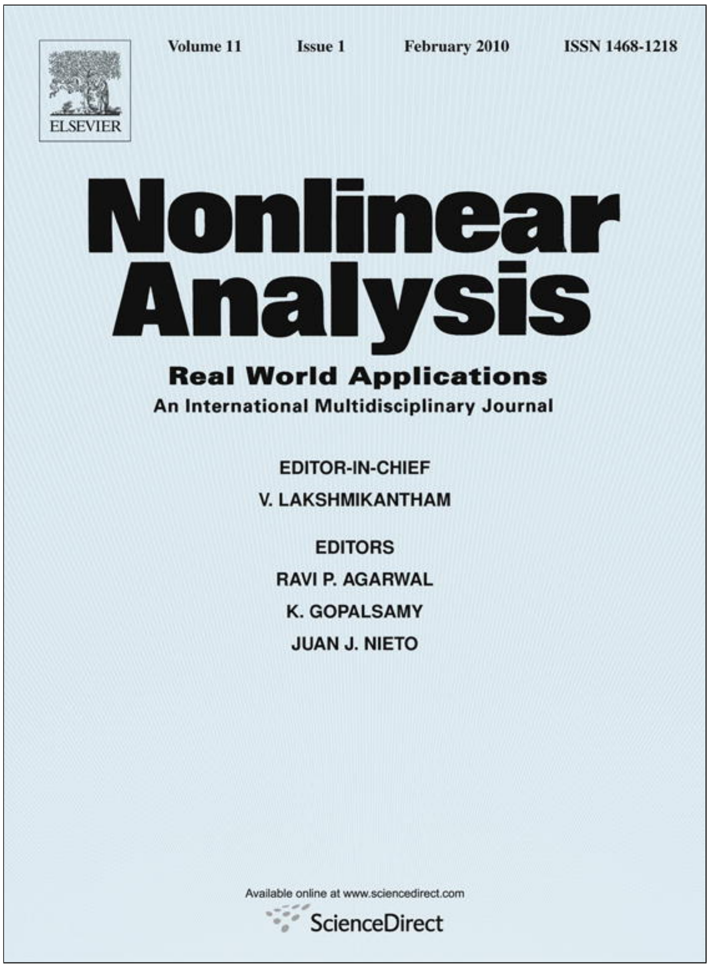
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Existence and global attractivity of positive periodic solutions of Lotka–Volterra predator–prey systems with deviating arguments[☆]

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ABSTRACT

The model discussed in this paper is described by the following periodic 3-species Lotka–Volterra predator–prey system with several deviating arguments:

$$\begin{cases} x_1'(t) = x_1(t)(r_1(t) - a_{11}(t)x_1(t-\tau_{11}(t)) - a_{12}(t)x_2(t-\tau_{12}(t)) \\ \quad -a_{13}(t)x_3(t-\tau_{13}(t))) \\ x_2'(t) = x_2(t)(-r_2(t) + a_{21}(t)x_1(t-\tau_{21}(t)) - a_{22}(t)x_2(t-\tau_{22}(t)) \\ \quad -a_{23}(t)x_3(t-\tau_{23}(t))) \\ x_3'(t) = x_3(t)(-r_3(t) + a_{31}(t)x_1(t-\tau_{31}(t)) - a_{32}(t)x_2(t-\tau_{32}(t)) \\ \quad -a_{33}(t)x_3(t-\tau_{33}(t))), \end{cases} \tag{$*$}$$

where $x_1(t)$ denotes the density of prey species at time t, $x_2(t)$ and $x_3(t)$ denote the density of predator species at time t, $r_i, a_{ij} \in C(R, [0, \infty))$ and $\tau_{ij} \in C(R, R)$ are w-periodic functions with

$$\bar{r}_i = \frac{1}{w}\int_0^w r_i(s)\mathrm{d}s > 0; \qquad \bar{a}_{ij} = \frac{1}{w}\int_0^w a_{ij}(s) > 0, \quad i, j = 1, 2, 3.$$

By using Krasnoselskii's fixed point theorem and the construction of Lyapunov function, a set of easily verifiable sufficient conditions are derived for the existence and global attractivity of positive periodic solutions of (*).

