ENTIRE SOLUTIONS OF THE FISHER-KPP EQUATION ON THE HALF LINE§

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Abstract. In this paper we study the entire solutions of the Fisher-KPP equation $u_t = u_{xx} + f(u)$ on the half line $[0, \infty)$ with Dirichlet boundary condition at $x = 0$. (1). For any $c \geq 2\sqrt{f'(0)}$, we show the existence of an entire solution $\mathcal{U}^c(x, t)$ which connects the traveling wave solution $\phi^c(x + ct)$ at $t = -\infty$ and the unique positive stationary solution $V(x)$ at $t = +\infty$; (2). We also construct an entire solution $\mathcal{U}(x, t)$ which connects the solution of $\eta_t = f(\eta)$ at $t = -\infty$ and $V(x)$ at $t = +\infty$.

1. Introduction

Consider the following reaction-diffusion equation:

$$u_t = u_{xx} + f(u), \tag{1.1}$$

where $f \in C^1([0, 1])$ is a Fisher-KPP type of nonlinearity:

$$f(0) = f(1) = 0, \quad f'(1) < 0 < f'(0), \quad f(u) > 0 \text{ and } f'(u) \leq f'(0) \text{ for } u \in (0, 1). \tag{1.2}$$

It is well known that, for each real number $c \geq c_0 := 2\sqrt{f'(0)}$, the equation (1.1) admits a unique traveling wave solution $u = \phi^c(x + ct)$, where $\phi^c(z)$ satisfies

$$\phi_{zz} - c\phi_z + f(\phi) = 0, \quad \phi(-\infty) = 0, \quad \phi(\infty) = 1, \quad \phi_z(z) > 0 \text{ for } z \in \mathbb{R}. \tag{1.3}$$

Traveling wave solutions are a special kind of *entire solutions* (that is, solutions of (1.1) defined for all $t \in \mathbb{R}$), and play a key role in understanding the dynamics of the equation. In order to figure out the complete global dynamics, however, we need to investigate other types of entire solutions.

Since the pioneering works in [10] and [19], there are huge number of works on entire solutions to reaction-diffusion equations. In particular, in one space dimension, [2, 3, 7, 9, 10, 13], etc. studied entire solutions of the reaction-diffusion equations like (1.1) with monostable or bistable

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type of nonlinearities. The readers also see the survey paper [14] and the references cited in. Meanwhile, some authors studied the entire solutions for lattice differential equations (cf. [17, 18]) and for equations with nonlocal or delayed terms (cf. [12, 15, 16]), etc.

In this paper we are interested in the entire solutions of the equation (1.1) on the half line:

$$\begin{cases} u_t = u_{xx} + f(u), & x > 0,\ t > 0, \\ u(0,t) = 0, & t > 0. \end{cases} \tag{1.4}$$

This problem has no traveling wave solutions but a unique positive stationary solution $u = V(x)$ with

$$V(0) = 0, \quad V(\infty) = 1, \quad V'(x) > 0 \text{ for } x \geq 0$$

(see details in the next section) and, by a similar argument as in [1, 5] one sees that, any solution $u(x,t)$ of (1.4) starting from a nonnegative initial data $u(x,0)$ converges as $t \to \infty$ to $V(x)$ in the topology of $L^\infty_{loc}([0,\infty))$. Hence, any positive entire solution $\mathcal{U}(x,t)$, if it exists, also satisfies $\mathcal{U}(\cdot,t) \to V(\cdot)$ as $t \to \infty$. To distinguish entire solutions, we classify the α-limit of $\mathcal{U}(x, t+t_n)$ for any $t_n \to -\infty$. In fact, we construct two types of entire solutions for (1.4). Each of the first type connects a traveling wave solution $\phi^c(x+ct)$ at $t = -\infty$ with $V(x)$ at $t = \infty$, while the second type connects the solution $\eta(t)$ of the ordinary differential equation $\eta'(t) = f(\eta)$ at $t = -\infty$ with $V(x)$ at $t = \infty$.

On the first type of entire solutions we have the following result.

Theorem 1.1. *Assume* (1.2). *Then for each* $c \geq c_0 := 2\sqrt{f'(0)}$ *and any* $\theta \in \mathbb{R}$, *the problem* (1.4) *has an entire solution* $\mathcal{U}^c(x,t)$ *which satisfies*

$$\mathcal{U}^c_x(x,t) > 0 \text{ for } x > 0,\ t \in \mathbb{R}; \qquad \mathcal{U}^c(\infty,t) = 1 \text{ for } t \in \mathbb{R}, \tag{1.5}$$

$$\sup_{x\in[0,\infty)} |\mathcal{U}^c(x,t) - V(x)| \to 0 \text{ as } t \to \infty, \tag{1.6}$$

and

$$\sup_{x\in[0,\infty)} |\mathcal{U}^c(x,t) - \phi^c(x+ct-\theta)| \to 0 \text{ as } t \to -\infty. \tag{1.7}$$

To obtain the second type of entire solution we need a slightly stronger condition on f:

$$f \in C^2([0,1]), \quad f''(u) \leq 0 \text{ in } [0,1]. \tag{1.8}$$

Theorem 1.2. *Assume* (1.2) *and* (1.8). *Then the problem* (1.4) *has an entire solution* $\mathcal{U}(x,t)$ *with the following properties:*

$$\mathcal{U}_t(x,t) > 0, \quad \mathcal{U}_x(x,t) > 0, \quad \mathcal{U}_{xx}(x,t) < 0 \text{ for all } x > 0,\ t \in \mathbb{R}; \tag{1.9}$$

$$\mathcal{U}(\infty,t) = 1 \text{ for all } t \in \mathbb{R}; \tag{1.10}$$

$$\sup_{x\in[0,\infty)} |\mathcal{U}(x,t) - V(x)| \to 0 \text{ as } t \to \infty; \tag{1.11}$$

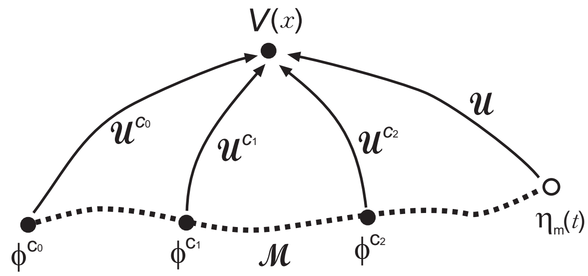

FIGURE 1. Orbits of the entire solutions on the phase space.

