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Hopf and homoclinic bifurcations for near-Hamiltonian systems

Yun Tian, Maoan Han *

Department of Mathematics, Shanghai Normal University, Shanghai, China

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Abstract

We study homoclinic bifurcation of limit cycles in perturbed planar Hamiltonian systems. Suppose that a homoclinic loop is defined by $H = h_s$. Our main result is that a new method is established for computing the coefficients of the expansion of Melnikov functions at $h = h_s$. Then by using those coefficients, more limit cycles would be found around homoclinic loops. An example is also provided to illustrate our method.

Keywords: Homoclinic loop; Hamiltonian system; Melnikov function; Limit cycle

1. Introduction

In the qualitative theory of differential systems, one well-known problem is to estimate the number of limit cycles in a planar polynomial differential system, and to investigate their distributions. It is the second part of Hilbert's sixteenth problem, posed by Hilbert at the Second International Congress of Mathematicians [10]. This problem is very difficult, and is still open even for quadratic systems (see the survey article [14] and the reference cited therein).

One weak version of the second part of Hilbert's sixteenth problem is to study the number of limit cycles in an analytic system of the form

* Corresponding author.
E-mail addresses: ytian22@shnu.edu.cn (Y. Tian), mahan@shnu.edu.cn (M. Han).

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$$\dot{x} = H_y + \varepsilon f(x, y, a), \ \ \dot{y} = -H_x + \varepsilon g(x, y, a), \tag{1}$$

where H, f and g are analytic functions on the plane $\mathbb{R}^2$, $a \in \mathbb{R}^m$ is a vector parameter, and $\varepsilon \in \mathbb{R}$ is small.

When $\varepsilon = 0$ the system (1) reduces to the Hamiltonian system

$$\dot{x} = H_y, \ \ \dot{y} = -H_x. \tag{2}$$

We suppose that (2) has a family of periodic orbits $L_h \subset \{(x, y)|\ H(x, y) = h\}$ with two boundaries: an elementary center C as the inner boundary and a homoclinic loop L_S as the outer boundary which passes through a hyperbolic saddle S. Without loss of generality, suppose $S = (0, 0)$, $C = (x_c, 0)$, $x_c \neq 0$ and $h \in (h_c, h_s)$ with $h_c = H(x_c, 0)$ and $h_s = H(0, 0)$.

The main task for (1) is to study the number of limit cycles in a open set containing the union $U = \bigcup\limits_{h_c \le h \le h_s} L_h$. One of main tools to study the problem is the so-called first order Melnikov function below

$$M(h, a) = \oint\limits_{L_h} g\mathrm{d}x - f\mathrm{d}y, \tag{3}$$

by which estimating the number of limit cycles bifurcating from periodic orbits L_h for $h \in (h_c, h_s)$ is converted to determining the number of isolated zeros of $M(h, a)$ in the interval (h_c, h_s). The problem of studying zeros of $M(h, a)$ is called the weakened Hilbert's 16th problem when the functions H, f and g are real polynomials. In recent years, many results on this problem have been obtained, for instance see [5,7,11–13,16,19,20,22].

For Hopf bifurcation, Melnikov function $M(h, a)$ is analytic at the end point h_c and has an expansion of the form

$$M(h, a) = \sum_{j \ge 0} b_j(a)(h - h_c)^{j+1}, \ \ 0 \le h - h_c \ll 1. \tag{4}$$

A general method was established in Han–Yang–Yu [7] for computation of the coefficients b_j in (4) and for finding limit cycles near the center C with the help of the coefficients. Thus, it is possible to find all the small-amplitude limit cycles in Hopf bifurcation if enough coefficients b_j are obtained and can be solved.

For homoclinic bifurcation, from [2,17] $M(h, a)$ has the following expansion

$$\begin{aligned} M(h, a) &= \sum_{j \ge 0} [c_{2j}(a) + c_{2j+1}(a)(h - h_s)\ln|h - h_s|](h - h_s)^j \\ &= c_0(a) + c_1(a)(h - h_s)\ln|h - h_s| + c_2(a)(h - h_s) \\ &\quad + c_3(a)(h - h_s)^2 \ln|h - h_s| + \cdots, \ \ 0 < h_s - h \ll 1. \end{aligned} \tag{5}$$

There are very few results about the computation of the coefficients c_j. Only the first four coefficients c_j were obtained in [8,9], but how to get the others is still an open problem. Obviously,

$$c_0(a) = M(h_s, a) = \oint_{L_{h_s}} g dx - f dy.$$

Han–Ye [9] obtained formulas of c_1 and c_2 as follows

$$\begin{gathered} c_1 = -\frac{1}{|\lambda|}\bar{c}_1, \quad c_2 = \bar{c}_2 + \beta\bar{c}_1, \\ \bar{c}_1 = (f_x + g_y)(S, a), \quad \bar{c}_2 = \oint_{L_{h_s}} (f_x + g_y - \bar{c}_1)dt, \end{gathered} \tag{6}$$

where β is a constant, and $\pm\lambda$ are the eigenvalues of the matrix

$$\mathrm{Hess}(S) = \begin{pmatrix} H_{xy}(S) & H_{yy}(S) \\ -H_{xx}(S) & -H_{xy}(S) \end{pmatrix}.$$

The formula of c_3 was given by Han et al. [8] in the form

$$\begin{aligned} c_3(a) = \frac{-1}{2|\lambda|\lambda}\{(-3a_{30} - b_{21} + a_{12} + 3b_{03}) - \\ \frac{1}{\lambda}[(2b_{02} + a_{11})(3h_{03} - h_{21}) + (2a_{20} + b_{11})(3h_{30} - h_{12})]\} + bc_1(a) \end{aligned} \tag{7}$$

for some constant b, when H, f and g can be written as

$$H(x, y) = h_s + \frac{\lambda}{2}(y^2 - x^2) + \sum_{i+j\geq 3} h_{ij}x^i y^j, \tag{8}$$

and

$$f(x, y, a) = \sum_{i+j\geq 0} a_{ij}x^i y^j, \quad g(x, y, a) = \sum_{i+j\geq 0} b_{ij}x^i y^j$$

for (x, y) near S. More information about homoclinic bifurcation can be found in [6, Chapter 3].

In this paper, we shall establish a way to compute coefficients c_j, $j > 3$. What is more, we first show how to use our method to find formulas of c_3 and c_4 under some conditions and c_5 and c_6 under further conditions in section 2. Then in section 3 we generalize our results to double homoclinic bifurcation in the centrally symmetric case. In section 4, we present some application to some polynomial systems.

2. Hopf and homoclinic bifurcation

From Han [4] we have the following lemma.

Lemma 1. *For the function M defined by (3), we have*

$$\frac{\partial M}{\partial h} = \oint_{L_h} (f_x + g_y)\mathrm{d}t.$$

By Lemma 1, (4) and (5) we have

$$b_0(a) = T_0\bar{b}_0(a), \quad \bar{b}_0(a) = (f_x + g_y)(C, a), \tag{9}$$

and

$$\lim_{h\to h_s^-} \oint_{L_h} (f_x + g_y)\mathrm{d}t = \oint_{L_{h_s}} (f_x + g_y)\mathrm{d}t \in \mathbb{R} \Leftrightarrow \bar{c}_1(a) = 0,$$

where $T_0 > 0$ is a constant, and $\bar{c}_1$ is given in (6).

