Analogue of De Giorgi's conjecture in heterogeneous media

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Abstract. We propose analogue of De Giorgi's Conjecture for partial differential equations in heterogeneous media.

Consider

$$u_t = \Delta u + u - u^3, \qquad \boldsymbol{x} = (\boldsymbol{x}', x_n) \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{1}$$

where $\boldsymbol{x}' = (x_1, \cdots, x_{n-1})$. A solution u of (1) is called a *traveling wave solution* in x_n-direction if

$$u(\boldsymbol{x}', x_n, t) = v(\boldsymbol{x}', x_n + ct), \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{2}$$

where c is the traveling speed. Clearly, a 0-speed-traveling wave solution satisfies

$$\Delta u + u - u^3 = 0, \qquad \boldsymbol{x} \in \mathbb{R}^n. \tag{3}$$

On the profile of the solution of (3), De Giorgi [2] proposed the following conjecture:

De Giorgi's Conjecture: *Suppose that u is an entire solution of* (3) *satisfying $|u| \leq 1$ and $\frac{\partial u}{\partial x_n} > 0$. Then, at least for $n \leq 8$, the level sets of u must be hyperplanes.*

This conjecture was proved recently in [3], [8] and references therein.

It is natural to expect that some analogue of De Giorgi's conjecture in heterogeneous media should be also true. We present such a conjecture in the following. First, consider

$$u_t = \nabla(A(\boldsymbol{x})\nabla u) + \frac{1}{\varepsilon^2} B(\boldsymbol{x}) u \left(Z^2(\boldsymbol{x}) - u^2\right), \qquad \boldsymbol{x} \in \mathbb{R}^n, \ t \in \mathbb{R}, \tag{4}$$

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where $\varepsilon > 0$ is a small parameter, A, B and Z are bounded, smooth functions with positive infimums. When $Z(\boldsymbol{x}) \equiv Z_0 > 0$, term $u\,(Z^2 - u^2)$ is the derivative of a typical double-equal-well potential.

By the theory of monotone dynamical systems (see, for example, [5]), it is easily seen that, when ε is sufficiently small, (4) has at least three stationary solutions: Z_+, 0 and Z_- such that $Z_\pm \approx \pm Z$, and $\inf Z_+(\boldsymbol{x}) > 0 > \sup Z_-(\boldsymbol{x})$.

We discuss three cases:

(P1) A, B and Z are periodic in $\boldsymbol{x}$.

(P2) A, B and Z are periodic in x_n, not periodic but bounded in $\boldsymbol{x}'$.

(AP) A, B and Z are almost periodic in $\boldsymbol{x}$.

In case (P1) or (P2) holds, one can define a *periodic traveling wave solution* u of (4), which means that

$$u(\boldsymbol{x}', x_n, t+T) = u(\boldsymbol{x}', x_n - L, t), \qquad (\boldsymbol{x}', x_n) \in \mathbb{R}^n, \quad t \in \mathbb{R},$$

for some $T > 0$, $L > 0$. [9] and references therein studied periodic traveling wave solutions of (4).

Recently, Matano [6] studied the *almost periodic traveling wave solutions* of (4) in case $Z(\boldsymbol{x}) \equiv 1$ and (AP) holds, including the definition, existence, uniqueness, asymptotic stability, speed estimate of such solutions.

From these works, we know that periodic traveling wave solutions (as well as some almost periodic traveling wave solutions) which have average speed 0 must be stationary ones. (In almost periodic case, there indeed exist propagating solutions which have 0-average speed but not stationary ones). So we consider analogue of De Giorgi's conjecture for stationary solution of (4):

Conjecture. *Assume $\varepsilon > 0$ is sufficiently small,* (P1)*, or* (P2)*, or* (AP) *holds. If u is an entire stationary solution of* (4) *satisfying*

$$Z_-(\boldsymbol{x}) \le u(\boldsymbol{x}) \le Z_+(\boldsymbol{x}), \qquad \lim_{x_n \to \pm\infty} u(\boldsymbol{x}) = Z_\pm(\boldsymbol{x}). \tag{5}$$

Then at least for $n \le 8$, the 0*-level set of u: $\{\boldsymbol{x} \mid u(\boldsymbol{x}) = 0\}$ must be the graph of a function $x_n = x_n(\boldsymbol{x}')$.*

Moreover, function $x_n(\boldsymbol{x}')$ is periodic when (P1) *holds, it is bounded when* (P2) *holds, it is almost periodic when* (AP) *holds.*

In case (P1) or (P2) holds, denote by X the period of A, B and Z in x_n-direction. From (5), one see that $\frac{\partial u}{\partial x_n} > 0$ as in De Giorgi's conjecture may be not always true, instead of that

$$u(\boldsymbol{x}', x_n) < u(\boldsymbol{x}', x_n + X), \qquad\qquad \boldsymbol{x} \in \mathbb{R}^n$$

may play a similar role and may be used as additional condition for the Conjecture.