and, with $\xi_m(s)$ denoting the m-level set of $\mathcal{U}(\cdot, s)$ for each given $m \in (0,1)$,

$$\begin{cases} \xi_m(s) \to \infty \text{ and } -\xi_m'(s) \to \infty \text{ as } s \to -\infty, \\ \mathcal{U}(x+\xi_m(s), t+s) \to \eta_m(t) \text{ as } s \to -\infty, \quad \text{in } C^{2,1}_{loc}(\mathbb{R}\times\mathbb{R}) \text{ topology,} \end{cases} \tag{1.12}$$

where $\eta_m(t)$ is the unique solution of the initial value problem:

$$\eta'(t) = f(\eta), \quad \eta(0) = m. \tag{1.13}$$

These two theorems show that, in a phase space (see Figure 1), the orbit of a first type of entire solution $\mathcal{U}^c(\cdot, t)$ connects that of $\phi^c(\cdot + ct + \theta)$ at $t = -\infty$ and that of $V(\cdot)$ at $t = \infty$; while the orbit of the second type of entire solution $\mathcal{U}$ connects that of $\eta_m(t)$ at $t = -\infty$ and that of $V(\cdot)$ at $t = \infty$. Since, in the phase space, the points $\phi^c(\cdot)$ form a "one-dimensional continuous manifold" as c increasing from c_0 to ∞ (see $\mathcal{M}$ in Figure 1), and since $\phi^c(x+\hat{\xi}_m) \to m = \eta_m(0)$ as $c \to \infty$, locally uniformly in $x \in \mathbb{R}$, where $\hat{\xi}_m$ is the m-level set of $\phi^c(\cdot)$, we can roughly say that $\phi^c(x + ct + \hat{\xi}_m) \to \eta_m(t)$ as $c \to \infty$.

We note that our setting, the Dirichlet boundary condition on the Fisher-KPP equation in the half-line, imposes a strong restriction on the structure of the solutions. In fact, the readers could find various types of entire solutions in [10, 14] for reaction-diffusion equations in the whole space while in our problem it seems that all the possible entire solutions consist of the stationary solutions (0 and V), the two types of entire solutions ($\mathcal{U}^c$ and $\mathcal{U}$) and their temporal translations. It, however, not so simple to determine all the entire solution even in our simple setting and a further study would be required for the desired assertion.

Our approach to these theorems are quite different. To prove Theorem 1.1 we directly construct a pair of sub- and supersolutions: $\underline{u}(x,t) = \phi^c(x+ct+\theta) - \rho(t)$ and $\overline{u}(x,t) = \phi^c(x+ct+\theta)$ over $t \le 0$, for some $\theta \in \mathbb{R}$ and $\rho(t) \searrow 0$ $(t \to -\infty)$. Then the entire solution can be constructed in between as in [7, 9] (see details in section 3). To prove Theorem 1.2 we consider a sequence of initial-boundary value problems with initial data $\approx \frac{1}{n}$, each solution $u_n(x,t)$ is concave in x. A subsequence of $\{u_n\}$ is proved to converge to a second type of entire solution $\mathcal{U}(x,t)$, whose concavity is a main feature to distinguish it from the first type of entire solutions. In particular, to prove the property $\mathcal{U}(\infty, t) \equiv 1$ (despite of the fact $u_n(\infty, 0) = \frac{1}{n}$), we have to do some precise estimate for the solutions of several related linear problems (see details in subsection 4.2). This is quite different from the approach in section 3.

This paper is arranged as the following. In section 2, as preliminaries, we present the positive stationary solution $V(x)$ of (1.4) and traveling wave solutions $\phi^c(x+ct)$ of (1.1). In section 3, we show the existence of the first type of entire solutions $\mathcal{U}^c(x,t)$ for any $c \geq c_0$, and prove Theorem 1.1. In section 4 we construct the second type of entire solution $\mathcal{U}(x,t)$ and prove Theorem 1.2.

2. Stationary Solutions and Traveling Wave Solutions

In this section we present the positive stationary solution $V(x)$ and traveling wave solutions of $(1.4)_1$. Consider the following equation

$$q''(z) - cq'(z) + f(q) = 0, \quad q(z) \geq 0 \quad \text{for } z \in J, \tag{2.1}$$

where J is some interval in $\mathbb{R}$. Note that a nonnegative stationary solution u of $(1.4)_1$ solves (2.1) with $c = 0$ in $J = (0, \infty)$, and a nonnegative traveling wave solution $u(x,t) = q(x+ct)$ of (1.1) solves (2.1) in $\mathbb{R}$. The equation (2.1) is equivalent to the system

$$\begin{cases} q'(z) = p, \\ p'(z) = cp - f(q). \end{cases} \tag{2.2}$$

A solution $(q(z), p(z))$ of this system traces out a trajectory in the q-p phase plane. It is easily seen that $(0,0)$ and $(1,0)$ are two equilibrium points of the system (2.2). The eigenvalues of the corresponding linearizations at these points are

$$\lambda_0^{\pm}(c) = \frac{c \pm \sqrt{c^2 - 4f'(0)}}{2} \quad (\text{at } (0,0)) \quad \text{and} \quad \lambda_1^{\pm}(c) = \frac{c \pm \sqrt{c^2 - 4f'(1)}}{2} \quad (\text{at } (1,0)), \tag{2.3}$$

respectively. Since $f'(0) > 0$ and $f'(1) < 0$, $(1,0)$ is always a saddle point, however, $(0,0)$ is a center or a focus when $0 \leq c < c_0 = 2\sqrt{f'(0)}$, and it is an unstable node when $c \geq c_0$. Using the phase plane analysis (cf. [1, 4, 8]), it is not difficult to give the solutions of (2.1). We list two types of them, which will be used in this paper.

(I) **Positive stationary solution $V(z)$ on the half line**. When $c = 0$, the system (2.2) can be solved explicitly. In particular, the trajectory tending to $(1,0)$ in the domain $\{(q,p) \mid 0 < q < 1,\ p > 0\}$ is given by $p = \sqrt{2\int_q^1 f(s)ds}$ (see Γ_1 in Figure 2 (a)), which corresponds to a solution $q = V(z)$ of (2.1) with $c = 0$. It satisfies (by shifting its zero to $z = 0$)

$$V(0) = 0, \quad V(\infty) = 1, \quad V'(z) > 0 \text{ for } z \in [0, +\infty).$$

(II) **Strictly increasing solutions $\phi^c(z)$ in $\mathbb{R}$ in case $c \geq c_0$**. It is well known that (cf. [1, 11]) for any $c \geq c_0$, the equation (2.1) has a solution $q = \phi^c(z)$ satisfying

$$\phi^c(-\infty) = 0, \quad \phi^c(\infty) = 1, \quad \phi^c(0) = \frac{1}{2}, \quad \phi^c_z(z) > 0 \text{ for } z \in \mathbb{R} \tag{2.4}$$