Theorem 1. *Suppose there exist analytic functions $P_1(x, y, a)$ and $Q_1(x, y, a)$ such that for $\bar{b}_0 = \bar{c}_1 = 0$, the following equality holds*

$$f_x + g_y = H_x(x, y)P_1(x, y, a) + H_y(x, y)Q_1(x, y, a), \quad (x, y) \in U \tag{10}$$

where U is defined in the first section. Then when $\bar{b}_0 = \bar{c}_1 = 0$, we have for b_1 in (4) and c_3 and c_4 in (5)

$$b_1 = T_1\bar{b}_1(a), \quad c_3(a) = -\frac{1}{2|\lambda|}\bar{c}_3(a), \quad c_4(a) = \frac{1}{2}\bar{c}_4 + \beta_1\bar{c}_3(a), \tag{11}$$

where T_1 and β_1 are constants with $T_1 > 0$, and

$$\begin{aligned} \bar{b}_1(a) &= (P_{1x} + Q_{1y})(C, a), \\ \bar{c}_3(a) &= (P_{1x} + Q_{1y})(S, a), \\ \bar{c}_4(a) &= \oint_{L_{h_s}} (P_{1x} + Q_{1y} - \bar{c}_3(a))\mathrm{d}t. \end{aligned} \tag{12}$$

Further, if we let $H(x, y)$ satisfy (8), and

$$P_1(x, y, a) = \sum_{i+j\geq 0} \tilde{a}_{ij}x^i y^j, \quad Q_1(x, y, a) = \sum_{i+j\geq 0} \tilde{b}_{ij}x^i y^j$$

for (x, y) near S, then $c_5(a) = \dfrac{-1}{6|\lambda|}\bar{c}_5 + b\bar{c}_3$ for some constant b under $\bar{b}_0 = \bar{c}_1 = 0$, where

$$\begin{aligned} \bar{c}_5(a) = \frac{1}{\lambda}\{&(-3\tilde{a}_{30} - \tilde{b}_{21} + \tilde{a}_{12} + 3\tilde{b}_{03}) - \frac{1}{\lambda}[(2\tilde{b}_{02} + \tilde{a}_{11}) \\ &\cdot (3h_{03} - h_{21}) + (2\tilde{a}_{20} + \tilde{b}_{11})(3h_{30} - h_{12})]\}. \end{aligned} \tag{13}$$

Proof. Let $\bar{b}_0 = \bar{c}_1 = 0$. We have $b_0 = c_1 = 0$ from (6) and (9). Then by (4) and (5), it is easy to get

$$\frac{\partial M}{\partial h} = 2b_1(h - h_c) + 3b_2(h - h_c)^2 + 4b_3(h - h_c)^3 + \cdots, \tag{14}$$

for $0 \le h - h_c \ll 1$, and

$$\begin{aligned}\frac{\partial M}{\partial h} &= c_2 + 2c_3(h - h_s)\ln|h - h_s| + (c_3 + 2c_4)(h - h_s)\\ &\quad + 3c_5(h - h_s)^2\ln|h - h_s| + (c_5 + 3c_6)(h - h_s)^2\\ &\quad + 4c_7(h - h_s)^3\ln|h - h_s| + \cdots\end{aligned} \tag{15}$$

for $0 < h_s - h \ll 1$.

On the other hand, by Lemma 1 and (10) we have

$$\begin{aligned}\frac{\partial M}{\partial h} &= \oint_{L_h} (H_x P_1 + H_y Q_1)\mathrm{d}t\\ &= \oint_{L_h} Q_1 \mathrm{d}x - P_1 \mathrm{d}y \equiv M_1(h, a).\end{aligned} \tag{16}$$

Applying formulas (4), (5), (6) and (9) to the function M_1 above we obtain

$$M_1(h, a) = b_{10}(a)(h - h_c) + b_{11}(a)(h - h_c)^2 + \cdots$$

for $0 \le h - h_c \ll 1$, and

$$\begin{aligned}M_1(h, a) = c_{10}(a) + c_{11}(a)(h - h_s)\ln|h - h_s| + c_{12}(a)(h - h_s) +\\ c_{13}(a)(h - h_s)^2\ln|h - h_s| + c_{14}(a)(h - h_s)^2 + \cdots\end{aligned}$$

for $0 < h_s - h \ll 1$, where

$$\begin{aligned}b_{10}(a) &= T_0(P_{1x} + Q_{1y})(C, a) = T_0\bar{b}_1(a),\\ c_{11} &= -\frac{1}{|\lambda|}(P_{1x} + Q_{1y})(S, a) = -\frac{1}{|\lambda|}\bar{c}_3,\\ c_{12} &= \oint_{L_{h_s}} (P_{1x} + Q_{1y} - \bar{c}_3)\mathrm{d}t + \beta\bar{c}_3 = \bar{c}_4 + \beta\bar{c}_3,\end{aligned}$$

where T_0 and β are constants as before. Then comparing the two expansions of M_1 above with (14) and (15) respectively, one obtains $2b_1 = b_{10}$, $2c_3 = c_{11}$ and $c_3 + 2c_4 = c_{12}$, or

$$b_1 = \frac{T_0}{2}\bar{b}_1, \quad c_3 = -\frac{1}{2|\lambda|}\bar{c}_3, \quad c_4 = \frac{1}{2}\bar{c}_4 + \frac{1}{2}(\beta + \frac{1}{|\lambda|})\bar{c}_3.$$

Then (11) follows. Similarly, by (7), we can find $c_5(a) = \frac{-1}{6|\lambda|}\bar{c}_5 + b\bar{c}_3$, where $\bar{c}_5$ is given by (13). The proof is completed.

From the above proof, one can easily see that

$$\bar{c}_2 = \oint_{L_{h_s}} Q_1 \mathrm{d}x - P_1 \mathrm{d}y.$$

This formula can be used to calculate $\bar{c}_2$, simpler than the one in (6) in some cases.

By Lemma 1 and (16) we have

$$\frac{\partial^2 M}{\partial h^2} = \frac{\partial M_1}{\partial h} = \oint_{L_h} (P_{1x} + Q_{1y}) \mathrm{d}t.$$

If there exist analytic functions $P_2(x, y, a)$ and $Q_2(x, y, a)$ such that for $\bar{b}_0 = \bar{b}_1 = 0$, $\bar{c}_1 = \bar{c}_3 = 0$

$$P_{1x} + Q_{1y} = H_x(x, y)P_2(x, y, a) + H_y(x, y)Q_2(x, y, a), \quad (x, y) \in U, \tag{17}$$

then, as before, we have further

$$\frac{\partial^2 M}{\partial h^2} = \oint_{L_h} Q_2 \mathrm{d}x - P_2 \mathrm{d}y \equiv M_2(h, a).$$

