}^2}}{r} \tag{2.1}$$

for $t \geq 0$ *and* r *with* $(r, u_1(r,t)) \in \Omega_m$,

$$\begin{aligned} u_{1r}(r,t) &\geq \frac{h'(u_1)}{r^2} \qquad \text{for } t \geq 0 \quad \text{and} \quad r \text{ with } (r, u_1(r,t)) \in \partial_1\Omega_m, \\ u_{1r}(r,t) &\leq \frac{-g'(u_1)}{r^2} \qquad \text{for } t \geq 0 \quad \text{and} \quad r \text{ with } (r, u_1(r,t)) \in \partial_2\Omega_m. \end{aligned} \tag{2.2}$$

u_2 *is said to be an upper solution of* (1.2)–(1.3) *if the opposite inequalities hold.*

Lemma 2.2. *Assume* $u_1(r,t)$ *and* $u_2(r,t)$ *are the lower solution and the upper solution of* (1.2)–(1.3) *for* $0 \leq t < t_1$, *respectively. If* $u_1(r,0) \leq u_2(r,0)$, *then* $u_1(r,t) \leq u_2(r,t)$ *for* $0 \leq t < t_1$ *and* r *with* $(r, u_i(r,t)) \in \overline{\Omega}_m$. *If* $u_1(r,0) \leq u_2(r,0)$ *and* $u_1(r,0) \not\equiv u_2(r,0)$, *then* $u_1(r,t) < u_2(r,t)$ *for* $0 < t < t_1$ *and* r *with* $(r, u_i(r,t)) \in \overline{\Omega}_m$.

This lemma follows from the maximum principle.

2.1.2. Appropriate initial data. Assume $\theta_1 \in (0, 2\pi\varepsilon)$ such that $h'(\theta_1) = 0$ and $h'(\theta) \leq 0$ for $\theta_1 \leq \theta < \theta_1 + \delta_1$ for small $\delta_1 > 0$. Since the period of the boundaries is $2\pi\varepsilon$, there exists $\theta_2 \in (-2\pi\varepsilon + \theta_1, 2\pi\varepsilon + \theta_1)$ such that $g'(\theta_2) = 0$ and $g'(\theta) \leq 0$ for $\theta_2 \leq \theta < \theta_2 + \delta_1$, provided $\delta_1 > 0$ is small. Denote $P_1 = (H - h(\theta_1), \theta_1)$ and $P_2 = (G + g(\theta_2), \theta_2)$. It is easy to connect P_1 and P_2 by a smooth curve Γ_0 (denote its function by $u_0(r)$) such that the parts of Γ_0 near the boundaries are straight line segments, it contacts $\partial_1\Omega_m$ (resp., $\partial_2\Omega_m$) at P_1 (resp., P_2) vertically, and

$$|u_0(r)| = O(\varepsilon), \quad |u_{0r}(r)| = O(\varepsilon), \quad |u_{0rr}(r)| = O(\varepsilon).$$

Choose $\delta_2 > 0$ small. Then function $u_0(r)+\delta_2 t$ satisfies (2.1). Moreover, for small $\sigma_1 > 0$ and $t \in [0, \sigma_1]$, this function also satisfies (2.2), which is equivalent to saying that the graph of $u_0(r) + \delta_2 t$ contacts $\partial_i \Omega_m$ with angles not less than $\frac{\pi}{2}$. Therefore, $u_0(r) + \delta_2 t$ is a lower solution of (1.2)–(1.3) on $t \in [0, \sigma_1)$.

Assume $u(r,t)$ is a solution of (1.2)–(1.3) with initial data $u_0(r)$ on time-interval $0 \le t < \sigma_2$. Denote $\sigma_3 := \min\{\sigma_1, \sigma_2\}$. Then using the comparison principle to solution $u(r,t)$ and lower solution $u_0(r) + \delta_2 t$ we have

$$u(r,t) \ge u_0(r) + \delta_2 t, \qquad 0 \le t < \sigma_3.$$

For any given $t_1, t_2 \in [0, \sigma_3)$ with $t_2 > t_1$, the above inequality implies that

$$u(r, t_2 - t_1) \ge u_0(r) + \delta_2(t_2 - t_1) > u_0(r).$$

Note that a solution starting at $u(r, t_2 - t_1)$ is nothing but $u(r, t + t_2 - t_1)$. Applying again the comparison principle again to $u(r, t + t_2 - t_1)$ and to solution $u(r,t)$ (which starts at $u_0(r)$), we have

$$u(r, t + t_2 - t_1) > u(r,t), \qquad 0 \le t < \sigma_2 - (t_2 - t_1).$$

Especially, at $t = t_1$, it is $u(r, t_2) > u(r, t_1)$. Since $t_1, t_2 \in (0, \sigma_3)$ with $t_2 > t_1$ can be chosen arbitrarily, we have

$$u_t(r,t) \ge 0, \qquad 0 \le t < \sigma_3.$$

Finally, it is easily seen that this inequality holds indeed on $[0, \sigma_2)$.

2.1.3. Rotating waves in trivial annuli. In this part we study rotating waves in trivial annuli. We shall select two such rotating waves as the lower solution and the upper solution, and then we can give an a priori estimate of $|u|$ for finite time T by using the lower and upper solutions.

Let $\widetilde{H}$ and $\widetilde{G}$ be constants satisfying $\widetilde{H} = H + O(\varepsilon), \widetilde{G} = G + O(\varepsilon)$. Let $\gamma_1, \gamma_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and consider the two-point boundary value problem of the ODE

$$\text{(2.3)} \qquad \begin{cases} \omega = \dfrac{\varphi''}{1 + r^2 \varphi'^2} + \dfrac{2\varphi' + r^2 \varphi'^3}{r(1 + r^2\varphi'^2)} + A\dfrac{\sqrt{1 + r^2 \varphi'^2}}{r}, & \widetilde{H} < r < \widetilde{G}, \\ \varphi'(\widetilde{H}) = \dfrac{\tan \gamma_1}{\widetilde{H}}, \quad \varphi'(\widetilde{G}) = \dfrac{-\tan \gamma_2}{\widetilde{G}}, \quad \varphi(\widetilde{H}) = 0. \end{cases}$$

If there exist ω and $\varphi(r) = \varphi(r; \omega, \widetilde{H}, \widetilde{G}, \gamma_1, \gamma_2)$ satisfying (2.3), then we call the pair $(\omega, \varphi(r))$ a solution of (2.3). This solution determines a rotating wave $\varphi(r) + \omega t$ of (1.2) in annulus $\{(r,\theta) | \widetilde{H} < r < \widetilde{G}\}$, and its graph contacts $r = \widetilde{H}$ (resp., $r = \widetilde{G}$) with angle $\frac{\pi}{2} + \gamma_1$ (resp., $\frac{\pi}{2} + \gamma_2$).

LEMMA 2.3. (i) *Assume* (1.9) *holds and* ε *is small. Let* $\gamma_i \in [0, \alpha_i + \zeta_i \varepsilon)$ $(i = 1, 2)$ *for* $\zeta_i = O(1)$. *Then* (2.3) *has a unique solution* $(\omega_l, \varphi(r; \omega_l))$, *and* $\omega_l > 0$.

(ii) *Let* $\gamma_i = -\beta_i + O(\varepsilon)$ $(i = 1, 2)$. *Then* (2.3) *has a unique solution* $(\omega_u, \varphi(r; \omega_u))$, *and* $\omega_u > 0$.

(iii) $\omega_l < \omega_u$.

Proof. Set $\psi(r) = \varphi'(r)$, and consider the following initial value problem:

$$\text{(2.4)} \qquad \begin{cases} \psi' = \omega(1 + r^2 \psi^2) - \dfrac{2\psi + r^2 \psi^3}{r} - A\dfrac{(1 + r^2\psi^2)^{3/2}}{r}, & r \ge \widetilde{H}, \\ \psi(\widetilde{H}) = \dfrac{\tan \gamma_1}{\widetilde{H}}. \end{cases}$$

For each ω, denote the solution of (2.4) by $\psi(r;\omega)$. It is clear that $\psi(r;\omega)$ is strictly increasing in ω.

(i) First, when $\omega \geq A/\widetilde{H}$, we have $\psi(r) \geq 0$ $(r > \tilde{H})$.

Next, when $\omega = 0$, the solution of (2.4) is

$$\psi(r;0) = \frac{\frac{L}{r} - \frac{Ar}{2}}{r\sqrt{1-(\frac{L}{r}-\frac{Ar}{2})^2}} \qquad \text{for} \quad r \in [\widetilde{H}, \tilde{r}),$$

where $L = \frac{A}{2}\widetilde{H}^2 + \widetilde{H}\sin\gamma_1$ and $\tilde{r} = \frac{1+\sqrt{1+2AL}}{A}$. If $\tilde{r} \leq \widetilde{G}$, then for some $\omega_1 > 0$, $\psi(r;\omega_1)$ is defined on $[\widetilde{H}, \widetilde{G}]$ and $\psi(\widetilde{G};\omega_1) < \frac{-\tan\gamma_2}{\widetilde{G}}$. If $\tilde{r} > \widetilde{G}$, then $\psi(\widetilde{G};0) < \frac{-\tan\gamma_2}{\widetilde{G}}$ if and only if the following holds:

$$A > \frac{2\widetilde{H}\sin\gamma_1 + 2\widetilde{G}\sin\gamma_2}{\widetilde{G}^2 - \widetilde{H}^2}. \tag{2.5}$$

This is true since (1.9) holds and ε is small. Then we always have $\omega_2 \geq 0$ such that $\psi(\widetilde{G};\omega_2) < \frac{-\tan\gamma_2}{\widetilde{G}}$.