(see Γ_2 in Figure 2 (b)). For each $\phi^c(z)$, it is clear that $u=\phi^c(x+ct)$ is a traveling wave solution of (1.1). Moreover, when $c=c_0=2\sqrt{f'(0)}$, we have

$$\phi^{c_0}(z)\sim |z|e^{c_0 z/2} \text{ as } z\to-\infty;$$

when $c>c_0$, we have

$$\phi^c(z)\sim e^{\lambda_c z} \text{ as } z\to-\infty, \qquad \text{with } \lambda_c:=\lambda_0^-(c)=\frac{1}{2}(c-\sqrt{c^2-4f'(0)})>0.$$

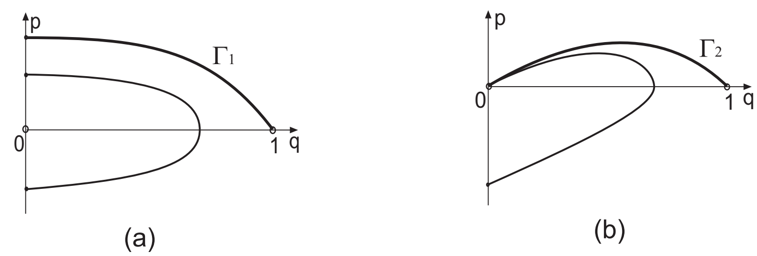

FIGURE 2. Trajectories of the system (2.2). (a) $c=0$; (b) $c\geq 2\sqrt{f'(0)}$.

3. The First Type of Entire Solutions

We prove the existence of $\mathcal{U}^c$ and (1.7) in Theorem 1.1. We extend $f(u)$ outside the interval $[0,1]$ so that

$$f'(u)=f'(0), \qquad u<0 \tag{3.1}$$

In fact, since we will be able to obtain the entire solution taking the values in $(0,1)$, this modification does not affect the desired entire solution. We define

$$\mathcal{L}[u]:=u_t-u_{xx}-f(u).$$

Then $c\lambda_c=f'(0)b_c$, where λ_c is defined as above and

$$b_c:=\frac{2c}{c+\sqrt{c^2-4f'(0)}} \quad (2\sqrt{f'(0)}\leq c<\infty)$$

is strictly monotone decreasing in c and $f'(0)<c\lambda_c\leq 2f'(0)$. Indeed,

$$\begin{aligned}\frac{d}{dc}\left(\frac{c}{c+\sqrt{c^2-4f'(0)}}\right)&=\frac{1}{c+\sqrt{c^2-4f'(0)}}-\frac{c(1+c/\sqrt{c^2-4f'(0)})}{(c+\sqrt{c^2-4f'(0)})^2}\\ &=\frac{1}{(c+\sqrt{c^2-4f'(0)})^2}\left(\sqrt{c^2-4f'(0)}-c^2/\sqrt{c^2-4f'(0)}\right)\\ &=\frac{-4f'(0)}{(c+\sqrt{c^2-4f'(0)})^{5/2}}<0.\end{aligned}$$

We note that $b_{c_0}=2$.

We first consider the case $c\in(c_0,\infty)$. Then there is a positive number A_c such that

$$0<\phi^c(z)\leq A_c e^{\lambda_c z}, \quad -\infty<z\leq 0.$$

For arbitrarily given $\theta \in \mathbb{R}$, we put

$$\overline{u}(x,t) := \phi^c(x+ct-\theta), \tag{3.2}$$

$$\underline{u}(x,t) := \phi^c(x+ct-\theta) - \rho(t), \tag{3.3}$$

where

$$\rho(t) := A_c e^{-\lambda_c \theta} e^{c\lambda_c t} \qquad (-\infty < t \le 0). \tag{3.4}$$

It is clear that $\overline{u}$ of (3.2) is a supersolution of (1.4) not only for $c > c_0$ but also for $c = c_0$. Plug $\underline{u}$ of (3.3) into $\mathcal{L}[u]$ to yield

$$\begin{aligned}
\mathcal{L}[\underline{u}] &= c(\phi^c)' - \dot\rho(t) - (\phi^c)'' - f(\phi^c - \rho(t)) \\
&= -\dot\rho(t) + f(\phi^c) - f(\phi^c - \rho(t)) \\
&= -\dot\rho(t) - \int_0^1 f'(\phi^c - s\rho(t))ds(-\rho(t)) \\
&= \rho(t)\Big[-c\lambda_c + \int_0^1 f'(\phi^c - s\rho(t))ds \Big] \\
&\le \rho(t)[-c\lambda_c + f'(0)] < 0.
\end{aligned}$$

Moreover,

$$\underline{u}(0,t) \;=\; \phi^c(ct-\theta) - \rho(t) \le A_c e^{\lambda_c(ct-\theta)} - \rho(t) = 0 \qquad (t \le \theta/c),$$

which implies that $\underline{u}$ is a subsolution in $t \in (-\infty, \theta/c]$.

Next in the case $c = c_0$ we have

$$c_0 \lambda_{c_0} = 2f'(0),$$

and

$$0 < \phi^{c_0}(z) \le A_{c_0}|z| e^{\lambda_{c_0} z}, \quad -\infty < z \le 0,$$

for a positive constant A_{c_0}. We set

$$\rho_*(t) := \rho_*^0 e^{pt} \quad (t \le 0), \quad p := 2f'(0) - \delta, \quad 0 < \delta < f'(0).$$

Then, in a similar way as in the previous case $\underline{u}(x,t) := \phi^{c_0}(x + c_0 t - \theta) - \rho_*(t)$ enjoys

$$\mathcal{L}[\underline{u}] \le \rho_*(t)[-p + f'(0)] = \rho_*(t)[-f'(0) + \delta] < 0.$$

In order to show $\underline{u}(0,t) \le 0$, we compute

$$\begin{aligned}
\underline{u}(0,t) &\le A_{c_0}|c_0 t - \theta| e^{\lambda_{c_0}(c_0 t - \theta)} - \rho_*^0 e^{pt} \\
&\le e^{pt}[A_{c_0}(|c_0 t| + |\theta|) e^{2f'(0)t - pt} e^{-\lambda_{c_0}\theta} - \rho_*^0] \\
&= e^{pt} A_{c_0}[|c_0 t| e^{\delta t} e^{-\lambda_{c_0}\theta} + |\theta| e^{-\lambda_{c_0}\theta} e^{\delta t} - \rho_*^0/A_{c_0}] \\
&\le e^{pt} A_{c_0}[\{\sup_{t \le 0} |c_0 t| e^{\delta t}\} e^{-\lambda_{c_0}\theta} + |\theta| e^{-\lambda_{c_0}\theta} - \rho_*^0/A_{c_0}], \quad (t \le \theta/c_0).
\end{aligned}$$

Thus, taking ρ_*^0 as

$$\rho_*^0 = A_{c_0} e^{-\lambda_{c_0}\theta}(\sup_{t\le 0} |c_0 t| e^{\delta t} + |\theta|)$$

yields $\underline{u}(0,t) \le 0$ $(t \le \theta/c_0)$.