Therefore, applying formulas (4), (5), (6) and (9) again we obtain

$$M_2(h, a) = b_{20}(a)(h - h_c) + b_{21}(a)(h - h_c)^2 + \cdots \tag{18}$$

for $0 \le h - h_c \ll 1$, and

$$\begin{aligned} M_2(h, a) &= c_{20}(a) + c_{21}(a)(h - h_s)\ln|h - h_s| + c_{22}(a)(h - h_s) \\ &\quad + c_{23}(a)(h - h_s)^2 \ln|h - h_s| + \cdots \end{aligned} \tag{19}$$

for $0 < h_s - h \ll 1$, where

$$b_{20}(a) = T_0\bar{b}_2(a), \quad c_{21}(a) = -\frac{1}{|\lambda|}\bar{c}_5(a), \quad c_{22}(a) = \bar{c}_6(a) + \beta\bar{c}_5(a), \tag{20}$$

and

$$\begin{aligned} &\bar{b}_2 = (P_{2x} + Q_{2y})(C, a), \ \bar{c}_5 = (P_{2x} + Q_{2y})(S, a), \\ &\bar{c}_6 = \oint_{L_{h_s}} (P_{2x} + Q_{2y} - \bar{c}_5) \mathrm{d}t. \end{aligned} \tag{21}$$

Under $\bar{b}_0 = \bar{b}_1 = 0$ and $\bar{c}_1 = \bar{c}_3 = 0$, we also have by (14) and (15)

$$\frac{\partial^2 M}{\partial h^2} = 6b_2(h - h_c) + 12b_3(h - h_c)^2 + \cdots \tag{22}$$

for $0 \le h - h_c \ll 1$, and

$$\begin{aligned}\frac{\partial^2 M}{\partial h^2} = 2c_4 + 6c_5(h - h_s)\ln|h - h_s| + (5c_5 + 6c_6)(h - h_s) \\ + 12c_7(h - h_s^2)\ln|h - h_s| + \cdots\end{aligned} \tag{23}$$

for $0 < h_s - h \ll 1$. Hence it follows from (18), (19), (22) and (23) that

$$6b_2 = b_{20},\ \ 2c_4 = c_{20},\ \ 6c_5 = c_{21},\ \ 5c_5 + 6c_6 = c_{22},\ 12c_7 = c_{23}$$

which, together with (20), gives

$$b_2 = T_2\bar{b}_2,\ c_5 = -\frac{1}{6|\lambda|}\bar{c}_5,\ c_6 = \frac{1}{6}\bar{c}_6 + \beta_2\bar{c}_5,\ c_7 = \frac{1}{12}c_{23}, \tag{24}$$

and

$$\bar{c}_4 = \oint_{L_{h_s}} Q_2 \mathrm{d}x - P_2 \mathrm{d}y,$$

where $T_2 = \frac{T_0}{6} > 0$, $\beta_2 = \frac{\beta}{6} + \frac{5}{36|\lambda|}$. Hence, $\bar{c}_5$ in (13) is the same as in (21), and the formula of it in (21) is easy to use. Similar to $\bar{c}_2$, the formula of $\bar{c}_4$ above is also easy to use relatively comparing to the one in (12). If $H(x, y)$ satisfies (8), and

$$P_2(x, y, a) = \sum_{i+j\ge 0} \bar{a}_{ij}x^i y^j, \quad Q_2(x, y, a) = \sum_{i+j\ge 0} \bar{b}_{ij}x^i y^j$$

for (x, y) near S, then by (7)

$$\begin{aligned} c_7(a) &= \frac{-1}{24|\lambda|}\bar{c}_7(a) + O(\bar{c}_5), \\ \bar{c}_7(a) &= \frac{1}{\lambda}\{(-3\bar{a}_{30} - \bar{b}_{21} + \bar{a}_{12} + 3\bar{b}_{03}) - \frac{1}{\lambda}[(2\bar{b}_{02} + \bar{a}_{11}) \\ &\quad \cdot (3h_{03} - h_{21}) + (2\bar{a}_{20} + \bar{b}_{11})(3h_{30} - h_{12})]\}. \end{aligned} \tag{25}$$

Thus, we have verified the following theorem.

Theorem 2. *Suppose there exist analytic functions P_1, Q_1, P_2 and Q_2 such that (10) and (17) are satisfied for $\bar{b}_0 = \bar{b}_1 = 0$ and $\bar{c}_1 = \bar{c}_3 = 0$. Then (21), (24) and (25) hold.*

Remark 1. Set $\bar{c}_0(a) = c_0(a)$. By (6), (9), (11), (21), (24) and (25), we can use $\bar{b}_i$, $i = 0, 1, 2$ and $\bar{c}_j$, $0 \le j \le 7$ to study the bifurcation problem of limit cycles. More specifically, let (10) hold for $\bar{b}_0 = \bar{c}_1 = 0$. Suppose for some $a_0 \in R^m$ we have

$$\bar{b}_0(a_0) = 0, \quad \bar{c}_j(a_0) = 0, \quad j = 0, \cdots, k-1, \quad \mu \equiv \bar{b}_1(a_0)\bar{c}_k(a_0) \neq 0,$$

and

$$\operatorname{rank} \frac{\partial(\bar{b}_0, \bar{c}_0, \bar{c}_1, \cdots, \bar{c}_{k-1})}{\partial(a_1, \cdots, a_m)}(a_0) = k + 1, \tag{26}$$

where $k \le 5$. First, by (4) and (5), it can be determined from the sign of μ whether $M(h)$ has opposite signs for h near h_c and h_s. If so, $M(h)$ has a zero in (h_c, h_s). Then by (26) we take $\bar{b}_0$ and $\bar{c}_j$, $j = 0, \cdots, k-1$, as free parameters, and vary them suitably near zero to produce k zeros near h_s and one zero near h_c. Similarly, if further (17) holds for $\bar{b}_1 = \bar{c}_3 = 0$, $\bar{b}_2$, $\bar{c}_6$ and $\bar{c}_7$ can be also obtained and used for the study of the bifurcation of limit cycles.

We remark that following the method above we can give further conditions to obtain formulas of more coefficients c_8, c_9 and others. Here, we describe the idea in general. First, we use the formulas (6), (7) and (9) to compute c_1, c_2, c_3 in (5) and b_0 in (4). Then, we suppose (10) is satisfied under $b_0 = c_1 = 0$, and obtain (16). Thus,

$$M_1(h, a) = b_{10}(a)(h - h_c) + b_{11}(a)(h - h_c)^2 + \cdots$$

for $0 < h - h_c \ll 1$, and

$$\begin{aligned} M_1(h, a) = c_{10}(a) + c_{11}(a)(h - h_s)\ln|h - h_s| + c_{12}(a)(h - h_s) + \\ c_{13}(a)(h - h_s)^2 \ln|h - h_s| + c_{14}(a)(h - h_s)^2 + \cdots \end{aligned}$$

for $0 < h_s - h \ll 1$. Here, we use the formulas (6), (7) and (9) to compute c_{11}, c_{12}, c_{13} and b_{10}. On the other hand, by (4) and (5) we have

$$M_1(h, a) = \sum_{j \ge 0} (j + 2) b_{j+1} (h - h_c)^{j+1}$$

for $0 < h - h_c \ll 1$, and

$$M_1(h, a) = \sum_{j \ge 0} [c_{2j+1} + (j + 1)c_{2j+2} + (j + 2)c_{2j+3}(h - h_s)\ln|h - h_s|](h - h_s)^j$$

for $0 < h_s - h \ll 1$ under $b_0 = c_1 = 0$. Therefore, we have the following relations

$$\begin{aligned} c_{1,2j} = c_{2j+1} + (j + 1)c_{2j+2}, \quad c_{1,2j+1} = (j + 2)c_{2j+3}, \\ b_{1,j} = (j + 2)b_{j+1}, \quad j \ge 0 \end{aligned} \tag{27}$$

which especially yield the formulas of b_1, c_4 and c_5.