Therefore, there is a unique $\omega_l > 0$ such that the solution $\psi(r;\omega_l)$ of (2.4) is defined on $[\widetilde{H}, \widetilde{G}]$ and $\psi(\widetilde{G};\omega_l) = \frac{-\tan\gamma_2}{\widetilde{G}}$, which determines a solution of (2.3): $\varphi(r) = \int_{\widetilde{H}}^r \psi(\zeta;\omega_l)d\zeta$.

(ii) can be proved in a way similar to (i) above, and (iii) is verified by an easy analysis of (2.4). ◻

Now we use this lemma to construct lower and upper solutions. We show that for appropriate choice of $\widetilde{H}$, $\widetilde{G}$, γ_1, and γ_2, the rotating wave $\varphi(r;\omega,\widetilde{H},\widetilde{G},\gamma_1,\gamma_2) + \omega t$, given by the unique solution of (2.3), is a lower/upper solution of (1.2)–(1.3).

Remark 2.1. We remark that it is complicated to state the *optimal* lower solution (cf. section 2.5). Here by *optimal* lower solution we mean a rotating wave whose graph contacts $\partial\Omega_m$ with angles not smaller than $\frac{\pi}{2}$ and, at some points, exactly $\frac{\pi}{2}$. For example, even if we choose a rotating wave whose graph contacts $\partial_1\Omega_m$ perpendicularly at point $(H - h(s_1), s_1)$ where $h'(s_1) = H\tan\alpha_1$, and contacts $\partial_2\Omega_m$ perpendicularly at point $(G + g(s_2), s_2)$ where $g'(s_2) = G\tan\alpha_2$, we are not yet clear whether the angles between the graph of the rotating wave and $\partial\Omega_m$ are larger than $\frac{\pi}{2}$ at other places, because this depends on the geometry of $\partial\Omega_m$ and the shape of the rotating wave. So for simplicity, instead of constructing the *optimal* lower solution, we construct a *good* lower solution, which is a rotating wave whose graph contacts $\partial\Omega_m$ with angles not smaller than $\frac{\pi}{2}$, and equals $\frac{\pi}{2} + O(\varepsilon)$ at some points.

In what follows, we shall use many positive constants μ, C, ζ_1, ζ_2, etc., which may be different from line to line and may depend on some of h, g, H, G, A, m and sometimes on t. When such a constant depends only on h', g', H, G, A, or even if it depends on h, g, and m but can be replaced by another constant independent of h, g, and m (as $m \to \infty$), then we will omit the dependence on h, g, h', g', H, G, A, m and just write it simply as μ or C; only when μ or C really depends on m and t do we write it out clearly, i.e., $\mu(t)$, $C(t,m)$, etc.

LEMMA 2.4. *Assume* (1.9) *holds and* ε *is small. Then* (1.2)–(1.3) *has a lower solution* $\widehat{\varphi}(r) + \widehat{\omega}t$ *and upper solution* $\widetilde{\varphi}(r) + \widetilde{\omega}t$.

Let $u(r,t)$ $(t \in [0,T))$ *be the solution of* (1.2)–(1.3) *with initial data* $u_0(r)$ *as in subsection* 2.1.2. *Then for* $t_0 \in [0,T)$ *and* $t \in [0, T-t_0)$ *we have*

$$\widehat{\omega}t - C \leq u(r, t+t_0) - u(r,t_0) \leq \widetilde{\omega}t + C \tag{2.6}$$

for some $C > 0$.

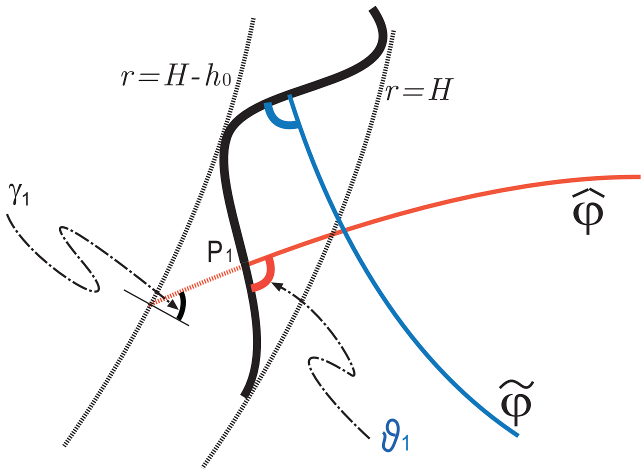

FIG. 2.1. Good *lower solution.*

Proof. Denote $h_0 = \max_s h(s)$, $g_0 = \max_s g(s)$. Consider

$$\text{(2.7)}\qquad \begin{cases} \omega = \dfrac{\varphi''}{1+r^2\varphi'^2} + \dfrac{2\varphi' + r^2\varphi'^3}{r(1+r^2\varphi'^2)} + A\dfrac{\sqrt{1+r^2\varphi'^2}}{r}, \quad H - h_0 < r < G + g_0, \\ \varphi'(H-h_0) = \dfrac{\tan\gamma_1}{H-h_0}, \qquad \varphi'(G+g_0) = \dfrac{-\tan\gamma_2}{G+g_0}, \qquad \varphi(H-h_0) = 0, \end{cases}$$

with $\gamma_i = \alpha_i + \zeta_i\varepsilon$ $(i = 1, 2)$. Denote the unique solution of this problem by $(\widehat{\omega}, \widehat{\varphi}(r))$. Suppose that the graph of $\widehat{\varphi}(r)$ contacts $\partial_1\Omega_m$ (resp., $\partial_2\Omega_m$) at P_1 (resp., P_2) with angle ϑ_1 (resp., ϑ_2) (see Figure 2.1).

Then a careful analysis shows that, for large ζ_i $(i = 1, 2)$ with order $O(1)$, we have $\vartheta_i \geq \frac{\pi}{2}$ $(i = 1, 2)$, and there exist s_1 and s_2 such that

$$\text{(2.8)}\qquad \vartheta_1 = \frac{\pi}{2} + O(\varepsilon) \text{ at } (H - h(s_1), s_1), \qquad \vartheta_2 = \frac{\pi}{2} + O(\varepsilon) \text{ at } (G + g(s_2), s_2).$$

Hence $\widehat{\varphi}(r) + \widehat{\omega}t$ is a *good* lower solution of (1.2)–(1.3).

In a similar way, using (ii) in Lemma 2.3 one can find a solution $(\widetilde{\omega}, \widetilde{\varphi}(r))$ of (2.3) with $\gamma_i = -\beta_i - \zeta_i\varepsilon$. It is easy to see that when $\zeta_i > 0$ is large, the graph of $\widetilde{\varphi}(r) + \widetilde{\omega}t$ contacts $\partial\Omega_\varepsilon$ with angles smaller than $\frac{\pi}{2}$, and equal to $\frac{\pi}{2} + O(\varepsilon)$ at some points. This means that $\widetilde{\varphi}(r) + \widetilde{\omega}t$ is an upper solution of (1.2)–(1.3).

Now if $u(r, t)$ is the solution of (1.2)–(1.3) on $[0, T)$, then we denote $\operatorname{osc} u(r, t) := \max_r u(r, t) - \min_r u(r, t)$ to be the oscillation of $u(r, t)$. By (i) of Lemma 2.5 below, which has nothing to do with this lemma, there exists C such that

$$\operatorname{osc} \widehat{\varphi}(r), \quad \operatorname{osc} \widetilde{\varphi}(r), \quad \operatorname{osc} u(r, t) \leq C.$$

For any $t_0 \in [0, T)$ and $t \in [0, T - t_0)$, we have

$$\widehat{\varphi}(r) + u(0, t_0) - 2C \leq u(r, t_0) \leq \widetilde{\varphi}(r) + u(0, t_0) + 2C,$$

$$\widehat{\varphi}(r) + \widehat{\omega}t + u(0, t_0) - 2C \leq u(r, t + t_0) \leq \widetilde{\varphi}(r) + \widetilde{\omega}t + u(0, t_0) + 2C.$$

Hence

$$\widehat{\omega}t - 6C \leq u(r, t + t_0) - u(r, t_0) \leq \widetilde{\omega}t + 6C. \qquad \square$$

2.1.4. A priori estimates and global existence. In this subsection, we give some a priori estimates and then prove the global existence of solutions of (1.2)–(1.3). Since the proofs are very long and are not our main purpose in this paper, we move them to the appendices.

For any $T > 0$, denote $Q_T := \{(r,t) \mid \eta_1(t) < r < \eta_2(t) \text{ and } 0 < t \le T\}$.