In the sequel, for any $c \ge c_0$ and arbitrarily given θ we have obtained the sub-super solution pairs, by which the exisitence of a solution $\mathcal{U}^c(x,t)$ sandwiched by $\underline{u}(x,t)$ and $\overline{u}(x,t)$ in $t \in (-\infty, \theta/c]$ is shown in a similar way as in [7, 9]. This solution satisfies the desired asymptotic behavior as $t \to -\infty$ and can be extended to the whole time by the theorem of Cauchy problem. This concludes the proof. □

Remark 3.1. We can find a similar subsolution to $\underline{u}$ of (3.3) in [6], where they utilize it to prove the asymptotic stability of the traveling front solution to the bistable reaction-diffusion equation. On the other hand the present study is related to the asymptotic behavior as $t \to -\infty$. We, however, see that this type of subsolution is quite useful.

4. The Second Type of Entire Solution

In this section, we always assume (1.2) and (1.8). We first construct a second type of entire solution in subsection 4.1, and then study its properties in subsection 4.2. Finally, in subsection 4.3 we study the limit of $\mathcal{U}$ as $t \to -\infty$ and prove Theorem 1.2. For simplicity, in what follows we write

$$\mu := \sqrt{f'(0)} = \frac{c_0}{2}. \tag{4.1}$$

4.1. **Construction of the second type of entire solution.** By (1.2), there exists a large integer N such that

$$f'(u) > \frac{1}{2} f'(0) = \frac{1}{2}\mu^2, \quad u \in \Big[0, \frac{1}{N}\Big].$$

For each positive integer n, define

$$\psi_n(x) := \begin{cases} \dfrac{1}{(n+N)\pi}\Big[\sin\dfrac{\mu x}{\sqrt{2}} + \dfrac{\mu x}{\sqrt{2}}\Big], & 0 \le x \le \dfrac{\sqrt{2}\pi}{\mu}, \\ \dfrac{1}{n+N}, & x \ge \dfrac{\sqrt{2}\pi}{\mu}. \end{cases}$$

Clearly, $\psi_n(x) \in C^2([0,\infty))$ and

$$\psi_n'(x) \ge 0, \quad \psi_n''(x) \le 0, \quad \psi_n''(x) + f(\psi_n(x)) \ge \psi_n''(x) + \frac{\mu^2}{2}\psi_n(x) \ge 0 \quad \text{for } x \ge 0. \tag{4.2}$$

We construct the second type of entire solution of (1.4) by using the solutions of the following initial-boundary value problems:

$$\begin{cases} u_t = u_{xx} + f(u), & x > 0,\ t > 0, \\ u(0,t) = 0, & t > 0, \\ u(x,0) = \psi_n(x), & x \ge 0. \end{cases} \tag{4.3}$$

Lemma 4.1. *For each positive integer n, the solution $u_n(x,t)$ of (4.3) exists for all $t>0$, and it satisfies*

$$(u_n)_t(x,t)>0, \quad (u_n)_x(x,t)>0, \quad (u_n)_{xx}(x,t)<0 \ \text{ for } x>0,\ t>0; \tag{4.4}$$

$$u_n(\cdot,t)\to V(\cdot) \text{ as } t\to\infty, \quad \text{in } C^2_{loc}([0,\infty)) \text{ topology}; \tag{4.5}$$

and

$$u_n(x,t)\le \eta_{\frac{1}{n+N}}(t) \text{ for all } x\ge 0,\ t>0, \tag{4.6}$$

where $\eta_{\frac{1}{n+N}}(t)$ is the solution of the initial value problem (1.13) with $m=\frac{1}{n+N}$.

Proof. By the standard parabolic theory, the classical solution $u_n(x,t)$ of the problem (4.3) exists globally. The first two inequalities in (4.4) follow from (4.2) and the strong maximum principle easily. To show the third inequality, we see that $\zeta(x,t):=(u_n)_{xx}(x,t)$ satisfies

$$\zeta_t=\zeta_{xx}+f'(u_n)\zeta+f''(u_n)(u_n)_x^2\le \zeta_{xx}+f'(u_n)\zeta$$

by the assumption (1.8). Moreover,

$$\zeta(0,t)=(u_n)_{xx}(0,t)=(u_n)_t(0,t)-f(u_n(0,t))=0, \quad t>0.$$

Hence

$$\zeta(x,t)<0 \quad \text{for } x>0 \text{ and } t>0$$

by the strong maximum principle and the fact $\psi_n''(x)\le,\not\equiv 0$ in (4.2).

Since $(u_n)_t(x,t)>0$, by parabolic estimates, $u_n(\cdot,t)$ converges as $t\to\infty$ to a positive stationary solution $\widetilde{V}(x)$. The uniqueness of stationary solutions to our equation implies $\widetilde{V}(x)\equiv V(x)$ and so

$$u_n(\cdot,t)\to V(\cdot) \text{ as } t\to\infty, \tag{4.7}$$

in the topology of $C^2_{loc}([0,\infty))$.

Finally, (4.6) follows from the fact that $\eta_{\frac{1}{n+N}}(t)$ is a supersolution of (4.3). □

Set

$$t_n:=\min\{t>0 \mid (u_n)_x(0,t)=V'(0)/2\}. \tag{4.8}$$

We claim that $t_n\to\infty$ as $n\to\infty$. In fact, from (4.6) it is easily to know that $u_n(x,t)\to 0$ as $n\to\infty$, locally uniformly in $t\in[0,+\infty)$ and uniformly in $x\ge 0$. This holds true for $(u_n)_x$, thanks to parabolic estimates. The claim is then proved. Define

$$U_n(x,t):=u_n(x,t+t_n) \text{ for } x\ge 0,\ t>-t_n.$$

By Lemma 4.1 we have

$$(U_n)_x(0,0)=\frac{1}{2}V'(0), \quad (U_n)_t(x,t)>0, \quad (U_n)_x(x,t)>0, \quad (U_n)_{xx}(x,t)<0 \ \text{ for } x>0,\ t>-t_n. \tag{4.9}$$