Further, we suppose (17) holds under $b_0 = b_{10} = 0$ and $c_1 = c_{11} = 0$. Then we obtain

$$\frac{\partial^2 M}{\partial h^2} = \oint_{L_h} Q_2 \mathrm{d}x - P_2 \mathrm{d}y \equiv M_2(h, a)$$

and expansions (18) and (19). As before, we can use the formulas (6), (7) and (9) to compute c_{21}, c_{22}, c_{23} and b_{20}. On the other hand, by (27) we have the following relations

$$\begin{gathered} c_{2,2j} = c_{1,2j+1} + (j+1)c_{1,2j+2}, \quad c_{2,2j+1} = (j+2)c_{1,2j+3}, \\ b_{2,j} = (j+2)b_{1,j+1}, \quad j \geq 0 \end{gathered} \tag{28}$$

which, together with (27), yield the formulas of b_2, c_6 and c_7.

If we let

$$P_{2x} + Q_{2y} = H_x(x, y)P_3(x, y, a) + H_y(x, y)Q_3(x, y, a), \quad (x, y) \in U \tag{29}$$

under $b_0 = b_{10} = b_{20} = 0$ and $c_1 = c_{11} = c_{21} = 0$, then we have further

$$\begin{aligned} &\frac{\partial^3 M}{\partial h^3} = \oint_{L_h} Q_3 \mathrm{d}x - P_3 \mathrm{d}y \equiv M_3(h, a), \\ &M_3(h, a) = b_{30}(a)(h - h_c) + b_{31}(a)(h - h_c)^2 + \cdots \end{aligned}$$

for $0 \leq h - h_c \ll 1$, and

$$\begin{aligned} M_3(h, a) = {} & c_{30}(a) + c_{31}(a)(h - h_s)\ln|h - h_s| + c_{32}(a)(h - h_s) + \\ & c_{33}(a)(h - h_s)^2 \ln|h - h_s| + c_{34}(a)(h - h_s)^2 + \cdots \end{aligned}$$

for $0 < h_s - h \ll 1$. Also as before, we can use the formulas (6), (7) and (9) to compute c_{31}, c_{32}, c_{33} and b_{30}, and use (27) to obtain the following relations

$$\begin{gathered} c_{3,2j} = c_{2,2j+1} + (j+1)c_{2,2j+2}, \quad c_{3,2j+1} = (j+2)c_{2,2j+3}, \\ b_{3,j} = (j+2)b_{2,j+1}, \quad j \geq 0 \end{gathered}$$

which, together with (27) and (28), yield the formulas of b_3, c_8 and c_9. In fact, we can obtain $\bar{b}_3$, $\bar{c}_8$ and $\bar{c}_9$ by replacing (P_2, Q_2) by (P_3, Q_3) in the formulas of $\bar{b}_2$, $\bar{c}_6$ and $\bar{c}_7$ given in (21) and (25).

3. Hopf and double homoclinic bifurcation

In this section, we suppose that the equation $H(x, y) = h_s$ defines a double homoclinic loop $L = L_1 \bigcup L_2$ consisting of two homoclinic loops L_1 and L_2 with a hyperbolic saddle S. Then the equation $H(x, y) = h$ defines a family of periodic orbits $L(h)$ for h on one side of h_s, and two families of periodic orbits $L_1(h)$ and $L_2(h)$ for h on the other side of h_s. For definiteness, let $L(h)$ exist for $h \in (h_s, \bar{h})$, and both $L_1(h)$ and $L_2(h)$ exist for $h \in (h_c, h_s)$, where $L_1(h)$ and

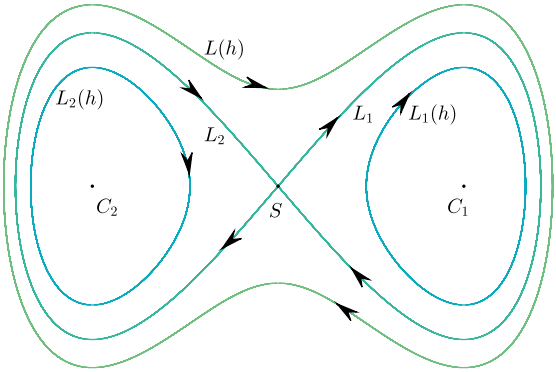

Fig. 1. Phase portrait of a double homoclinic loop.

$L_2(h)$ approach center points C_1 and C_2 respectively as $h \to h_c^+$. The phase portrait is shown in Fig. 1.

We use our method presented in section 2 to study bifurcation of limit cycles from orbits $L_1(h)$ and $L_2(h)$, respectively. We shall show that our method can be also applied to study double homoclinic bifurcation for orbits $L(h)$. For simplicity, assume that the vector field defined by (1) is centrally symmetric. For the case where (1) is not centrally symmetric, some results can be found in [6, Chapter 3].

For system (1), we have three Melnikov functions as follows

$$\begin{aligned} M(h,a) &= \oint_{L(h)} g\mathrm{d}x - f\mathrm{d}y, \ h \in (h_s, \bar{h}), \\ M_i(h,a) &= \oint_{L_i(h)} g\mathrm{d}x - f\mathrm{d}y, \ h \in (h_c, h_s), \ i = 1, 2. \end{aligned} \tag{30}$$

Because of the symmetry, $M_1(h,a) = M_2(h,a)$. As before, we have

$$M_1(h,a) = \sum_{j\geq 0} b_j(a)(h-h_c)^{j+1}, \ \ 0 \leq h - h_c \ll 1, \tag{31}$$

and

$$\begin{aligned} M_1(h,a) &= \sum_{j\geq 0} [c_{2j}(a) + c_{2j+1}(a)(h-h_s)\ln|h-h_s|](h-h_s)^j \\ &= c_0(a) + c_1(a)(h-h_s)\ln|h-h_s| + c_2(a)(h-h_s) \\ &\quad + c_3(a)(h-h_s)^2\ln|h-h_s| + \cdots, \ \ 0 < h_s - h \ll 1. \end{aligned} \tag{32}$$

By [21] we have

$$\begin{aligned} M(h,a) &= \sum_{j\geq 0}[\tilde{c}_{2j}(a)+\tilde{c}_{2j+1}(a)(h-h_s)\ln|h-h_s|](h-h_s)^j \\ &= \tilde{c}_0(a)+\tilde{c}_1(a)(h-h_s)\ln|h-h_s|+\tilde{c}_2(a)(h-h_s) \\ &\quad +\tilde{c}_3(a)(h-h_s)^2\ln|h-h_s|+\cdots, \quad 0<h-h_s\ll 1, \end{aligned} \tag{33}$$

where

$$\tilde{c}_0(a)=2c_0(a),\ \tilde{c}_1(a)=2c_1(a),\ \tilde{c}_2(a)=2c_2(a)+O(c_1),\ \tilde{c}_3(a)=2c_3(a). \tag{34}$$