LEMMA 2.5. *Let $u(r,t) \in C^{2,1}(\overline{Q_T})$ be a solution of* (1.2)–(1.3) *with initial data $u_0(r)$. Then for $(r,t) \in \overline{Q_T}$ we have*

(i) $|u_r(r,t)| < \overline{M}$;

(ii) *there exist $\mu(T) > 0$ and $C > 0$ (independent of T) such that $|u_r|_\mu \le C$;*

(iii) *there exist $\mu(T) > 0$ and $C > 0$ (independent of T) such that $|u|_\mu \le C$.*

Based on Lemmas 2.4 and 2.5, we have the following result.

LEMMA 2.6. *Assume* (1.9) *holds. Then* (1.2)–(1.3) *with initial data u_0 has a unique, global solution $u(r,t)$ which satisfies $u_t(\cdot,t) \ge 0$.*

Moreover, for any $T > 0$, there exist positive constants $\mu(T)$, C_1, and C_2 (C_1, C_2 are independent of T) such that

$$u \in C^{2+\mu,1+\frac{\mu}{2}}(\overline{Q_T}) \quad \text{and} \quad \|u(r,t)\|_{C^{2+\mu,1+\frac{\mu}{2}}(\overline{Q_T})} \le C_1 T + C_2. \tag{2.9}$$

2.2. Existence of entire solutions. A solution defined on $t \in (-\infty,\infty)$ is called an *entire solution*. We use the *renormalization method* to show the existence of entire solutions.

LEMMA 2.7. *Equations* (1.2)–(1.3) *have an entire solution $U(r,t)$ such that $U_t(r,t) \ge 0$ and*

$$\widehat{\omega} t - \widehat{C} \le U(r,t+t_0) - U(r,t_0) \le \widetilde{\omega} t + \widetilde{C} \qquad \text{for } t_0 \in \mathbf{R} \text{ and } t \ge 0, \tag{2.10}$$

for some $\widehat{C}, \widetilde{C} > 0$.

Proof. Let u be the global solution of (1.2)–(1.3) obtained in Lemma 2.6. Take $t_n \to \infty$ in the following way:

$$\max_r u(r,t_n) = n \cdot 2\pi\varepsilon \qquad (n = n_0, n_0+1, \ldots),$$

where n_0 is a large integer. Set

$$u_n(r,t) := u(r,t+t_n) - n \cdot 2\pi\varepsilon.$$

Then u_n also satisfies (1.2)–(1.3) for $-t_n \le t < \infty$, and

$$\max_r u_n(r,0) = 0, \qquad \frac{\partial u_n}{\partial t} \ge 0 \qquad (n = n_0, n_0+1, \ldots).$$

By (2.6), there exist $\widehat{C}, \widetilde{C} > 0$ such that

$$\widehat{\omega} t - \widehat{C} \le u_n(r,t) \le \widetilde{\omega} t + \widetilde{C} \qquad \text{for} \quad -t_n \le t < \infty. \tag{2.11}$$

For any given $T > 0$, consider the problem about u_n (for large n) on $[-T,T]$. One can see that (i)–(iii) of Lemma 2.5 and (2.9) remain valid for u_n, the constant μ depends on T, and neither μ nor C depend on n. Therefore there exist $\mu = \mu(T) > 0$, $U(r,t) \in C^{2+\mu,1+\frac{\mu}{2}}(\overline{Q_T^U})$, and a sequence $n_j \to \infty$ ($j \to \infty$) such that

$$u_{n_j}(r,t) \to U(r,t) \qquad \text{in} \quad C^{2+\mu,1+\frac{\mu}{2}}(\overline{Q_T^U}),$$

where $Q_T^U := \{(r,t) \mid t \in [-T,T], r \text{ with } (r,U(r,t)) \in \Omega_m\}$.

Taking $T \to \infty$ and using Cantor's diagonal argument, one finds that there exists a sequence, still writing it as n_j ($n_j \to \infty$ as $j \to \infty$) and $U(r,t) \in C^{2,1}(\overline{Q^U_\infty})$ with $Q^U_\infty = \lim_{T\to\infty} Q^z_T$, such that for any $T > 0$, $U(r,t) \in C^{2+\mu(T),1+\frac{\mu(T)}{2}}(\overline{Q^U_T})$ for some $\mu(T) > 0$ and

$$u_{n_j}(r,t) \to U(r,t) \qquad \text{in} \quad C^{2+\mu(T),1+\frac{\mu(T)}{2}}(\overline{Q^U_T}).$$

Hence U is an entire solution of (1.2)–(1.3). It is also easy to see that U satisfies (2.10), $\max_r U(r,0) = 0$, and satisfies all the conclusions for u in Lemmas 2.5 and 2.6. □

2.3. Uniqueness of entire solution. Assume $U(r,t)$ and $W(r,t)$ are two entire solutions of (1.2)–(1.3) satisfying (2.10) and the conclusions for u in Lemmas 2.5 and 2.6. We shall prove that U is a time-shift of W. Define

$$\Lambda_{U,W}(t) := \inf\{\Lambda > 0 \mid \exists\ a \in \mathbf{R} \text{ such that } U(r,t+a) \le W(r,t) \le U(r,t+a+\Lambda)\}.$$

LEMMA 2.8. (i) *$\Lambda_{U,W}(t)$ is monotone decreasing, and there exists $M > 0$ such that $0 \le \Lambda_{U,W}(t) \le M$ for $t \in \mathbf{R}$.*

(ii) *If $\Lambda_{U,W}(t_0) = 0$ for some t_0, then there exists $a \in \mathbf{R}$ such that $U(r,t+a) \equiv W(r,t)$ for $t \ge t_0$. If $\Lambda_{U,W}(t_0) > 0$ for some t_0, then $\Lambda_{U,W}(t) > 0$ and is strictly decreasing for $t < t_0$.*

Proof. (i) For any $t \in \mathbf{R}$, by the definition of $\Lambda_{U,W}(t)$ there exist r_1 and r_2 such that

$$U(r_1, t+a) = W(r_1,t), \quad W(r_2,t) = U(r_2, t+a+\Lambda_{U,W}(t)),$$

so

$$\begin{aligned}&\max_r U(r,t+a+\Lambda_{U,W}(t)) - \min_r U(r,t+a)\\ &\le U(r_2,t+a+\Lambda_{U,W}(t)) + \operatorname{osc} U - (U(r_1,t+a) - \operatorname{osc} U)\\ &\le W(r_2,t) - W(r_1,t) + 2\overline{M}(G-H) \le 3\overline{M}(G-H).\end{aligned}$$

On the other hand, (2.10) implies that

$$\begin{aligned}&\max_r U(r,t+a+\Lambda_{U,W}(t)) - \min_r U(r,t+a)\\ &\ge U(r,t+a+\Lambda_{U,W}(t)) - U(r,t+a) \ge \widehat{\omega}\cdot\Lambda_{U,W}(t) - \widehat{C}.\end{aligned}$$

Hence

$$0 \le \Lambda_{U,W}(t) \le \frac{3\overline{M}(G-H)+\widehat{C}}{\widehat{\omega}}.$$

(ii) The first statement in (ii) is clear by the uniqueness. If $\Lambda_{U,W}(t_0) > 0$ for some t_0, then for any $t < t_0$ we have

$$U(r,t+a) \le W(r,t) \le U(r,t+a+\Lambda_{U,W}(t)) \qquad \text{for some } a = a(t).$$

By the strong comparison principle, after time $\tau = t_0 - t > 0$ we have

$$U(r,t+\tau+a(t)) < W(r,t+\tau) < U(r,t+\tau+a(t)+\Lambda_{U,W}(t)),$$

i.e.,

$$U(r, t_0 + a(t)) < W(r, t_0) < U(r, t_0 + a(t) + \Lambda_{U,W}(t)).$$

By the definition of $\Lambda_{U,W}(t_0)$ we have

$$U(r, t_0 + a(t_0)) \le W(r, t) \le U(r, t_0 + a(t_0) + \Lambda_{U,W}(t_0))$$

for some $a(t_0)$; each of the two equalities holds at some r. Since $U_t \ge 0$ we have

$$a(t) < a(t_0) \quad \text{and} \quad a(t) + \Lambda_{U,W}(t) > a(t_0) + \Lambda_{U,W}(t_0),$$

and so $\Lambda_{U,W}(t) > \Lambda_{U,W}(t_0)$. ∎

It is also easy to know the following.

LEMMA 2.9. *Let U_n and W_n be two sequences of entire solutions of* (1.2)–(1.3). *If $U_n \to U_\infty$ for $t \in \mathbf{R}$ and r with $(r, U_\infty(r,t)) \in \Omega_m$, $W_n \to W_\infty$ for all $t \in \mathbf{R}$ and r with $(r, W_\infty(r,t)) \in \Omega_m$, then $\Lambda_{U_n,W_n}(t) \to \Lambda_{U_\infty,W_\infty}(t)$ for every t.*

Our aim in this section is to prove the following.

LEMMA 2.10. *$W(r,t)$ is a time-shift of $U(r,t)$.*

Proof. We need to show only that $\Lambda_{U,W}(t) = 0$ for all $t \in \mathbf{R}$. If this is not true, then $\Lambda_{U,W}(t_0) > 0$ for some t_0.

By the monotonicity and boundedness of $\Lambda_{U,W}(t)$, we have $\lim_{t\to-\infty} \Lambda_{U,W}(t) =: \bar{\Lambda}$ for some $\bar{\Lambda}$ satisfying $0 < \Lambda_{U,W}(t_0) < \bar{\Lambda} \le M$.