For any $\alpha \in (0,1)$, $T > 0$, $X > 0$ and any integer n large such that $t_n > T$, by the L^p estimate we have

$$\|U_n(\cdot,\cdot)\|_{W_4^{2,1}([0,X]\times[-T,T])} \leq C_1,$$

for some C_1 depending on T and X but not on n, where, for any bounded domain $Q \subset \{(x,t) \mid x \geq 0, t \in \mathbb{R}\}$, $W_4^{2,1}(Q)$ denotes the Sobolev space $\{u \in L^4(Q) \mid \|u\|_{L^4(Q)} + \|u_x\|_{L^4(Q)} + \|u_{xx}\|_{L^4(Q)} + \|u_t\|_{L^4(Q)} < \infty\}$. Using the embedding theorem we have

$$\|U_n(\cdot,\cdot)\|_{C^{\alpha,\alpha/2}([0,X]\times[-T,T])} \leq C_2\|U_n(\cdot,\cdot)\|_{W_4^{2,1}([0,X]\times[-T,T])} \leq C_2C_1,$$

for some C_2 independent of n. By the Schauder estimate we derive

$$\|U_n(\cdot,\cdot)\|_{C^{2+\alpha,1+\alpha/2}([0,X]\times[-T,T])} \leq C_3, \tag{4.10}$$

for some C_3 depending on α, T, X but not on n. Therefore, there exists a sequence $\{n_i\}$ of $\{n\}$ and a function $\mathcal{U}(x,t) \in C^{2+\alpha,1+\alpha/2}([0,X]\times[-T,T])$ such that

$$\|U_{n_i}(\cdot,\cdot) - \mathcal{U}(\cdot,\cdot)\|_{C^{2,1}([0,X]\times[-T,T])} \to 0, \text{ as } i \to \infty.$$

Taking X and T larger and larger, and using Cantor's diagonal argument, we can find a subsequence of $\{n\}$ (denoted it again by $\{n_i\}$) and a function in $C_{loc}^{2+\alpha,1+\alpha/2}([0,\infty)\times\mathbb{R})$ (denoted it again by $\mathcal{U}(x,t)$) such that

$$U_{n_i}(x,t) \to \mathcal{U}(x,t) \text{ as } i \to \infty, \quad \text{in } C_{loc}^{2,1}([0,\infty)\times\mathbb{R}) \text{ topology.} \tag{4.11}$$

Thus we obtain an entire solution $\mathcal{U}$ of (1.4).

4.2. **Properties of $\mathcal{U}$.** In this part we study the properties of $\mathcal{U}$ and show that it is the desired solution in Theorem 1.2.

4.2.1. *Monotonicity and concavity.*

Proposition 4.2. *Let $\mathcal{U}$ be the entire solution obtained in* (4.11). *Then*

$$\mathcal{U}_x(0,0) = \frac{1}{2}V'(0), \tag{4.12}$$

$$\mathcal{U}_t(x,t) > 0 \text{ for } x > 0,\ t \in \mathbb{R}, \tag{4.13}$$

$$\mathcal{U}_x(x,t) > 0,\ \ \mathcal{U}_{xx}(x,t) < 0 \text{ for } x \geq 0,\ t \in \mathbb{R}, \tag{4.14}$$

and

$$\mathcal{U}(\cdot,t) \to V(\cdot) \text{ as } t \to \infty, \quad \text{in } L^\infty([0,\infty)) \text{ topology.} \tag{4.15}$$

Proof. The conclusions in (4.12), (4.13) and (4.14) follow from (4.9) and the strong maximum principle easily.

Now we prove (4.15). For any small $\varepsilon > 0$, since $V(\infty) = 1$, there exists $X > 0$ such that

$$1 - \varepsilon \leq V(x) \leq 1 \text{ for } x \geq X.$$

Since $\mathcal{U}_t(x,t) > 0$, as proving (4.7) one can show that $\mathcal{U}(\cdot,t) \to V(\cdot)$ as $t \to \infty$, in the topology of $C^2_{loc}([0,\infty))$. Hence for some $T > 0$ we have

$$\|\mathcal{U}(\cdot,t) - V(\cdot)\|_{L^\infty([0,X])} \leq \varepsilon, \quad \text{when } t \geq T. \tag{4.16}$$

This implies that

$$|\mathcal{U}(X,t) - 1| \leq |\mathcal{U}(X,t) - V(X)| + |V(X) - 1| \leq 2\varepsilon, \quad \text{when } t \geq T.$$

Therefore, when $t \geq T$ and $x \geq X$ we have

$$|\mathcal{U}(x,t) - V(x)| \leq |\mathcal{U}(x,t) - 1| + |V(x) - 1| \leq |\mathcal{U}(X,t) - 1| + |V(X) - 1| \leq 3\varepsilon.$$

Combining with (4.16), we obtain (4.15). □

Remark 4.3. Note that the first type of entire solutions $\mathcal{U}^c$ have all the properties of $\mathcal{U}$ except for (4.12) and $\mathcal{U}_{xx}(x,t) < 0$. The concavity is the main difference between them.

Furthermore, we can show the following properties for $\mathcal{U}$.

Proposition 4.4. *Let $\mathcal{U}(x,t)$ be the entire solution obtained as above. Then $\mathcal{U}(x,t+s) \to 0$ as $s \to -\infty$, in $C^{2,1}_{loc}([0,\infty) \times \mathbb{R})$ topology.*

Proof. For any time sequence $\{s_k\}$ decreasing to $-\infty$, by the parabolic estimate as in (4.10) we have, for any $X, T > 0$,

$$\|\mathcal{U}(\cdot,\cdot + s_k)\|_{C^{2+\alpha,1+\alpha/2}([0,X]\times[-T,T])} \leq C_2, \tag{4.17}$$

where C_2 depends on X and T, but not on k. Hence there exist a subsequence $\{s_{k_j}\}$ of $\{s_k\}$ and an entire solution $W(x,t)$ of (1.4) such that $\mathcal{U}(\cdot,\cdot+s_{k_j}) \to W(\cdot,\cdot)$ as $j \to \infty$, in $C^{2,1}_{loc}([0,\infty)\times\mathbb{R})$ topology.