Let

$$\begin{aligned} &\bar{b}_0(a)=(f_x+g_y)(C_1,a),\ \bar{c}_0(a)=\oint_{L_1} g\,dx-f\,\mathrm{d}y \\ &\bar{c}_1=(f_x+g_y)(S,a),\ \ \bar{c}_2=\oint_{L_1}(f_x+g_y-\bar{c}_1)\mathrm{d}t. \end{aligned} \tag{35}$$

And, as before, suppose there exist analytic functions $P_1(x,y,a)$ and $Q_1(x,y,a)$ such that (10) holds for $\bar{b}_0=\bar{c}_1=0$, where the region U in (10) is given by

$$U=U_1\bigcup U_2\bigcup U_3,\ U_j=\bigcup_{h_c\leq h\leq h_s}L_j(h),\ j=1,2,\ U_3=\bigcup_{h_s\leq h<\bar{h}}L(h).$$

Under (10), let further

$$\begin{aligned} \bar{b}_1(a)&=(P_{1x}+Q_{1y})(C_1,a), \\ \bar{c}_3(a)&=(P_{1x}+Q_{1y})(S,a), \\ \bar{c}_4(a)&=\oint_{L_1}(P_{1x}+Q_{1y}-\bar{c}_3(a))\mathrm{d}t, \end{aligned} \tag{36}$$

and $\bar{c}_5$ is given by (13).

Further suppose there exist analytic functions $P_2(x,y,a)$ and $Q_2(x,y,a)$ such that (17) holds for $\bar{b}_0=\bar{c}_1=0$ and $\bar{b}_1=\bar{c}_3=0$. In this case, we have $\bar{c}_5(a)=(P_{2x}+Q_{2y})(S,a)$. Let

$$\begin{aligned} \bar{b}_2(a)&=(P_{2x}+Q_{2y})(C_1,a), \\ \bar{c}_6(a)&=\oint_{L_1}(P_{2x}+Q_{2y}-\bar{c}_5(a))\mathrm{d}t, \end{aligned} \tag{37}$$

and $\bar{c}_7$ is given in (25). Then by (31), (32) and (33), and the discussion in section 2, we have

$$\begin{aligned} &b_0=T_0\bar{b}_0,\ c_0=\bar{c}_0,\ c_1=-\frac{1}{|\lambda|}\bar{c}_1,\ \ c_2=\bar{c}_2+O(\bar{c}_1), \\ &b_1=T_1\bar{b}_1,\ c_3=-\frac{1}{2|\lambda|}\bar{c}_3,\ \ c_4=\frac{1}{2}\bar{c}_4+O(\bar{c}_3),\ \ c_5=-\frac{1}{6|\lambda|}\bar{c}_5, \end{aligned} \tag{38}$$

and

$$b_2 = T_2\bar{b}_2, \quad c_6 = \frac{1}{6}\bar{c}_6 + O(\bar{c}_5), \quad c_7 = \frac{-1}{24|\lambda|}\bar{c}_7 + O(\bar{c}_5), \tag{39}$$

where T_0, T_1 and T_2 are positive constants. Then by (34) we further have

$$\tilde{c}_0 = 2\bar{c}_0, \ \tilde{c}_1 = -\frac{2}{|\lambda|}\bar{c}_1, \ \tilde{c}_2 = 2\bar{c}_2 + O(\bar{c}_1). \tag{40}$$

Similar to Theorems 1 and 2, we can prove

$$\tilde{c}_3 = -\frac{1}{|\lambda|}\bar{c}_3, \ \tilde{c}_4 = 2\bar{c}_4 + O(\bar{c}_3), \ \tilde{c}_5 = -\frac{1}{3|\lambda|}\bar{c}_5 \tag{41}$$

under (10), and

$$\tilde{c}_6 = \frac{1}{3}\bar{c}_6 + O(\bar{c}_5), \quad \tilde{c}_7 = 2c_7 = \frac{-1}{12|\lambda|}\bar{c}_7 + O(\bar{c}_5) \tag{42}$$

under (17).

Thus, we have proved the following theorem.

Theorem 3. *Suppose the system (1) is centrally symmetric. That is, the right-hand side functions of it are odd in (x, y). Then the coefficients in the expansions (31), (32) and (33) satisfy (38), (40) and (41) under (10), and furthermore (39) and (42) under (17).*

Remark 2. By $\bar{b}_i$ and $\bar{c}_j$ given by (13), (25), (35), (36) and (37), $i = 0, 1, 2$, $j = 0, \cdots, 7$, we can study limit cycles of system (1) produced from the interiors of L_1 and L_2, and the exterior of L. For example, suppose that there exists $a_0 \in R^m$ such that

$$\bar{b}_0(a_0) = 0, \quad \bar{c}_j(a_0) = 0, \quad j = 0, 1, 2, 3, \quad \mu \equiv \bar{b}_1(a_0)\bar{c}_4(a_0) > 0,$$

and

$$\operatorname{rank}\frac{\partial(\bar{b}_0, \bar{c}_0, \bar{c}_1, \bar{c}_2, \bar{c}_3)}{\partial(a_1, \cdots, a_m)}(a_0) = 5.$$

As discussed in Remark 1, we can find 5 limit cycles surrounding C_1 by varying $\bar{b}_0$ and $\bar{c}_j$, $j = 0, 1, 2, 3$, and simultaneously find 2 limit cycles surrounding both center points C_1, C_2 and the saddle point S by (40) and (41). Because the symmetry of system (1), there are 5 limit cycles surrounding C_2. Therefore, (1) has 12 limit cycles in U with distribution $(5, 5, 2)$ for some (ε, a) near $(0, a_0)$.

4. Application to a class of smooth systems

In sections 2 and 3, we have illuminated the basic idea of computing coefficients c_j for bifurcations of limit cycles near a homoclinic loop and a double homoclinic loop, respectively. It is required that analytic functions P_j and Q_j exist, $j = 1, 2$, such that (10) and (17) hold. If the Hamiltonian is given by

$$H(x, y) = \int_0^x q(x)\mathrm{d}x + \int_0^y p(y)\mathrm{d}y$$

in system (1), it is easy to find functions

$$P_1(x, y) = \frac{F(x, 0)}{q(x)}, \quad Q_1(x, y) = \frac{F(x, y) - F(x, 0)}{p(y)}, \tag{43}$$

satisfying (10) when $\bar{b}_0 = \bar{c}_1 = 0$, where $F(x, y) = f_x(x, y) + g_y(x, y)$.

In this section, we shall present an example to apply our method. Consider perturbed Liénard systems given by

$$\dot{x} = y + \varepsilon f(x, y), \quad \dot{y} = -q(x) + \varepsilon g(x, y), \tag{44}$$

where $0 < \varepsilon \ll 1$, and $f(x, y)$ and $g(x, y)$ are polynomials in x and y. The corresponding Hamiltonian is $H = y^2/2 + \int_0^x q(x)\mathrm{d}x$.

There are lots of papers studying the bifurcation of limit cycles in system (44). Most of them consider the cases of $f(x, y) = 0$ and $g(x, y) = g_1(x)y$, where $g_1(x)$ is a polynomial in x, for instance see books [1,6]. Here, we focus on the cases where $f(x, y)$ and $g(x, y)$ are arbitrary polynomials with $n = \max(\deg(f(x, y)), \deg(g(x, y)))$.