Set $l_n = \left[\max_r U(r, -n) \cdot \frac{m}{2\pi}\right]$ and define

$$U_n(r,t) := U(r, t-n) - \frac{2\pi}{m} l_n, \qquad W_n(r,t) := W(r, t-n) - \frac{2\pi}{m} l_n.$$

Then both of U_n and W_n satisfy the inequalities for u_n in (2.11), and so a discussion similar to that in the proof of Lemma 2.7 shows that there exists a sequence $n_j \to \infty$ $(j \to \infty)$ and $U_\infty, W_\infty \in C^{2,1}$, which are entire solutions of (1.2)–(1.3), such that as $j \to \infty$,

$$U_{n_j}(r,t) \to U_\infty(r,t) \qquad \text{for } t \in \mathbf{R} \quad \text{and} \quad r \text{ with } (r, U_\infty(r,t)) \in \Omega_m,$$

$$W_{n_j}(r,t) \to W_\infty(r,t) \qquad \text{for } t \in \mathbf{R} \quad \text{and} \quad r \text{ with } (r, W_\infty(r,t)) \in \Omega_m.$$

It follows from Lemma 2.9 that $\Lambda_{U_\infty,W_\infty}(t) = \lim_{j\to\infty} \Lambda_{U_{n_j},W_{n_j}}(t)$.

On the other hand, $\Lambda_{U_{n_j},W_{n_j}}(t) = \Lambda_{U,W}(t - n_j)$, so $\Lambda_{U_\infty,W_\infty}(t) = \lim_{j\to\infty} \Lambda_{U,W}(t - n_j) = \bar{\Lambda}$, that is, $\Lambda_{U_\infty,W_\infty}(t) \equiv \bar{\Lambda}$ $(t \in \mathbf{R})$. Applying (ii) of Lemma 2.8 to functions U_∞ and W_∞ we see that this is true only if $\bar{\Lambda} = 0$, a contradiction to $\bar{\Lambda} > \Lambda_{U,W}(t_0) > 0$.

Therefore, $\Lambda_{U,W}(t) = 0$ (for all $t \in \mathbf{R}$), and so there exists a_0 such that $U(r, t + a_0) \equiv W(r,t)$ for $t \in \mathbf{R}$. ∎

2.4. Existence and uniqueness of periodic rotating wave.

Proof for the sufficiency part of Theorem 1.1. In previous sections we obtained an entire solution $U(r,t)$ of (1.2)–(1.3). Clearly, $U(r,t) + \frac{2\pi}{m}$ is also an entire solution; Lemma 2.10 implies that $U(r,t) + \frac{2\pi}{m}$ is a time-shift of $U(r,t)$, i.e., there exists $T_m > 0$ such that

$$U(r,t) + \frac{2\pi}{m} = U(r, t + T_m) \qquad \text{for } t \in \mathbf{R} \quad \text{and} \quad r \text{ with } (r, U(r,t)) \in \Omega_m.$$

In other words, $U(r,t)$ is a periodic rotating wave.

The uniqueness follows from the uniqueness of the entire solution.

2.5. Necessity of (1.9). We show that (1.9) is a necessary condition for the existence of counterclockwise periodic rotating waves with average rotating speeds $O(1)$.

(i) Assume $A = \frac{2H\sin\alpha_1 + 2G\sin\alpha_2}{G^2 - H^2}$.

From the proof of Lemma 2.3, one sees that when $\widetilde{H} = H, \widetilde{G} = G, \gamma_1 = \alpha_1, \gamma_2 = \alpha_2$, (2.3) has the solution $(0, \varphi(r; 0, H, G, \alpha_1, \alpha_2))$.

Extend this solution $\varphi(r)$ a little beyond $[H, G]$; suppose its graph contacts $\partial_1\Omega_m$ (resp., $\partial_2\Omega_m$) at a point $P_1' = (H - h(s_1'), s_1')$ for some s_1' (resp., $P_2' = (G + g(s_2'), s_2')$ for some s_2'). Choose $s_i \in (s_i', s_i' + 2\pi\varepsilon)$ $(i = 1, 2)$ such that $h'(s_1) = H\tan\alpha_1$, $g'(s_2) = G\tan\alpha_2$. Denote $P_1 = (H - h(s_1), s_1)$, $P_2 = (G + g(s_2), s_2)$. Then when we move the extended graph of $\varphi(r)$ such that it contacts $\partial_1\Omega_m$ at P_1, the angle ϑ_1 between the graph of $\varphi(r)$ and $\partial_1\Omega_m$ will be $\frac{\pi}{2} + O(\varepsilon)$. A similar conclusion is true at $P_2 \in \partial_2\Omega_m$.

Let us see the shape of $\partial_i\Omega_m$ near P_i $(i = 1, 2)$. First, at P_1, $h'(s_1) = \max_s h'(s) = H\tan\alpha_1$, $h''(s_1) = 0$ and $h'''(s_1) = O(\frac{1}{\varepsilon^2})$. Then for $\Delta s \in (-\varepsilon^{3/2}, \varepsilon^{3/2})$, we have

$$\begin{aligned} h'(s_1 + \Delta s) &= H\tan\alpha_1 + \frac{h'''(s_1)}{2}(\Delta s)^2 + h^{(4)}(s^*)(\Delta s)^3 \\ &= H\tan\alpha_1 + O\left(\left(\frac{\Delta s}{\varepsilon}\right)^2\right) = h'(s_1) + O(\varepsilon). \end{aligned}$$

A similar discussion is valid on $\partial_2\Omega_m$ near P_2.

Since the solution of (2.7) depends on γ_i continuously, we know that when $\gamma_i = \alpha_i - \zeta_i\varepsilon$ ($\zeta_i > 0$ are large), there is a unique $\omega^s > 0, \omega^s = O(\varepsilon)$ and $\varphi^s(r; \omega^s)$ solves (2.7). Moreover, the graph of $\varphi^s(r) + \omega^s t$ contacts $\partial_1\Omega_m$ at points $(H - h(s), s)$ for $s \in (s_1 - \varepsilon^{3/2}, s_1 + \varepsilon^{3/2})$ with angles less than $\frac{\pi}{2}$, and contacts $\partial_2\Omega_m$ at points $(G + g(s), s)$ for $s \in (s_2 - \varepsilon^{3/2}, s_2 + \varepsilon^{3/2})$ with angles less than $\frac{\pi}{2}$.

Suppose the end points of $\varphi^s(r) + \omega^s t$ reach $s_i + \varepsilon^{3/2}$ at time t_i $(i = 1, 2)$ and denote $t_3 := \min\{t_1, t_2\}$; then the discussion in the previous paragraph shows that $\varphi^s(r) + \omega^s t$ is an upper solution on $t \in [0, t_3]$. In this period the speed of $\varphi^s(r) + \omega^s t$ is $\omega^s = O(\varepsilon)$, and the θ-distance of this period is $O(\varepsilon^{3/2})$. Hence the upper solution $\varphi^s(r) + \omega^s t$ uses more time than $C\varepsilon^{1/2}$ (for some $C > 0$) to pass this period. Any rotating wave will be blocked by this kind of upper solution in each period of Ω_m such that the rotating wave passes one period in time greater than $C\varepsilon^{1/2}$. In other words, its average speed is at most $O(\varepsilon^{1/2})$. This is not the case we are interested in here.

(ii) In case $A < \frac{2H\sin\alpha_1 + 2G\sin\alpha_2}{G^2 - H^2}$, one can even find a curve (like $\varphi^s(r)$ above) such that it rotates *clockwise temporarily* in a small period, and it blocks *counterclockwise* rotation.

Consequently, (1.9) is a necessary condition.

2.6. Stability. One has the following stability result by general theory in [19].

THEOREM 2.11. *Let $U(r, t)$ be a counterclockwise periodic rotating wave of* (1.2)–(1.3). *Then $U(r, t)$ is asymptotically stable in the C^1 sense. More precisely, any solution $u(r, t)$ of* (1.2)–(1.3) *whose initial data $u(r, 0)$ is sufficiently close to $U(r, t_0)$ for some $t_0 \in \mathbf{R}$ satisfies*

$$\lim_{t\to\infty} \|U(\cdot, t + t_1) - u(\cdot, t)\|_{C^1} = 0$$

for some $t_1 \in \mathbf{R}$, where $\|\cdot\|_{C^1}$ is understood in the following sense: $U(r, t)$ and $u(r, t)$ correspond to $V(z, t)$ $(z \in (0, 1))$ and $v(z, t)$ $(z \in (0, 1))$, respectively, by using (A.2) *in the appendices. The above limit is understood as*

$$\lim_{t\to\infty} \|V(z,\, t+t_1) - v(z,\, t)\|_{C^1} = 0.$$

3. Estimate of average speed: Proof of Theorem 1.2.

3.1. Proof of (i) in Theorem 1.2. Recall that in section 2.1 we construct a *good* lower solution $\widehat{\varphi}(r) + \widehat{\omega}t$, so the average rotating speed ω_m satisfies $\omega_m \geq \widehat{\omega}$. Denote by $(\omega^*, \varphi^*(r))$ the solution of (1.11) (that is, (2.3) with $\widetilde{H} = H,\ \widetilde{G} = G,\ \gamma_1 = \alpha_1,\ \gamma_2 = \alpha_2$). Then it is easy to see by the proofs of Lemmas 2.3 and 2.4 that

$$\omega^* = \widehat{\omega} + O(\varepsilon), \quad \varphi^* = \widehat{\varphi} + C + O(\varepsilon), \quad {\varphi^*}' = \widehat{\varphi}\,' + O(\varepsilon). \tag{3.1}$$

Therefore $\omega_m > \omega^* - \frac{C}{m}$ for some $C > 0$.