We now show that $W_t \equiv 0$, and so $W(x,t) \equiv W(x)$ is a stationary solution of (1.4). For, otherwise, $W_t(x^*,t^*) \neq 0$ for some $(x^*,t^*) \in (0,\infty)\times\mathbb{R}$. Since $\mathcal{U}_t(x^*,t^*+s_{k_j}) > 0$ for all j we may assume that $W_t(x^*,t^*) = 3\delta_1$ for some $\delta_1 > 0$. Then, for sufficiently large j we have $\mathcal{U}_t(x^*,t^*+s_{k_j}) > 2\delta_1$. By the uniform estimate in (4.17) we see that, when $\varepsilon_1 \in (0,\frac{1}{2})$ is small,

$$\mathcal{U}_t(x^*,t^*+t) > \delta_1 \text{ if } |t - s_{k_j}| < \varepsilon_1.$$

Without loss of generality, we assume that $s_{k_j} - s_{k_{j+1}} > 1$, then

$$\begin{aligned}
\mathcal{U}(x^*,t^*+s_{k_j}+\varepsilon_1) &> \mathcal{U}(x^*,t^*+s_{k_j}-\varepsilon_1) + 2\delta_1\varepsilon_1 \\
&\geq \mathcal{U}(x^*,t^*+s_{k_{j+1}}+\varepsilon_1) + 2\delta_1\varepsilon_1 \geq \cdots \\
&\geq \mathcal{U}(x^*,t^*+s_{k_{j'}}-\varepsilon_1) + 2(j'-j)\delta_1\varepsilon_1 \to \infty, \quad \text{as } j' \to \infty,
\end{aligned}$$

contradicting the fact $\mathcal{U}(x,t) \in (0,1)$. Thus $W_t \equiv 0$ and so $W(x,t) \equiv W(x)$ solves

$$W_{xx} + f(W) = 0 \ (x > 0), \qquad W(0) = 0.$$

This problem has only two nonnegative solutions 0 and $V(x)$. Clearly $W(x)\neq V(x)$ since $\mathcal{U}_t>0$ and $\mathcal{U}(x,t)\to V(x)$ as $t\to\infty$. Hence $W(x)\equiv 0$, and $\mathcal{U}(\cdot,\cdot+s_{k_j})\to 0$ as $j\to\infty$.

Since $\{s_k\}$ is an arbitrary sequence tending to $-\infty$, we conclude that $\mathcal{U}(\cdot,\cdot+s)\to 0$ as $s\to-\infty$, in $C^{2,1}_{loc}([0,\infty)\times\mathbb{R})$ topology. □

4.2.2. *Uniform upper bound of $\mathcal{U}$: $\mathcal{U}(\infty,t)\equiv 1$.* Define

$$\kappa(t):=\mathcal{U}(\infty,t),\quad \beta(t):=\mathcal{U}_x(0,t),\quad t\in\mathbb{R}.$$

Since the entire solution $\mathcal{U}$ is obtained by taking limit for a subsequence of $\{u_n(x,t)\}$ and each $u_n(x,t)$ takes supermum in $(0,1)$ for any $t>0$, one may guess that $\kappa(t)$ also takes values in $(0,1)$. We will see that this in not true. In fact, in what follows we can prove a surprising result: $\kappa(t)=1$ for all $t\in\mathbb{R}$.

Lemma 4.5. *Assume $\kappa(t_1)\in(0,1)$ for some $t_1\in\mathbb{R}$. Then*

(i) *$\kappa'(t)>0$ for $t\in\mathbb{R}$, and $\kappa(t)\to 1$ as $t\to\infty$, $\kappa(t)\to 0$ as $t\to-\infty$;*
(ii) *for any $\nu\in(0,1)$, $\beta(t)/[\kappa(t)]^{1+\nu}\to\infty$ as $t\to-\infty$.*

Proof. (i). For any $\alpha\in(0,1)$ and $T>0$, by parabolic estimate we have $\|\mathcal{U}(x,\cdot)\|_{C^{1+\alpha/2}([-T,T])}\leq C$ for some C independent of x. Hence, $\mathcal{U}(x,t)\to\kappa(t)$ as $x\to\infty$ in $C^1_{loc}(\mathbb{R})$ topology. In the equation of $\mathcal{U}$, if we take limit as $x\to\infty$ in $C^1_{loc}(\mathbb{R})$ topology, then we have

$$\kappa'(t)=f(\kappa(t)).$$

Using $\kappa(t_1)\in(0,1)$ as the initial data, we derive the conclusions.

(ii). For any $\nu\in(0,1)$, set

$$z(x,t;\nu):=\frac{\mathcal{U}(x,t)}{[\kappa(t)]^{1+\nu}},\quad x\geq 0,\ t\in\mathbb{R}.$$

Then z solves

$$\begin{cases} z_t=z_{xx}+c(x,t)z, & x>0,\ t\in\mathbb{R},\\ z(0,t;\nu)=0, & t\in\mathbb{R},\end{cases}$$

where

$$c(x,t):=\frac{f(\mathcal{U})}{\mathcal{U}}-(1+\nu)\frac{f(\kappa)}{\kappa}\leq -\nu f'(0)+O(1)\kappa(t)<0,\qquad \text{when } t\ll -1.$$

Since $z_{xx}=\mathcal{U}_{xx}/[\kappa(t)]^{1+\nu}<0$, we conclude that

$$z_t(x,t;\nu)<0,\quad x>0,\ t\ll -1.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}[z_x(0,t;\nu)] &= z_{xt}(0,t;\nu)=\lim_{x\to 0+}\frac{z_t(x,t;\nu)-z_t(0,t;\nu)}{x}\\ &= \lim_{x\to 0+}\frac{z_t(x,t;\nu)}{x}\leq 0,\quad t\ll -1.\end{aligned}$$

This means that $z_x(0,t;\nu)$ in a non-increasing function when $t \ll -1$. Note that $z_x(0,t;\nu) > 0$ for all $t \in \mathbb{R}$, we have

$$z_x(0,t;\nu) \geq \vartheta(\nu), \quad t \leq 0,$$

for some positive real number $\vartheta(\nu)$. For any $\nu_1 > \nu$ we have

$$\frac{\beta(t)}{[\kappa(t)]^{1+\nu_1}} = \frac{\mathcal{U}_x(0,t)}{[\kappa(t)]^{1+\nu}} \cdot \frac{1}{[\kappa(t)]^{\nu_1-\nu}} = \frac{z_x(0,t;\nu)}{[\kappa(t)]^{\nu_1-\nu}} \geq \frac{\vartheta(\nu)}{[\kappa(t)]^{\nu_1-\nu}} \to \infty, \quad \text{as } t \to -\infty.$$

Since $\nu \in (0,1)$ and $\nu_1 > \nu$ are arbitrary, we obtain the conclusion in (ii). □

Using this lemma we can prove the uniform upper bound of $\mathcal{U}$:

Proposition 4.6. *Assume* (1.2) *and* (1.8). *Let* $\mathcal{U}$ *be the entire solution obtained in* (4.11). *Then*

$$\kappa(t) := \mathcal{U}(\infty,t) = 1 \text{ for all } t \in \mathbb{R}. \tag{4.18}$$

Proof. Set

$$M := \max_{u\in[0,1]} |f''(u)|, \quad M_1 := \max\left\{ \frac{e^t}{\sqrt{t}} \;\middle|\; t \in J := \left[\frac{1}{\mu}, \frac{2}{\mu}\right] \right\}.$$