When $q(x)$ is a quadratic polynomial, by a proper linear transformation, system $(44)|_{\varepsilon=0}$ with a center can be changed into the so-called Bogdanov–Takens Hamiltonian system, where the Hamiltonian $H = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3$ has a homoclinic loop. Proved by Petrov [18] and Mardešić [15], the sharp upper bound is $n - 1$ for the total number of zeros of $M_1(h)$ in any finite interval of h.

For $\deg(q(x)) = 3$, the problem becomes much more difficult. Recently, Gavrilov and Iliev [3] studied system (44) with the normal form of Hamiltonian $H = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{a}{4}x^4$, $a \in \mathbb{R}/\{0, \frac{8}{9}\}$. In this case, the Hamiltonian can have four topological structures: saddle-loop $a < 0$, eight-loop $0 < a < 1$, cuspidal loop $a = 1$ and global center $a > 1$. In [3], several upper bounds were found for the number of zeros of Abelian integrals for cases of saddle-loop and eight-loop, respectively.

In our example, we shall consider Hopf bifurcation and homoclinic bifurcation for the case $a = \frac{8}{9}$. To study homoclinic bifurcation of the other cases mentioned above, it will involve much more computation, and it is beyond the scope of this manuscript.

After moving the saddle point to the origin, we consider the following perturbed Liénard system

$$\dot{x} = y + \varepsilon f(x, y), \quad \dot{y} = x - x^3 + \varepsilon g(x, y), \tag{45}$$

with the Hamiltonian $H(x,y)=\frac{1}{2}y^2-\frac{1}{2}x^2+\frac{1}{4}x^4$, where $0<\varepsilon\ll 1$, $f(x,y)$ and $g(x,y)$ are polynomials satisfying $-f(x,y)=f(-x,-y)$ and $-g(x,y)=g(-x,-y)$. Then system (45) is centrally symmetric.

It is easy to see that $C_1=(1,0)$ and $C_2=(-1,0)$ are centers, and an eight-loop is defined by $H(x,y)=0$ with the saddle point S at the origin. The two families of periodic orbits $L_1(h)$ and $L_2(h)$ are given by $H(x,y)=h$, $h\in(-1/4,0)$, and another family of periodic orbits $L(h)$ is defined by $H(x,y)=h$, $h\in(0,\infty)$. We have the following theorem.

Theorem 4. *Suppose* $n=\max(\deg(f(x,y)),\deg(g(x,y)))$. *For* $n=3,5,7,9$, *system* (45) *can have* $[\frac{7n-6}{3}]$ *limit cycles under proper perturbations with distribution,* $(2,2,1)$ *for* $n=3$, $(4,4,1)$ *for* $n=5$, $(6,6,2)$ *for* $n=7$ *and* $(8,8,3)$ *for* $n=9$, *where* $[\cdot]$ *denotes the integer part function.*

In system (45), the Melnikov function for the exterior eight-loop case is

$$M(h)=\oint_{L(h)} g\mathrm{d}x-f\mathrm{d}y=\oint_{L(h)}(g+\int_0^y f_x\mathrm{d}y)\mathrm{d}x,\ h\in(0,\infty),$$

and similarly Melnikov functions for the interior eight-loop case are given by

$$M_i(h)=\oint_{L_i(h)}(g+\int_0^y f_x\mathrm{d}y)\mathrm{d}x,\ h\in(-\frac{1}{4},0),\ i=1,2.$$

Obviously, $M_1(h)=M_2(h)$. Then for the homoclinic bifurcation of limit cycles, we only need to study zeros of $M(h)$ for $h\in(0,\infty)$ and $M_1(h)$ for $h\in(-\frac{1}{4},0)$.

Let $G(x,y)=g(x,y)+\int_0^y f_x(x,y)\mathrm{d}y$. For any $n=2m+1$, $m=1,2,3,4$, we suppose

$$G(x,y)=\sum_{k=0}^{m}\sum_{i+j=2k+1}^{i\geq 0,\ j\geq 0} a_{ij}x^iy^j. \tag{46}$$

Before we give the proof of Theorem 4, we have the following lemma.

Lemma 2. *Let*

$$A_{ij}=\oint_{L_1} x^iy^j\mathrm{d}x$$

for any integers $i\geq 0$ *and* $j\geq -1$. *Then if* j *is even, then* $A_{ij}=0$, *otherwise*

$$A_{ij}=2^{\frac{i+j+1}{2}}\frac{\Gamma(1+\frac{j}{2})\Gamma(\frac{1}{2}+\frac{i}{2}+\frac{j}{2})}{\Gamma(\frac{3}{2}+\frac{i}{2}+j)}, \tag{47}$$

where $\Gamma(\cdot)$ *is the Gamma function.*

Proof. It is direct that

$$\begin{aligned} A_{ij} &= \int_0^{\sqrt{2}} x^i (y(x))^j \mathrm{d}x - \int_0^{\sqrt{2}} x^i (-y(x))^j \mathrm{d}x \\ &= (1-(-1)^j) \int_0^{\sqrt{2}} x^{i+j} \left(\frac{\sqrt{-2x^2+4}}{2} \right)^j \mathrm{d}x, \end{aligned} \tag{48}$$

where $y(x) = x\sqrt{-2x^2+4}/2$. Then $A_{ij} = 0$ when j is even. Using the substitution $x = \sqrt{2}\sin\theta$, we have

$$\int_0^{\sqrt{2}} x^{i+j} \left(\frac{\sqrt{-2x^2+4}}{2} \right)^j \mathrm{d}x = 2^{\frac{i+j+1}{2}} \int_0^{\frac{\pi}{2}} \sin^{i+j}\theta \cos^{j+1}\theta \mathrm{d}\theta. \tag{49}$$

By the method of integration by part, for $j \geq 1$ it is easy to get

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \sin^{i+j}\theta \cos^{j+1}\theta \mathrm{d}\theta \\ &= \frac{1}{i+j+1} \sin^{i+j+1}\theta \cos^j\theta \Big|_0^{\frac{\pi}{2}} + \frac{j}{i+j+1} \int_0^{\frac{\pi}{2}} \sin^{i+j+2}\theta \cos^{j-1}\theta \mathrm{d}\theta \\ &= \frac{j}{i+j+1} \int_0^{\frac{\pi}{2}} (1-\cos^2\theta) \sin^{i+j}\theta \cos^{j-1}\theta \mathrm{d}\theta. \end{aligned}$$

Then

$$\int_0^{\frac{\pi}{2}} \sin^{i+j}\theta \cos^{j+1}\theta \mathrm{d}\theta = \frac{j}{i+2j+1} \int_0^{\frac{\pi}{2}} \sin^{i+j}\theta \cos^{j-1}\theta \mathrm{d}\theta, \ j \geq 1,$$

from which it is obtained that if j is an odd positive integer,

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin^{i+j}\theta \cos^{j+1}\theta \mathrm{d}\theta &= \frac{j}{i+2j+1} \cdot \frac{j-2}{i+2j-1} \cdots \frac{1}{i+j+2} \int_0^{\frac{\pi}{2}} \sin^{i+j}\theta \mathrm{d}\theta \\ &= \frac{j!!(i+j)!!}{(i+2j+1)!!} \int_0^{\frac{\pi}{2}} \sin^{i+j}\theta \mathrm{d}\theta. \end{aligned} \tag{50}$$