In the following we discuss mean curvature flow equations and give another version of our conjecture.

It is known that, taking singular limit in some reaction diffusion equations one may obtain mean curvature flow equations (cf. [1], [7], etc.). For example, in [4], we derived from (4) a mean curvature flow equation as $\varepsilon \to 0$, which describes the motion of the level set of u:

$$V = (n-1)a(\boldsymbol{x})\kappa + c(\boldsymbol{x})\nabla d(\boldsymbol{x}) \cdot \boldsymbol{n} \qquad \text{for} \quad \boldsymbol{x} \in \Gamma_t \tag{6}$$

where, $\Gamma_t := \{\boldsymbol{x} \mid u(\boldsymbol{x}, t) = 0\}$ is a hypersurface (also called *interface*) in $\mathbb{R}^n$, $\boldsymbol{n}$ is the normal direction on Γ_t, V denotes the normal velocity of Γ_t, κ is the mean curvature, and a, b, c, d are some smooth functions.

Since the dimension of the interface Γ_t is $n-1$, smaller than the dimension of $\boldsymbol{x}$, if one studies (6) instead of (4), more information about the interface such as the shape should be easier to be known. Especially, in case $n = 2$, the interface reduces to a simple plane curve. Thus, we can transform the above conjecture for reaction diffusion equations to a conjecture for mean curvature flow equations.

We call a hypersurface $\Gamma \subset \mathbb{R}^n$ as an *entire stationary solution* of (6) if it satisfies (6) with $V \equiv 0$ and it does not intersect itself and it separates the whole $\mathbb{R}^n$ into two parts. In other words, Γ is a $(n-1)$-dimensional manifold without boundary.

CONJECTURE*. *Assume a, b and c are periodic (or, almost periodic) functions with $\inf a(\boldsymbol{x}) > 0$. If Γ is an entire stationary solution of (6), then at least for $n \le 8$, Γ is the graph of a function $x_n = x_n(\boldsymbol{x}')$.*

Moreover, function x_n is periodic (resp. almost periodic) in case a, b and c are periodic (resp. almost periodic).

In case $n = 2$, hypersurface Γ is a curve in the plane. Conjecture* states that any entire stationary curve of (6) must be periodic (or, almost periodic) oscillation of a straight line.

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References

[1] X. Chen, Generation and propagation of interfaces for reaction-diffusion equations, *J. Differential Equations*, **96** (1992), 116-141.

[2] E. De Giorgi, Convergence problems for functionals and operators, in "Proc. Internat. Meeting on Recent Methods in Nonlinear Analysis" (Rome, 1978) (E. De Giorgi et al., eds.), Pitagora, Bologna, 1979, 131-188.

[3] N. Ghoussoub and C. Gui, On De Giorgi's conjecture in dimensions 4 and 5, *Ann. of Math.*, **157** (2003), 313-334.

[4] B.D. Lou, Singular limits of spatially inhomogeneous reaction diffusion equations, preprint.

[5] H. Matano, Asymptotic behavior and stability of solutions of semilinear diffusion equations, *Publ. RIMS, Kyoto Univ.*, **15** (1979), 401-454.

[6] H. Matano, Traveling waves in spatically inhomogeneous media, preprint.

[7] K.I. Nakamura, H. Matano, D. Hilhorst and R. Schatzle, Singular limit of a reaction-diffusion equations with a spatially inhomogeneous reaction term, *J. Statistical Phys.*, **95** (1999), 1165-1185.

[8] O. Savin, Phase transitions: regularity of flat level sets, *Ann. of Math.*, (to appear).

[9] J. Xin, Front Propagation in Heterogeneous Media, *SIAM Rev.*, **42** (2000), 161-230.