If the conclusion is not true, then $\kappa(t_1) \in (0,1)$ for some $t_1 \in \mathbb{R}$. By the above lemma, there exists $\tau < 0$ with $|\tau|$ sufficiently large and $\epsilon := \kappa(\tau) = \mathcal{U}(\infty,\tau) > 0$ sufficiently small such that

$$M\epsilon < \mu^2 = f'(0), \quad 2 < e^{\frac{2\mu}{M\epsilon}}, \qquad \frac{\beta(\tau)}{[\kappa(\tau)]^{3/2}} > 2\sqrt{M}M_1. \tag{4.19}$$

We continue to define a series of parameters. Set

$$\begin{cases} b := f'(0) - \dfrac{M\epsilon}{2} \in \Big(\dfrac{1}{2}f'(0), f'(0)\Big); \\ \epsilon_1 := 2\epsilon e^{-\frac{2\mu}{M\epsilon}} \in (0,\epsilon); \\ \tau_1 := \kappa^{-1}(\epsilon_1) \quad (\Leftrightarrow\ \epsilon_1 = \kappa(\tau_1)\ ); \\ T_1 := \dfrac{1}{b} \ln \dfrac{2\epsilon}{\epsilon_1} = \dfrac{2\mu}{bM\epsilon}. \end{cases} \tag{4.20}$$

Choosing $X_1 > 0$ large so that

$$\mathcal{U}(x,\tau_1) \geq \frac{1}{2}\mathcal{U}(\infty,\tau_1) = \frac{\epsilon_1}{2}, \quad x \geq X_1,$$

we compare $\mathcal{U}$ with the solutions of the following problems:

$$\begin{cases} v_t = v_{xx} + bv, & x > X_1,\ t > 0, \\ v(X_1,t) = 0, & t > 0, \\ v(x,0) = \frac{\epsilon_1}{2}, & x \geq X_1, \end{cases} \tag{4.21}$$

and

$$\begin{cases} w_t = w_{xx} + \mu^2 w, & x > 0,\ t > 0, \\ w(0,t) = 0, & t > 0, \\ w(x,0) = \epsilon_1, & x \geq 0. \end{cases} \tag{4.22}$$

Both problems are linear ones and can be solved explicitly. In particular,

$$w(x,t) = \frac{e^{\mu^2 t}}{2\sqrt{\pi t}} \int_0^\infty \left[e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right] \epsilon_1 dy, \quad x \geq 0,\ t > 0, \tag{4.23}$$

and so

$$w_x(0,t) = \frac{e^{\mu^2 t}}{2t\sqrt{\pi t}} \int_0^\infty y e^{-\frac{y^2}{4t}} \epsilon_1 dy = \frac{\epsilon_1 e^{\mu^2 t}}{\sqrt{\pi t}}, \quad t > 0. \tag{4.24}$$

Clearly, w is a supersolution of (1.4) since $f(w) \le \mu^2 w$ for $w \ge 0$, and so

$$\mathcal{U}_x(0, t+\tau_1) \le w_x(0,t) \le \frac{\epsilon_1 e^{\mu^2 t}}{\sqrt{\pi t}} \text{ for all } t > 0. \tag{4.25}$$

On the other hand, as in (4.23), v can be expressed as

$$\begin{aligned} v(x,t) &= \frac{e^{bt}}{2\sqrt{\pi t}} \int_0^\infty \left[e^{-\frac{(x-X_1-y)^2}{4t}} - e^{-\frac{(x-X_1+y)^2}{4t}} \right] \frac{\epsilon_1}{2} dy \\ &= \frac{\epsilon_1 e^{bt}}{2\sqrt{\pi}} \int_{\frac{X_1-x}{2\sqrt{t}}}^{\frac{x-X_1}{2\sqrt{t}}} e^{-s^2} ds, \quad x \ge X_1,\ t > 0, \end{aligned}$$

and so

$$v(x,t) \le v(\infty, t) = \frac{\epsilon_1 e^{bt}}{2} \le \frac{\epsilon_1}{2} e^{bT_1} = \epsilon, \quad x \ge X_1,\ 0 < t \le T_1.$$

Hence, when $x \ge X_1$ and $t \in (0, T_1]$ we have

$$f(v) \ge f'(0)v - \frac{M}{2}v^2 \ge \left[f'(0) - \frac{M\epsilon}{2} \right] v = bv.$$

This implies that v is a subsolution, and so

$$\mathcal{U}(x, t+\tau_1) \ge v(x,t), \quad x \ge X_1,\ 0 < t \le T_1.$$

In particular, at $t = T_1$ we have

$$\mathcal{U}(\infty, T_1 + \tau_1) \ge v(\infty, T_1) = \frac{\epsilon_1 e^{bT_1}}{2} = \epsilon = \mathcal{U}(\infty, \tau).$$

Thanks to $\mathcal{U}_t(\infty, t) = \kappa'(t) > 0$ we have $T_1 + \tau_1 \ge \tau$. Combining this inequality with (4.20) and (4.25) we have

$$\begin{aligned} \beta(\tau) &= \mathcal{U}_x(0,\tau) \le \mathcal{U}_x(0, T_1 + \tau_1) \\ &\le w_x(0, T_1) \le \frac{\epsilon_1 e^{\mu^2 T_1}}{\sqrt{\pi T_1}} \\ &= \frac{\epsilon_1 e^{bT_1} e^{\frac{M\epsilon}{2} T_1}}{\sqrt{\pi T_1}} = \frac{2\epsilon e^{\frac{M\epsilon}{2} T_1}}{\sqrt{\pi T_1}} \\ &\le 2\sqrt{M} \epsilon^{3/2} \cdot \frac{e^{\frac{M\epsilon}{2} T_1}}{\sqrt{\frac{M\epsilon}{2} T_1}}. \end{aligned}$$

Since

$$\frac{M\epsilon}{2} T_1 = \frac{\mu}{b} \in J := \left[\frac{1}{\mu}, \frac{2}{\mu} \right],$$

we have

$$\frac{e^{\frac{M\epsilon}{2} T_1}}{\sqrt{\frac{M\epsilon}{2} T_1}} \le M_1$$

by the definition of M_1, and so

$$\beta(\tau) \le 2\sqrt{M} M_1 \epsilon^{3/2} = 2\sqrt{M} M_1 [\kappa(\tau)]^{3/2},$$

contradicting (4.19). This proves Proposition 4.6. □

4.3. **The limit of $\mathcal{U}$ as $t \to -\infty$ and the proof of Theorem 1.2.** By the properties of $\mathcal{U}$ obtained above we see that, for any $m \in (0,1)$, the m-level set of $\mathcal{U}(\cdot, s)$ is a unique point $\xi_m(s)$, that is, $x = \xi_m(s)$ is the unique root of $\mathcal{U}(x,s) = m$ for each $s \in \mathbb{R}$. Note that, without the above proposition, that is, if $\kappa(s) \to 0$ $(s \to -\infty)$, then $\mathcal{U}(\cdot, s)$ may have no m-level set when $s \ll -1$.