Then when $j \geq -1$ is odd, from (48)–(50) we have

$$A_{ij} = 2^{\frac{i+j+3}{2}} \frac{j!!(i+j)!!}{(i+2j+1)!!} \int_0^{\frac{\pi}{2}} \sin^{i+j}\theta \mathrm{d}\theta, \tag{51}$$

where $-1!! = 1$. For any $k > 1$, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^k \theta \mathrm{d}\theta &= -\sin^{k-1}\theta\cos\theta\Big|_0^{\frac{\pi}{2}} + (k-1)\int_0^{\frac{\pi}{2}} \sin^{k-2}\theta\cos^2\theta \mathrm{d}\theta \\ &= (k-1)\int_0^{\frac{\pi}{2}} \sin^{k-2}\theta(1-\sin^2\theta)\mathrm{d}\theta,\end{aligned}$$

from which

$$\int_0^{\frac{\pi}{2}} \sin^k \theta \mathrm{d}\theta = \frac{k-1}{k}\int_0^{\frac{\pi}{2}} \sin^{k-2}\theta \mathrm{d}\theta$$

is derived. Similarly, then

$$\int_0^{\frac{\pi}{2}} \sin^{i+j}\theta \mathrm{d}\theta = \begin{cases} \dfrac{(i+j-1)!!}{(i+j)!!}\displaystyle\int_0^{\frac{\pi}{2}} \mathrm{d}\theta = \dfrac{\pi}{2}\dfrac{(i+j-1)!!}{(i+j)!!}, \text{ if } i+j \text{ is even}, \\ \dfrac{(i+j-1)!!}{(i+j)!!}\displaystyle\int_0^{\frac{\pi}{2}} \sin\theta \mathrm{d}\theta = \dfrac{(i+j-1)!!}{(i+j)!!}, \text{ if } i+j \text{ is odd}. \end{cases} \tag{52}$$

Because $\Gamma(n+1) = n\Gamma(n)$, $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, (47) is obtained from (51) and (52). The proof is completed.

Proof of Theorem 4. When $n = 3$, from (35), (46) and Lemma 2 we have

$$\begin{aligned}
&\bar{c}_0 = \oint_{L_1} G(x,y)\mathrm{d}x = \frac{4}{3}a_{01} + \frac{16}{35}a_{03} + \frac{16}{15}a_{21}, \\
&\bar{c}_1 = G_y(0,0) = a_{01}, \quad \bar{b}_0 = G_y(1,0) = a_{01} + a_{21}, \\
&\bar{c}_2 = \oint_{L_1} G_y(x,y) - \bar{c}_1 \mathrm{d}t = \oint_{L_1} (3a_{03}y^2 + 2a_{12}xy + a_{21}x^2)/y\mathrm{d}x = 4(a_{03} + a_{21}).
\end{aligned}$$

Then solving $c_0 = c_1 = 0$ above for a_{01} and a_{21} yields $a_{01} = 0$ and $a_{21} = -\frac{3}{7}a_{03}$, under which we have $\bar{c}_2 = \frac{16}{7}a_{03}$ and $\bar{b}_0 = -\frac{3}{7}a_{03}$. When $a_{03} \neq 0$, similarly like in Remark 2, we vary a_{01} and a_{21} properly such that

$$|c_0| \ll |c_1| \ll |c_2|, \ c_1c_2 > 0 \text{ and } c_0c_1 < 0$$

for (32). Then $M_1(h)$ have two zeros for $0 < -h \ll 1$ and $M(h)$ have one zero for $0 < h \ll 1$ by (33) and (34). Therefore, for $n = 3$ system (45) can have 5 limit cycles with distribution $(2, 2, 1)$.

For $n = 5, 7, 9$, we use the method presented in this paper, to find more limit cycles around homoclinic orbits in system (45).

Let

$$F_0(x, y) = \frac{\partial G}{\partial y}(x, y) = \sum_{k=0}^{m} \sum_{i+j=2k+1}^{i\geq 0,\ j>0} j a_{ij} x^i y^{j-1}, \ m = 2, 3, 4.$$

When $n = 5$, from (35) we have

$$\begin{aligned} &\bar{c}_0 = \frac{4}{3}a_{01} + \frac{16}{35}a_{03} + \frac{16}{15}a_{21} + \frac{128}{693}a_{05} + \frac{128}{315}a_{23} + \frac{128}{105}a_{41}, \\ &\bar{c}_1 = a_{01}, \quad \bar{b}_0 = a_{01} + a_{21} + a_{41}. \end{aligned} \tag{53}$$

Then one obtains $a_{01} = 0$ and $a_{21} = -a_{41}$ from $\bar{b}_0 = \bar{c}_1 = 0$. By (43), it is easy to get

$$\begin{aligned} P_1(x, y) &= \frac{F_0(x, 0)}{x(x^2 - 1)} = a_{41}x, \\ Q_1(x, y) &= \frac{F_0(x, y) - F_0(x, 0)}{y} = \sum_{i+j=3,5}^{i\geq 0,\ j\geq 2} j a_{ij} x^i y^{j-2}. \end{aligned}$$

Then by (35) and (36) we have

$$\begin{aligned} &\bar{c}_2 = \oint_{L_1} Q_1 \mathrm{d}x - P_1 \mathrm{d}y = \oint_{L_1} (Q_1 + yP_1')\mathrm{d}x = 4a_{03} + \frac{16}{7}a_{05} + \frac{16}{5}a_{23} + \frac{4}{3}a_{41}, \\ &\bar{c}_3 = 3a_{03} + a_{41}, \quad \bar{b}_1 = 3a_{03} + 3a_{23} + a_{41}. \end{aligned} \tag{54}$$

From (53) and (54), we can get

$$a_{01} = 0, \quad a_{21} = -a_{41}, \quad a_{03} = -\frac{160}{77}a_{05} - \frac{1}{3}a_{41}, \quad a_{23} = \frac{145}{77}a_{05}, \tag{55}$$

from $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{b}_0 = 0$. Further, with (55) holding, $\bar{b}_1\bar{c}_3 = (-\frac{45}{77}a_{05}) \cdot (-\frac{480}{77}a_{05}) > 0$ when $a_{05} \neq 0$. When $a_{05} \neq 0$, we vary a_{01}, a_{21}, a_{03} and a_{23} properly such that

$$|b_0| \ll |b_1|, \ b_0b_1 < 0$$

for (31) and

$$|c_0| \ll |c_1| \ll |c_2| \ll |c_3|,\ c_2c_3 < 0,\ c_1c_2 > 0 \text{ and } c_0c_1 < 0$$

for (32). Then $M_1(h)$ have one zero for $0 < h + \frac{1}{4} \ll 1$ and three zeros for $0 < -h \ll 1$, and $M(h)$ have one zero for $0 < h \ll 1$ by (33) and (34). Therefore, system (45) can have 9 limit cycles with distribution $(4, 4, 1)$ when $n = 5$.