Also from the proofs of Lemmas 2.3 and 2.4 we have $\widehat{\omega} < \omega_0$ and so $\omega^* < \omega_0$.

This proves the first and the third inequalities of (1.10).

3.1.1. Upper solution. Now we use $\widehat{\varphi}(r) + \widehat{\omega}t$ to construct an upper solution. Let $U(r,t)$ be the periodic rotating wave of (1.2)–(1.3). We note that $U(r,t)\big|_{[H,G]}$ is nothing but the solution of

$$\begin{cases} \tilde{u}_t = \dfrac{\tilde{u}_{rr}}{1+r^2\tilde{u}_r^2} + \dfrac{2\tilde{u}_r + r^2\tilde{u}_r^3}{r(1+r^2\tilde{u}_r^2)} + A\dfrac{\sqrt{1+r^2\tilde{u}_r^2}}{r}, & t>0,\ H<r<G, \\ \tilde{u}(H,t) = U(H,t), \quad \tilde{u}(G,t) = U(G,t), & t>0, \\ \tilde{u}(r,0) = U(r,0), & H<r<G. \end{cases} \tag{3.2}$$

Without loss of generality, we may assume $U(r,0) \leq \widehat{\varphi}(r)$ for $r \in [H,G]$ and $U(r_0,0) = \widehat{\varphi}(r_0)$ for some $r_0 \in [H,G]$. Recall $\varepsilon = 1/m$ and define

$$w(r,t) = E\sqrt{\varepsilon}\,\left(1 - e^{-a^2 t}\, \sin(ar+b)\right) + aEF\sqrt{\varepsilon}\, t \quad \text{for } r\in[H,G],\ t\geq 0, \tag{3.3}$$

where $E = O(1)$ is determined later, $a = \frac{\pi}{G-H},\ b = \frac{-\pi H}{G-H}$, and

$$F = \max_{H-h_0\leq r\leq G+g_0} |\, F(r)\,| + 2, \tag{3.4}$$

where

$$F = \frac{2(r\widehat{\varphi}'' + 2\widehat{\varphi}' + r^2(\widehat{\varphi}')^3)r^2\widehat{\varphi}' - 2 - 2r^2(\widehat{\varphi}')^2 - 3r^2(\widehat{\varphi}')^2(1+r^2(\widehat{\varphi}')^2)}{r(1+r^2(\widehat{\varphi}')^2)^2} - \frac{Ar\widehat{\varphi}'}{\sqrt{1+r^2(\widehat{\varphi}')^2}}.$$

It is clear that

$$w - aEF\sqrt{\varepsilon}t \geq 0, \qquad w_t = w_{rr} + aEF\sqrt{\varepsilon} > 0, \qquad \min w(r,0) = 0.$$

LEMMA 3.1. *$\bar{u}(r,t) := w(r,t) + \widehat{\varphi}(r) + \widehat{\omega}t$ is an upper solution of* (3.2) *on* $t\in[0,1]$, *and hence*

$$\bar{u}(r,t) \geq U(r,t) \qquad \textit{for } r\in[H,G],\ t\in[0,1]. \tag{3.5}$$

Proof. To prove the lemma, it suffices to show that

$$\bar{u}_t \geq \frac{\bar{u}_{rr}}{1+r^2\bar{u}_r^2} + \frac{2\bar{u}_r + r^2\bar{u}_r^3}{r(1+r^2\bar{u}_r^2)} + A\frac{\sqrt{1+r^2\bar{u}_r^2}}{r} \qquad \text{for } H<r<G,\ t>0, \tag{3.6}$$

and

$$U(H,t) \le \bar{u}(H,t), \qquad t \in [0,1], \tag{3.7}$$

$$U(G,t) \le \bar{u}(G,t), \qquad t \in [0,1]. \tag{3.8}$$

We first prove (3.6). Direct calculation shows that

$$\begin{aligned} &\bar{u}_t - \frac{\bar{u}_{rr}}{1+r^2\bar{u}_r^2} - \frac{2\bar{u}_r + r^2\bar{u}_r^3}{r(1+r^2\bar{u}_r^2)} - A\frac{\sqrt{1+r^2\bar{u}_r^2}}{r} \\ &= w_t - \frac{w_{rr}}{1+r^2(w_r+\widehat{\varphi}')^2} + F_1(r)w_r + O(\varepsilon) \ge 0, \end{aligned}$$

where

$$\begin{aligned} F_1(r) = &\frac{2(r\widehat{\varphi}'' + 2\widehat{\varphi}' + r^2(\widehat{\varphi}')^3)r^2\widehat{\varphi}' - 2 - 2r^2(\widehat{\varphi}')^2 - 3r^2(\widehat{\varphi}')^2(1+r^2(\widehat{\varphi}')^2)}{r(1+r^2(\widehat{\varphi}')^2)(1+r^2(w_r+\widehat{\varphi}')^2)} \\ &- \frac{2Ar\widehat{\varphi}'}{\sqrt{1+r^2(\widehat{\varphi}')^2} + \sqrt{1+r^2(w_r+\widehat{\varphi}')^2}} = F(r) + O(\sqrt{\varepsilon}) \end{aligned}$$

satisfies

$$|F_1(r)| < F - 1 \qquad (\text{note that } \ |w_r| \le aE\sqrt{\varepsilon}\ ).$$

Next we prove (3.7) and (3.8). Suppose that they hold on $t \in [0,\tau]$ for some $\tau < 1$; we show that they hold in fact on $t \in [0,1]$.

Construct an arc $\theta = \zeta(r)$ as follows (see Figure 3.1). Assume that $h'(s_1) = H\tan\alpha_1$. Denote $P = (H - h(s_1), s_1) \in \partial_1\Omega_m$. Choose $\zeta(r)$ to be the arc with curvature $-A$ that contacts $\partial_1\Omega_m$ perpendicularly at P. Without loss of generality, assume that $\zeta(H) = \theta_0 \in (-2\pi\varepsilon, 2\pi\varepsilon)$. Then P and (H,θ_0) are on $\zeta(r)$.

By (2.8), we know that both $\widehat{\varphi}$ and $\zeta(r)$ have almost the same slope at P:

$$\widehat{\varphi}\,'(H-h(s_1)) - \zeta'(H-h(s_1)) = O(\varepsilon).$$

Hence, there exists $M > 0$ such that

$$|\widehat{\varphi}\,'(H+l\sqrt{\varepsilon}) - \zeta'(H+l\sqrt{\varepsilon})| \le (M-1)\sqrt{\varepsilon} \qquad \text{for any } \ l \in [0,1]. \tag{3.9}$$

Choose $D(\tau) > 0$, such that $\zeta(r) + D(\tau)$ intersects $\bar{u}(r,\tau)$ at $r = H + \sqrt{\varepsilon}$, that is, $\zeta(H+\sqrt{\varepsilon}) + D(\tau) = \bar{u}(H+\sqrt{\varepsilon},\tau)$ (see Figure 3.1). Then by (3.9) we have

$$\begin{aligned} D(\tau) &= w(H+\sqrt{\varepsilon},\tau) + \widehat{\varphi}(H+\sqrt{\varepsilon}) + \widehat{\omega}\tau - \zeta(H+\sqrt{\varepsilon}) \\ &= \bar{u}(H,\tau) + w_r(H,\tau)\sqrt{\varepsilon} + \widehat{\varphi}(H+\sqrt{\varepsilon}) - \widehat{\varphi}(H) - \zeta(H+\sqrt{\varepsilon}) + o(\varepsilon) \\ &< \bar{u}(H,\tau) - aEe^{-a^2}\varepsilon + (M+2\pi)\varepsilon. \end{aligned}$$

Therefore when we choose E satisfying $aEe^{-a^2} > 8\pi + M$ we have $D(\tau) < \bar{u}(H,\tau) - 6\pi\varepsilon$.