Lemma 4.7. *$\xi_m(s) \to \infty$ as $s \to -\infty$.*

Proof. By the definition we have

$$m = \mathcal{U}(\xi_m(s), s) - \mathcal{U}(0, s) = \mathcal{U}_x(\xi^*, s) \cdot \xi_m(s) \le \mathcal{U}_x(0, s) \cdot \xi_m(s), \quad \text{for some } \xi^* \in (0, \xi_m(s)).$$

Since $\mathcal{U}_x(0, s) \to 0$ $(s \to -\infty)$ by Proposition 4.4, we have $\xi_m(s) \to \infty$ as $s \to -\infty$. □

Proposition 4.8. *Let $\eta_m(t)$ be the solution of (1.13). Then $\mathcal{U}(x + \xi_m(s), t + s) \to \eta_m(t)$ as $s \to -\infty$, in $C^{2,1}_{loc}(\mathbb{R} \times \mathbb{R})$ topology. In addition, $\xi_m'(s) \to -\infty$ as $s \to -\infty$.*

Proof. For any time sequence $\{s_k\}$ decreasing to $-\infty$ and any $M > 0,\ T > 0$, by the parabolic estimate we have

$$\|\mathcal{U}(x + \xi_m(s_k), t + s_k)\|_{C^{2+\alpha, 1+\alpha/2}([-M,M]\times[-T,T])} \le C,$$

for some constant C depending on M and T but not on k. Hence there is a subsequence $\{s_{k_j}\}$ of $\{s_k\}$ and a function $W_m(x,t) \in C^{2+\alpha,1+\alpha/2}_{loc}(\mathbb{R} \times \mathbb{R})$ such that

$$\mathcal{U}(x + \xi_m(s_{k_j}), t + s_{k_j}) \to W_m(x,t) \text{ as } j \to \infty, \quad \text{in } C^{2,1}_{loc}(\mathbb{R} \times \mathbb{R}) \text{ topology.}$$

Clearly,

$$W_m(0,0) = \lim_{j \to \infty} \mathcal{U}(\xi_m(s_{k_j}), s_{k_j}) = m. \tag{4.26}$$

Note that, for any $x \in [-M, M]$ and $s \in [-T, T]$,

$$\begin{aligned} & \mathcal{U}(x + \xi_m(s_{k_j}), t + s_{k_j}) \\ = \ & \mathcal{U}(x + \xi_m(s_{k_j}), t + s_{k_j}) - \mathcal{U}(\xi_m(s_{k_j}), t + s_{k_j}) + \mathcal{U}(\xi_m(s_{k_j}), t + s_{k_j}) \\ = \ & \mathcal{U}_x(\rho x + \xi_m(s_{k_j}), t + s_{k_j}) \cdot x + \mathcal{U}(\xi_m(s_{k_j}), t + s_{k_j}), \quad \text{for some } \rho \in (0,1). \end{aligned}$$

Since

$$|\mathcal{U}_x(\rho x + \xi_m(s_{k_j}), t + s_{k_j}) \cdot x| \le \mathcal{U}_x(0, t + s_{k_j}) \cdot M \to 0 \text{ as } j \to \infty,$$

and

$$\mathcal{U}(\xi_m(s_{k_j}), t + s_{k_j}) \to W_m(0,t) \text{ as } j \to \infty,$$

we have

$$\mathcal{U}(x+\xi_m(s_{k_j}), t+s_{k_j}) \to W_m(0,t) \text{ as } j\to\infty.$$

Hence

$$W_m(x,t) \equiv W_m(0,t), \quad x,t\in\mathbb{R}.$$

In other words, $W_m(x,t)$ is actually independent of x. Consequently,

$$\mathcal{U}_{xx}(x+\xi_m(s_{k_j}), t+s_{k_j}) \to (W_m)_{xx}(x,t)\equiv 0 \text{ as } j\to\infty, \quad \text{in } L^\infty_{loc}(\mathbb{R}\times\mathbb{R}) \text{ topology.}$$

Now we show that $W_m(x,t)\equiv W_m(0,t)$ is nothing but the solution $\eta_m(t)$ of (1.13). In the equation

$$\mathcal{U}_t(\xi_m(s_{k_j}), t+s_{k_j}) = \mathcal{U}_{xx}(\xi_m(s_{k_j}), t+s_{k_j}) + f(\mathcal{U}(\xi_m(s_{k_j}), t+s_{k_j})),$$

by taking limit as $j\to\infty$ we have

$$[W_m(0,t)]_t = f(W_m(0,t)).$$

Combining with (4.26) we see that $W_m(x,t)\equiv W_m(0,t)$ is the unique solution $\eta_m(t)$ of (1.13).

Since the limit function $\eta_m(t)$ is unique we conclude that

$$\mathcal{U}(x+\xi_m(s), t+s) - \eta_m(t) \to 0 \text{ as } s\to-\infty, \quad \text{in } C^{2,1}_{loc}(\mathbb{R}\times\mathbb{R}) \text{ topology.} \tag{4.27}$$

In particular, at the point $(x,t)=(0,0)$, we have

$$\mathcal{U}_x(\xi_m(s),s)\to 0, \quad \mathcal{U}_t(\xi_m(s),s)\to \eta_m'(0) = f(\eta_m(0)) = f(m) > 0 \quad \text{as } s\to-\infty. \tag{4.28}$$

By differentiating $\mathcal{U}(\xi_m(s),s)\equiv m$ with respect to s we have

$$\mathcal{U}_x(\xi_m(s),s)\cdot\xi_m'(s) + \mathcal{U}_t(\xi_m(s),s)\equiv 0.$$

This reduces to $\xi_m'(s)\to-\infty$ $(s\to-\infty)$ by (4.28), and then the proposition is proved. □

Proof of Theorem 1.2. The conclusions follow from Propositions 4.2 and 4.8 directly. □

Remark 4.9. In the locally uniform topology, the entire solution $\mathcal{U}(x,t)$ obtained in this section can be regarded as a heteroclinic orbit connecting the identically zero and $V(x)$, while the spatially uniform solution $\eta_m(t)$ is a heteroclinic orbit connecting the constant solution 0 and 1 to the equation in the whole space. At $x=\infty$, however, $\mathcal{U}(x,t)$ always takes 1. It seems that this is caused by the effect of the Dirichlet boundary condition. On the other hand, the odd symmetric extension of the solutions around the origin give rise to the symmetric solutions to the Allen-Cahn equation (that is, the reaction-diffusion equation with balanced bistable nonlinearity) in the whole space. Hence, our result also contributes to the study of entire solutions of the Allen-Cahn equation.

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