Next, we consider the case $n = 7$. Similarly, we have

$$\bar{c}_0 = \sum_{k=0}^{3} \sum_{i+j=2k+1}^{i\geq 0,\ j\geq 0} A_{ij}a_{ij}, \quad \bar{c}_1 = a_{01}, \quad \bar{b}_0 = a_{01} + a_{21} + a_{41} + a_{61}. \tag{56}$$

In this case, P_1 and Q_1 are given by

$$P_1(x, y) = \frac{F_0(x, 0)}{x(x^2 - 1)} = x(a_{61}x^2 + a_{61} + a_{41}),$$
$$Q_1(x, y) = \frac{F_0(x, y) - F_0(x, 0)}{y} = \sum_{k=1}^{3} \sum_{i+j=2k+1}^{i\geq 0,\ j\geq 2} ja_{ij}x^i y^{j-2}.$$

Then by (35), (36) and (13),

$$\begin{aligned}
&\bar{c}_2 = 4a_{03} + \frac{16}{7}a_{05} + \frac{16}{5}a_{23} + \frac{4}{3}a_{41} + \frac{128}{99}a_{07} + \frac{128}{63}a_{25} + \frac{128}{35}a_{43} + \frac{68}{15}a_{61},\\
&\bar{c}_3 = 3a_{03} + a_{41} + a_{61},\ \bar{c}_4 = 20a_{05} + 12a_{23} + 16a_{07} + 16a_{25} + 16a_{43} + 12a_{61},\\
&\bar{c}_5 = -3a_{23} + 15a_{05} - 3a_{61}, \quad \bar{b}_1 = 3a_{03} + 3a_{23} + a_{41} + 3a_{43} + 4a_{61}.
\end{aligned} \tag{57}$$

From (56) and (57), if $\bar{b}_0 = 0$, $\bar{c}_i = 0$, $i = 0, \cdots, 4$, then we can have

$$\begin{aligned}
&a_{01} = 0, \quad a_{21} = -a_{41} - a_{61}, \quad a_{03} = -\frac{1}{3}a_{41} - \frac{1}{3}a_{61},\\
&a_{23} = \frac{1}{11}a_{07} - a_{43} - a_{61}, \quad a_{05} = -\frac{127}{55}a_{07} - \frac{1}{5}a_{43}, \quad a_{25} = \frac{20}{11}a_{07},
\end{aligned}$$

and $\bar{b}_1\bar{c}_5 = (\frac{3}{11}a_{07}) \cdot (-\frac{384}{11}a_{07}) < 0$ when $a_{07} \neq 0$. Similarly, we vary $a_{01}, a_{21}, a_{03}, a_{23}, a_{05}$ and a_{25} properly, and then system (45) can have 14 limit cycles with distribution $(6, 6, 2)$ for $n = 7$.

The last case is $n = 9$. Again, using (35) and Lemma 2 we have

$$\bar{c}_0 = \sum_{k=0}^{4} \sum_{i+j=2k+1}^{i\geq 0,\ j\geq 0} a_{ij}A_{ij}, \quad \bar{c}_1 = a_{01}, \quad \bar{b}_0 = a_{01} + a_{21} + a_{41} + a_{61} + a_{81}. \tag{58}$$

Let $\bar{b}_0 = \bar{c}_1 = 0$. Then $a_{01} = 0$ and $a_{21} = -(a_{41} + a_{61} + a_{81})$. By (43) we have

$$P_1(x,y)=\frac{F_0(x,0)}{x(x^2-1)}=x(a_{81}x^4+(a_{81}+a_{61})x^2+a_{81}+a_{61}+a_{41}),$$

$$Q_1(x,y)=\frac{F_0(x,y)-F_0(x,0)}{y}=\sum_{k=1}^{4}\sum_{i+j=2k+1}^{i\geq 0,\, j>2} ja_{ij}x^iy^{j-2},$$

and by (35) and (36) get

$$\begin{aligned}\bar{c}_2&=\sum_{k=1}^{4}\sum_{i+j=2k+1}^{i\geq 0,\, j>2} ja_{ij}A_{i,j-2}+5a_{81}A_{41}\\&\qquad+3(a_{81}+a_{61})A_{21}+(a_{81}+a_{61}+a_{41})A_{01},\\\bar{c}_3&=3a_{03}+a_{41}+a_{61}+a_{81},\\\bar{b}_1&=3a_{03}+3a_{23}+a_{41}+3a_{43}+4a_{61}+3a_{63}+9a_{81}.\end{aligned}\tag{59}$$

Let $F_1(x,y)=P_{1x}(x,y)+Q_{1y}(x,y)$. Setting $\bar{c}_3=\bar{b}_1=0$ yields

$$a_{03}=-\frac{1}{3}(a_{41}+a_{61}+a_{81}),\quad a_{23}=-a_{43}-a_{61}-a_{63}-\frac{8}{3}a_{81},$$

under which by (43) we have

$$P_2(x,y)=\frac{F_1(x,0)}{x(x^2-1)}=x(3a_{63}x^2+3a_{43}+3a_{63}+5a_{81}),$$

$$Q_2(x,y)=\frac{F_1(x,y)-F_1(x,0)}{y}=\sum_{k=2}^{4}\sum_{i+j=2k+1}^{i\geq 0,\, j>3} j(j-2)a_{ij}x^iy^{j-4}.$$

Furthermore, by (36), (37)

$$\begin{aligned}\bar{c}_4&=\oint Q_2\mathrm{d}x-P_2\mathrm{d}y=\oint(Q_2+yP_2')\mathrm{d}x\\&=\sum_{k=2}^{4}\sum_{i+j=2k+1}^{i\geq 0,\, j>3} j(j-2)a_{ij}A_{i,j-4}+9a_{63}A_{21}+(3a_{43}+3a_{63}+5a_{81})A_{01},\\\bar{c}_5&=15a_{05}+3a_{43}+3a_{63}+5a_{81},\\\bar{b}_2&=15a_{05}+15a_{25}+3a_{43}+15a_{45}+12a_{63}+5a_{81}.\end{aligned}\tag{60}$$

Solving $\bar{b}_0=\bar{b}_1=0$ and $\bar{c}_j=0,\ j=0,\cdots,5$, yields

$$
\begin{aligned}
&a_{01}=0,\ a_{21}=3a_{03}=-a_{41}-a_{61}-a_{81},\ a_{23}=-a_{43}-a_{61}-a_{63}-\frac{8}{3}a_{81},\\
&a_{05}=-\frac{1}{5}a_{43}-\frac{1}{5}a_{63}-\frac{1}{3}a_{81},\ a_{25}=\frac{63}{1045}a_{09}-a_{45}-\frac{3}{5}a_{63},\\
&a_{07}=-\frac{4599}{1045}a_{09}-\frac{1}{7}a_{45},\ a_{27}=\frac{4248}{1045}a_{09},
\end{aligned}
\tag{61}
$$

and $\bar{b}_2\bar{c}_6=(\frac{189}{209}a_{09})\cdot(-\frac{13824}{1045}a_{09})<0$ when $a_{09}\neq 0$. Again, we choose proper values for parameters, and system (45) can have 19 limit cycles with distribution $(8,8,3)$ when $n=9$. The proof is completed.

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