Since $\zeta(r)$ contacts $\partial_1\Omega_m$ at P perpendicularly, there exists $\delta \in [0, 2\pi\varepsilon)$ such that the graph of $\zeta(r) + D(\tau) + \delta$ contacts $\partial_1\Omega_m$ perpendicularly and hence $\zeta(r) + D(\tau) + \delta$ is stationary. So

$$U(H+\sqrt{\varepsilon},\tau) \le \bar{u}(H+\sqrt{\varepsilon},\tau) \le \zeta(H+\sqrt{\varepsilon}) + D(\tau) + \delta$$

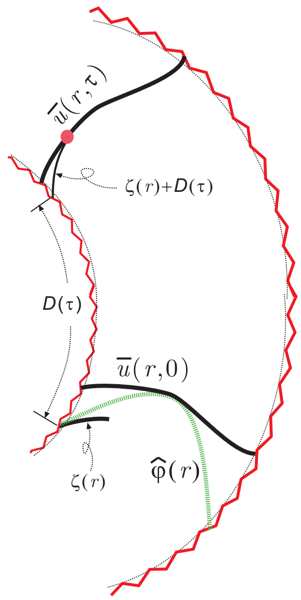

FIG. 3.1. *Upper solution.*

implies that $U(r,\tau) \leq \zeta(r) + D(\tau) + \delta$ for $H < r < H + \sqrt{\varepsilon}$. Especially,

$$U(H,\tau) \leq \zeta(H) + D(\tau) + \delta \leq \bar{u}(H,\tau) - 2\pi\varepsilon.$$

So

$$\bar{u}(H,\tau+t) \geq \bar{u}(H,\tau) \geq U(H,\tau) + 2\pi\varepsilon \geq U(H,\tau+t) \qquad \text{for} \quad t \in [0, T_m].$$

This means that (3.7) holds on $t \in [0, \tau + T_m]$ provided $\tau < 1$. Repeating the above discussion finite times, we obtain (3.7).

Similarly, we can show (3.8) provided that E in the definition of w is chosen large enough. ∎

3.1.2. Proof of the second inequality in (1.10).

$$U(r,1) - \widehat{\varphi}(r) \leq \bar{u}(r,1) - \widehat{\varphi}(r) \leq (E + aEF)\sqrt{\varepsilon} + \widehat{\omega} \leq \left[\frac{(E+aEF)\sqrt{\varepsilon} + \widehat{\omega}}{2\pi\varepsilon} + 1\right] \cdot 2\pi\varepsilon,$$

where $[\cdot]$ is a Gauss function. On the other hand, by the periodicity

$$U\left(r,\ \left[\frac{(E+aEF)\sqrt{\varepsilon} + \widehat{\omega}}{2\pi\varepsilon} + 1\right] \cdot T_m\right) \leq \widehat{\varphi}(r) + \left[\frac{(E+aEF)\sqrt{\varepsilon} + \widehat{\omega}}{2\pi\varepsilon} + 1\right] \cdot 2\pi\varepsilon,$$

where "$=$" holds at $r = r_0$. Hence time 1 is smaller than time $\left[\frac{(E+aEF)\sqrt{\varepsilon}+\widehat{\omega}}{2\pi\varepsilon} + 1\right] \cdot T_m$, and so

$$\omega_m = \frac{2\pi}{mT_m} \leq \widehat{\omega} + \frac{E + aEF + 1}{\sqrt{m}}. \tag{3.10}$$

Therefore (3.1) implies that

$$\omega_m \leq \omega^* + \frac{E + aEF + 2}{\sqrt{m}}.$$

This completes the proof of (i) in Theorem 1.2.

3.2. Proof of (ii) in Theorem 1.2. In this subsection, we write the unique periodic rotating wave constructed in Lemma 2.7 by $U_m(r,t)$. Clearly, the period T_m of $U_m(r,t)$ is smaller than 1 when $m > m_0$ for some m_0.

For any given $T > 0$, by Lemma 2.7, U_m $(m = m_0, m_0+1, \ldots)$ are bounded in $C^{2+\mu,1+\frac{\mu}{2}}(\overline{Q_T^m})$, where $\mu = \mu(T)$ is independent of m and $Q_T^m := \{(r,t)|t \in [-T, T+1],\ r \text{ with } (r, U_m(r,t)) \in \Omega_m\}$. So there exists a sequence $\{m_i\}_{i=0}^{\infty}$ and $\mathcal{U}(r,t) \in C^{2,1}([H,G]\times[-T,T+1])$ such that

$$\|U_{m_i} - \mathcal{U}\|_{C^{2,1}([H,G]\times[-T,T+1])} \to 0 \qquad \text{as } i \to \infty.$$

Thus, for any $(r,t) \in [H,G]\times[-T,T]$, when $i \to \infty$ we have

$$\omega^* \leftarrow \omega_{m_i} = \frac{2\pi}{m_i T_{m_i}} = \frac{1}{T_{m_i}} \int_t^{T_{m_i}+t} \frac{\partial U_{m_i}(r,t)}{\partial t} dt = \frac{\partial U_{m_i}(r,s)}{\partial t} \to \mathcal{U}_t(r,t)$$

with $s \in (t, T_{m_i} + t)$. This means that $\mathcal{U}(r,t) = \mathcal{U}(r) + \omega^* t$ in $[H,G]\times[-T,T]$ for some $\mathcal{U}(r) \in C^2([H,G])$.

U_{m_i} is the solution of (1.2)–(1.3); taking limit $i \to \infty$ in these equations we have

$$(3.11)\quad \begin{cases} \omega^* = \dfrac{\mathcal{U}_{rr}}{1+r^2\mathcal{U}_r^2} + \dfrac{2\mathcal{U}_r + r^2\mathcal{U}_r^3}{r(1+r^2\mathcal{U}_r^2)} + A\dfrac{\sqrt{1+r^2\mathcal{U}_r^2}}{r}, \qquad H \le r \le G, \\[2ex] \mathcal{U}_r(H) \in \left[\dfrac{-\tan\beta_1}{H},\ \dfrac{\tan\alpha_1}{H}\right], \qquad \mathcal{U}_r(G) \in \left[\dfrac{-\tan\alpha_2}{G},\ \dfrac{\tan\beta_2}{G}\right]. \end{cases}$$

Comparing it with (1.11) we find that when $\mathcal{U}_r(H) < \frac{\tan\alpha_1}{H}$, $\mathcal{U}_r(r) < \varphi_r^*(r;\omega^*)$ and so $\mathcal{U}_r(G) < \frac{-\tan\alpha_2}{G}$. Therefore, (3.11) has a solution if and only if

$$\mathcal{U}_r(H) = \frac{\tan\alpha_1}{H}, \qquad \mathcal{U}_r(G) = \frac{-\tan\alpha_2}{G},$$

and the solution $\mathcal{U}(r)$ is nothing but $\varphi^*(r;\omega^*) + C$ for some C. Recall that we require $\max_r U_m(r,0) = 0$ in the proof of Lemma 2.7; hence $\max_r \mathcal{U}(r) = 0$, which implies that $C = -\max_r \varphi^*(r;\omega^*)$.

For any sequence $\{U_{m_j}\}$, there is a subsequence $\{U_{m_{j_k}}\}$ that converges to the same homogenized limit $\varphi^*(r;\omega^*) + \omega^* t - \max_r \varphi^*(r;\omega^*)$ in $C^{2,1}([H,G]\times[-T,T])$ as $k \to \infty$. Consequently, $U_m(r,t) \to \varphi^*(r;\omega^*) + \omega^* t - \max_r \varphi^*(r;\omega^*)$ in $C^{2,1}([H,G]\times[-T,T])$ as $m \to \infty$. This proves (ii) of Theorem 1.2.

Appendix A. Proof of Lemma 2.5.

(i) *of Lemma* 2.5. Set $w = u_r$; then for $t \in (0,T]$ and $\eta_1(t) < r < \eta_2(t)$ we have

$$(\text{A.1})\quad \begin{cases} w_t = \dfrac{w_{rr}}{1+r^2w^2} + a w_r - \dfrac{2+5r^2w^2+r^4w^4}{r^2(1+r^2w^2)^2} w - \dfrac{A}{r^2\sqrt{1+r^2w^2}}, \\ w(\eta_1(t),t) = h'(u(\eta_1(t),t))/\eta_1^2(t), \qquad w(\eta_2,t) = -g'(u(\eta_2(t),t))/\eta_2^2(t), \\ w(r,0) = u_0'(r) \qquad\qquad \text{for } r \text{ with } (r, u_0(r)) \in \Omega_m, \end{cases}$$

where $a = a(r,u,w)$ is a smooth function. By the boundary condition we have

$$-\frac{\tan\beta_1}{H} - \frac{C}{m} < w(\eta_1(t),t) < \frac{\tan\alpha_1}{H} + \frac{C}{m}$$

for some $C > 0$. Hence $|w(\eta_1(t), t)| \leq M_1 + \frac{C}{m} < \overline{M} - \frac{1}{2}$. Similarly, $|w(\eta_2(t), t)| \leq M_1 + \frac{C}{m} < \overline{M} - \frac{1}{2}$.

On the other hand, if w takes the maximum at (r_1, t) with $r_1 \in (\eta_1(t), \eta_2(t))$ and $w(r_1, t) > \overline{M} - \frac{1}{2}$, then we have

$$w_t(r_1, t) < \frac{w_{rr}(r_1, t)}{1 + r^2 w^2(r_1, t)} + a w_r(r_1, t) \leq 0,$$

which implies that w will never be larger than $\overline{M} - \frac{1}{4}$. Similarly, if w takes the minimum at (r_2, t) with $r_2 \in (\eta_1(t), \eta_2(t))$ and $w(r_2, t) < -\overline{M} + \frac{1}{2}$, then, by the definition of M_2 in (1.8), we have

$$w_t(r_2, t) \geq -\frac{2 + 5r^2 w^2 + r^4 w^4}{r^2(1 + r^2 w^2)^2} w - \frac{A}{r^2 \sqrt{1 + r^2 w^2}} \bigg|_{r = r_2} > 0.$$

Therefore, w will never be smaller than $-\overline{M} + \frac{1}{4}$. Thus we obtain (i) of Lemma 2.5.

We prove (ii) and (iii) of Lemma 2.5 by converting the problem of u into a problem about $v(z, t)$ $(z \in (0, 1))$.

Introduce a new variable

$$z := \frac{r - H + h(\theta)}{J(\theta)}$$

with $J(\theta) = G - H + g(\theta) + h(\theta)$. Then in the new coordinates (z, θ), domain Ω_m is expressed as a domain $\{(z, \theta) \mid z \in (0, 1), \theta \in \mathbf{R}\}$. Now, given a solution $u(r, t)$ of (1.2), we define a new unknown $v(z, t)$ by

$$v(z, t) = u(r(z, t), t), \tag{A.2}$$

where $r(z, t)$ is the inverse function of

$$z(r, t) = \frac{r - H + h(u(r, t))}{J(u(r, t))}.$$

Such an inverse function exists if

$$\frac{\partial z}{\partial r} := \frac{J(u) + [(G + g)h' + (H - h)g' - r(g' + h')]u_r}{J^2(u)} \neq 0,$$

that is,

$$J(u) + [(G + g)h' + (H - h)g' - r(g' + h')]u_r \neq 0. \tag{A.3}$$

We will see later that (A.3) always holds for any solution of (1.2)–(1.3) with appropriate initial data. From $\frac{\partial z}{\partial r} \neq 0$, we have inverse function $r(z, t)$ and

$$\frac{\partial r}{\partial z} = J(v) + (g' + h')z v_z - h' v_z \neq 0.$$

As is easily seen, we have

$$u_r = v_z \frac{\partial z}{\partial r} = \frac{v_z}{J(v)} + \frac{[(G + g)h' + (H - h)g' - r(g' + h')]u_r v_z}{J^2(v)}$$

and hence

$$u_r = \frac{J(v)v_z}{J^2(v) - [(G+g)h' + (H-h)g' - r(g'+h')]v_z} = \frac{v_z}{J(v) + I(z,v)v_z},$$

where $I(z,v) = z(g'(v) + h'(v)) - h'(v)$. Similarly,

$$u_{rr} = \frac{Jv_{zz} - 2(g'+h')v_z^2 - (z(g''+h'') - h'')v_z^3}{(J+Iv_z)^3}, \quad u_t = \frac{Jv_t}{J+Iv_z}$$

since

$$\frac{\partial z}{\partial t} = \frac{h' - z(g'+h')}{J}u_t.$$

Hence the problem about u reduces to a problem about v,

$$v_t = \frac{v_{zz}}{K} + f(z,v,v_z) \qquad \text{for } 0 < z < 1,\ t > 0, \tag{A.4}$$

with

$$f(z,v,p) = \frac{(Jz + H - h)p^3 - 2(g'+h')p^2 - [z(g''+h'') - h'']p^3}{JK} + \frac{2p(J+Ip)^2}{JK(Jz+H-h)}$$

and

$$K(z,v,p) = (J+Ip)^2 + p^2(Jz+H-h)^2.$$

Boundary conditions (1.3) reduce to

$$v_z(0,t) = \frac{Jh'}{(H-h)^2 + (h')^2}, \qquad v_z(1,t) = \frac{-Jg'}{(G+g)^2 + (g')^2}. \tag{A.5}$$

Any solution $u(r,t)$ of (1.2)–(1.3) satisfying (A.3) defines a solution $v(z,t)$ of (A.4)–(A.5) by the relation (A.2). Conversely, if $v(z,t)$ is a solution of (A.4)–(A.5), then the function $u(r,t)$ defined by

$$u(r,t) = v\left(\frac{r - H + h(u)}{J(u)}, t\right) \tag{A.6}$$

is a solution of (1.2)–(1.3). For u to be well defined by (A.6), we need to assume that

$$\frac{\partial}{\partial u}\left(u - v\left(\frac{r - H + h(u)}{J(u)}, t\right)\right) \neq 0$$

or, equivalently,

$$J(v) + [z(g'+h') - h']v_z \neq 0. \tag{A.7}$$

We will see later that any solution of (A.4)–(A.5) with appropriate initial data satisfies (A.7) everywhere.

Using initial data $u_0(r)$ constructed in subsection 2.1.2, one can define a smooth function $v_0(z)$ by (A.2). Since

$$\frac{v_{0z}(z)}{J(v_0(z)) + I(z, v_0(z))v_{0z}(z)} \equiv u_{0r}(r) = O(\varepsilon) \qquad \text{for } z \in [0,1],$$

we have $v_{0z}(z) = O(\varepsilon)$. In what follows we consider problem (A.4)–(A.5) with initial data $v_0(z)$. First, we give an a priori estimate of v_z by (i) of Lemma 2.5.

LEMMA A.1. *Let $v(z,t)$ be a solution of* (A.4)–(A.5) *with initial data v_0 on some time-interval $0 \le t < t_1$. Then*

(i) *there exists $\sigma = \sigma(v_0) > 0$ such that*

$$J(v) + I(z,v)v_z \ge \sigma \qquad \textit{for } t \in [0,t_1),\ z \in [0,1];$$

(ii) *there exists $\varrho \in (0,1)$ such that*

$$|v_z(z,t)| \le \frac{G - H + \max(g(s) + h(s))}{\varrho} \cdot \overline{M} \qquad \textit{for } t \in [0,t_1),\ z \in [0,1].$$

Proof. (i) Suppose that

$$J(v(z,t)) + I(z,v(z,t))v_z(z,t) \ge \sigma_1 > 0 \qquad \text{for } \ 0 \le t < \tilde{t} < t_1,\ z \in [0,1];$$

then $u(r,t)$ is well defined by (A.6) on $[0,\tilde{t})$ with

$$u_r(r,t) = \frac{v_z(z,t)}{J(v(z,t)) + I(z,v(z,t))v_z(z,t)}.$$

$u(r,t)$ is a solution of (1.2)–(1.3) and satisfies (i) of Lemma 2.5 on $t \in [0,\tilde{t})$. This means that

$$\text{(A.8)} \qquad \left| \frac{v_z(z,t)}{J(v(z,t)) + I(z,v(z,t))v_z(z,t)} \right| < \overline{M} \qquad \text{for } t \in [0,\tilde{t}),\ z \in [0,1].$$

If there exists $z_0 \in [0,1]$ such that $J(v(z_0,t)) + I(z_0,v(z_0,t))v_z(z_0,t) \to 0$ as $t \nearrow \tilde{t}$, then (A.8) indicates that $v_z(z_0,t) \to 0$ as $t \nearrow \tilde{t}$. However, this implies that

$$J(v(z_0,t)) + I(z_0,v(z_0,t))v_z(z_0,t) \to J(v(z_0,t)) \ge G - H > 0 \qquad \text{as } t \nearrow \tilde{t},$$

a contradiction. This proves (i).

(ii) From (i), we can define a function $u(r,t)$ by (A.6) on $0 \le t < t_1$, which is a solution of (1.2)–(1.3) and satisfies (i) of Lemma 2.5 on $0 \le t < t_1$.

By (1.6), there exists a $\varrho > 0$ such that $|h'|\overline{M} < 1 - 2\varrho$, $|g'|\overline{M} < 1 - 2\varrho$. Therefore, for large m, (i) of Lemma 2.5 implies that

$$\left| \frac{(G+g)h' + (H-h)g' - r(g'+h')}{G - H + g + h} \cdot u_r \right| < \left| \frac{(G-r)h'}{G-H} - \frac{(r-H)g'}{G-H} \right| \overline{M} + \varrho < 1 - \varrho. \tag{A.9}$$

Hence $|u_r| = |v_z| \cdot \left|\frac{\partial z}{\partial r}\right| > \varrho |v_z| / J(v)$ and then

$$|v_z(z,t)| < \frac{G - H + \max(g(s) + h(s))}{\varrho} \overline{M} \qquad \text{for } z \in [0,1],\ t \in [0,t_1).$$

Remark A.1. In fact, from the proof we know that the left-hand side of (A.3) is positive, that is, $\frac{\partial z}{\partial r} > 0$, so $\frac{\partial r}{\partial z} > 0$.

Next we give the Hölder estimate for v_z by a result in [13].

THEOREM A.2 (see [13, Theorem 13.16]). *Let* $\Omega = \omega \times (0,T)$ *for some domain* $\omega \subset \mathbf{R}^n$ *with* $\partial\omega \in H_2$. *Let* $w \in C^{2,1}(\overline{\Omega})$ *be a solution of*

$$\text{(A.10)} \qquad \begin{cases} w_t = \operatorname{div} A(x,t,w,Dw) + B(x,t,w,Dw) & \text{in } \ \Omega, \\ A(x,t,w,Dw)\cdot\gamma + \psi(x,t,w) = 0 & \text{on } \ \partial\omega \times (0,T), \\ w = \varphi & \text{on } \ \omega \times \{0\}, \end{cases}$$

with A *a* C^1 *function of* (x,t,w,p), ψ *a* C^1 *function of* (x,t,w), *and* A *and* B *uniformly continuous with respect to* (x,t,w,p), γ *the inner normal to* $\partial\omega$. *Suppose that there are positive constants* L, λ_L, Λ_L, *and* μ_L *such that* $|w| + |Dw| \le L$ *and*

$$\text{(A.11)} \qquad \frac{\partial A^i}{\partial p_j}\xi_i\xi_j \ge \lambda_L|\xi|^2, \qquad |A_p| \le \Lambda_L,$$

$$\text{(A.12)} \qquad |A_w| + |A_x| + |A_t| + |B| \le \mu_L, \qquad |\psi_w| + |\psi_x| + |\psi_t| + |\psi| \le \mu_L$$

for (x,t,w,p) *with* $|w| + |p| \le L$. *Suppose also that*

$$\text{(A.13)} \qquad |A(x,t,w,p) - A(x,s,w,p)| + |\psi(x,t,w) - \psi(x,s,w)| \le \mu_L|t-s|^{\frac{1}{2}}$$

for $(x,w,p) \in \partial\omega \times \mathbf{R} \times \mathbf{R}^n$ *with* $|w| + |p| \le L$ *and all* $s,t \in (0,T)$. *If also* $\varphi \in H_2$ *and*

$$A(x,t,\varphi,D\varphi)\cdot\gamma + \psi(x,t,\varphi) = 0 \quad \text{on} \quad \partial\omega \times \{0\},$$

then there are positive constants $\alpha = \alpha(L,n,\lambda_L,\Lambda_L)$ *and* $C = C(n,\lambda_L,\Lambda_L,\mu_L,\Omega,|\varphi|_2)$ *such that* $|Du|_\alpha \le C$.

Define

$$A(z,v,p) = \frac{p}{K(z,v,p)} + \int_0^p \frac{pK_p(z,v,p)}{K^2(z,v,p)}dp,$$

$$\mathcal{H}(v) = \frac{J(v)h'(v)}{(H-h(v))^2 + h'(v)^2}, \quad \mathcal{G}(v) = \frac{-J(v)g'(v)}{(G+g(v))^2 + g'(v)^2}.$$

Then (A.4)–(A.5) with initial data $v_0(z)$ can be written in divergence form:

$$\text{(A.14)} \qquad \begin{cases} v_t = \dfrac{\partial}{\partial z}A(z,v,v_z) + B(z,v,v_z), & z \in (0,1), t > 0, \\ v_z(0,t) = \mathcal{H}(v(0,t)), \quad v_z(1,t) = \mathcal{G}(v(1,t)), & t > 0, \\ v(z,0) = v_0(z), & z \in (0,1), \end{cases}$$

where $\frac{\partial A}{\partial z}$ denotes the partial derivative of A on z when we regard A as a function of z and t, and a careful calculation shows that $B(z,v,p)$ is a smooth and bounded function provided v_z is bounded.

LEMMA A.3. *Let* $v(z,t)$ *be a solution of* (A.14) *on some time-interval* $0 \le t \le T$; *then there exist* $\mu(T) > 0$ *and* $C > 0$ (*independent of* T) *such that*

$$\text{(A.15)} \qquad |v_z|_\mu \le C \qquad \text{in} \quad [0,1]\times[0,T].$$

Proof. First, (2.6) gives the a priori estimate of $u \equiv v$:

$$\widehat{\omega}t - C \le v(z,t) < \widetilde{\omega}t + C \qquad \text{for } \ z \in [0,1], \ t \in [0,T].$$

As T goes to $+\infty$, v also goes to $+\infty$. However, one important thing should be noticed: in our problem (A.14), v appears always in a form such as $g(v), h(v), g'(v), h'(v)$, etc. Therefore, the unboundedness of v does not cause the unboundedness of the coefficients in (A.14).

Second, the a priori estimate for v_z is given by (ii) in Lemma A.1: $|v_z(z,t)| \le C$ for $z \in [0,1]$ and $t \in [0,T]$.

A trivial and careful calculation shows that Theorem A.2 stated above is applicable to (A.14) and none of the constants λ, Λ, and μ depend on T (since the bound of z and v_z are independent of T, and v appears in the form $g(v), h(v)$, etc.). So there exist $\mu = \mu(T)$ and $C > 0$ (independent of T) such that (A.15) holds. ∎

Proof of (ii) *of Lemma* 2.5. By $u_r = \frac{v_z}{J(v)+I(z,v)v_z}$ and the a priori estimates of v, v_z, and $|v_z|_\mu$ we have $\mu(T) > 0$ and $C > 0$ (independent of T) such that $|u_r|_{\mu(T)} \le C$ for $(r,t) \in \overline{Q_T}$.

Proof of (iii) *of Lemma* 2.5. Choose a constant $\sigma > 0$ such that

$$\sigma > \max\{\max_s |\mathcal{H}'(s)|, \ \max_s |\mathcal{G}'(s)|\} + 1$$

and define $w(z,t) := v(z,t)e^{\sigma z}$. Then w is a solution of

$$\text{(A.16)} \qquad \begin{cases} w_t = \dfrac{\partial}{\partial z}\widetilde{A}(z,w,w_z) + \widetilde{B}(z,w,w_z), & z \in (0,1), t > 0, \\ w_z(0,t) = \mathcal{H}(v(0,t)) + \sigma w(0,t), & t > 0, \\ w_z(1,t) = \mathcal{G}(v(1,t))e^{\sigma} + \sigma w(1,t), & t > 0, \\ w(z,0) = v_0(z)e^{\sigma z}, & z \in (0,1), \end{cases}$$

where $\widetilde{A}$ and $\widetilde{B}$ are smooth functions like A and B. Theorem A.2 stated above can also be used for this problem and so we have $|w_z|_{\mu(T)} \le C$ for $(z,t) \in \overline{Q_T}$, where $\mu(T) > 0$ and $C > 0$ (independent of T).

Especially, at $z = 0$ we have $|w_z(0,t) - w_z(0,s)| \le C|t-s|^{\mu(T)}$. On the other hand,

$$\begin{aligned} |w_z(0,t) - w_z(0,s)| &= |\mathcal{H}(v(0,t)) - \mathcal{H}(v(0,s)) + \sigma w(0,t) - \sigma w(0,s)| \\ &= |\sigma + \mathcal{H}'(\varsigma)| \cdot |w(0,t) - w(0,s)| \ge |w(0,t) - w(0,s)|. \end{aligned}$$

Hence $|w(0,t) - w(0,s)| \le C|t-s|^{\mu(T)}$. Similarly we have $|w(1,t) - w(1,s)| \le C|t-s|^{\mu(T)}$. Finally by Theorem 1.1 of Chapter V in [14] we have $|w|_{\mu(T)} \le C$. This implies that $|v|_{\mu(T)} \le C$ for another $C > 0$. ∎

Appendix B. Proof of Lemma 2.6. It is clear that to prove Lemma 2.6 we only need to prove similar conclusions for v.

LEMMA B.1. *Assume* (1.9) *holds; then* (A.14) *has a unique, global solution* $v(z,t)$ *satisfying* $v_t(\cdot,t) \ge 0$. *Moreover, for any* $T > 0$, *let* $Q_T^z := (0,1) \times (0,T]$; *then there exist positive constants* $\mu(T)$, C_1, *and* C_2 (C_1, C_2 *are independent of* T) *such that*

(i) $v \in C^{2+\mu,1+\frac{\mu}{2}}(\overline{Q_T^z})$;

(ii) $\|v(z,t)\|_{C^{2+\mu,1+\frac{\mu}{2}}(\overline{Q_T^z})} \le C_1 T + C_2$.

Proof. First, by Theorem 8.1 in [21] and the above a priori estimates for v, we have a unique, global solution v of (A.14). Moreover for any $T > 0$, $v \in C^\infty([0,T], H^\infty(0,1))$, where $H^\infty(0,1) := \bigcap_{k=0}^\infty H^k(0,1)$ is a Fréchet space with norms $(\|\cdot\|_k)_{k=0}^\infty$, and $H^k(0,1)$ is a Sobolev space with norm $\|\cdot\|_k$ (cf. [21]). Embedding theorem implies that we indeed obtain a global solution of (A.14): $v \in C^\infty(\overline{Q_\infty^z})$, where $Q_\infty^z = (0,1) \times (0,\infty)$. (i) is proved.

The result $u_t(\cdot,t) \geq 0$ for $t > 0$ in subsection 2.1.2 implies that $v_t(z,t) \geq 0$.

Now we use the a priori estimates in Lemma 2.5 and the interior estimate (see, for example, Theorem 5 of Chapter 3 in [7]) for problem (A.4)–(A.5) with initial data $v_0(z)$; then for any $T > 0$, there exist $\mu = \mu(T) > 0$ and $C_1 > 0$, $C_2 > 0$ (C_1, C_2 are independent of T) such that

$$\|v\|_{C^{2+\mu,1+\frac{\mu}{2}}(Q_T^z)} \leq C_1 T + C_2.$$

By the smoothness of v on $\overline{Q_T^z}$, we indeed obtain $\|v\|_{C^{2+\mu,1+\frac{\mu}{2}}(\overline{Q_T^z})} \leq C_1 T + C_2$. □

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