1. Introduction

In recent years, many authors have researched the theories of functional differential equations in mathematical ecology. Various mathematical models have been proposed in the study of population dynamics, ecology and epidemiology. One of the most famous models for the dynamics of population is the Lotka–Volterra system. Owing to its theoretical and practical significance, Lotka–Volterra systems have been studied extensively [1–15]. Particularly, [4,6,7,10–14] investigated the existence of positive periodic solutions of the following periodic n-species Lotka–Volterra competitive systems with several deviating arguments:

$$x_i'(t) = x_i(t)\left[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t-\tau_{ij}(t))\right], \quad i = 1, 2, \dots, n, \tag{1.1}$$

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where $r_i, a_{ij} \in C(R, [0, \infty))$ and $\tau_{ij} \in C(R, R)$ are w-periodic functions with

$$\bar{r}_i = \frac{1}{w}\int_0^w r_i(s)\mathrm{d}s > 0; \qquad \bar{a}_{ij} = \frac{1}{w}\int_0^w a_{ij}(s) > 0, \quad i, j = 1, 2, \ldots, n. \tag{1.2}$$

For example, by using the method of coincidence degree and Lyapunov function, Fan and Wang [7] studied the system (1.1) and derived a set of easily verifiable sufficient conditions for the existence and global attractivity of positive periodic solutions of system (1.1).

Recently Tang and Zou [14] investigated the existence of positive periodic solutions of system (1.1) and established the following result.

Theorem 1.1 ([14]). *Assume that*

(H1) *the linear system*

$$\sum_{j=1}^{n} \bar{a}_{ij} x_j = \bar{r}_i, \quad i = 1, 2, \ldots, n, \tag{1.3}$$

has a positive solution $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^{\mathrm{T}}$ *with* $x_i^* > 0, i = 1, 2, \ldots, n$. *Then system* (1.1) *has at least one positive* w*-periodic solution.*

Theorem 1.2 ([14]). *Assume that* $a_{ij}(t) \equiv a_{ij} > 0, \tau_{ij}(t) \equiv \tau_{ij}, i, j = 1, 2, \ldots, n$. *Then system* (1.1) *has at least one positive* w*-periodic solution if and only if the system of linear equations*

$$\sum_{j=1}^{n} a_{ij} x_j = \bar{r}_i, \quad i = 1, 2, \ldots, n, \tag{1.4}$$

has a positive solution.

In the proof of Theorem 1.1, the author took advantage of the fact that there are no negative terms in $\sum_{j=1}^{n} a_{ij}(t)x_j(t - \tau_{ij}(t)), i = 1, 2, \ldots, n$. But for the Lotka–Volterra predator–prey systems, it is more difficult to discuss.

Motivated by this problem and some ideas in [14]. In this paper, we conjectured the following Lotka–Volterra predator–prey system:

$$\begin{cases} x_1'(t) = x_1(t)\left(r_1(t) - a_{11}(t)x_1(t - \tau_{11}(t)) - a_{12}(t)x_2(t - \tau_{12}(t)) - a_{13}(t)x_3(t - \tau_{13}(t))\right) \\ x_2'(t) = x_2(t)\left(-r_2(t) + a_{21}(t)x_1(t - \tau_{21}(t)) - a_{22}(t)x_2(t - \tau_{22}(t)) - a_{23}(t)x_3(t - \tau_{23}(t))\right) \\ x_3'(t) = x_3(t)\left(-r_3(t) + a_{31}(t)x_1(t - \tau_{31}(t)) - a_{32}(t)x_2(t - \tau_{32}(t)) - a_{33}(t)x_3(t - \tau_{33}(t))\right) \end{cases} \tag{1.5}$$

may have the similar result.

The purpose of this paper is to give a positive answer to the above conjecture. In Section 2, we assume that (H1) ($n = 3$) and a set of easily verifiable hypotheses hold, then system (1.5) has at least one positive w-periodic solution. In Section 3, we will explore the global attractivity of positive periodic solutions of system (1.5).

Throughout of this paper, we say a vector $x = (x_1, x_2, x_3)^{\mathrm{T}}$ is positive if $x_i > 0, i = 1, 2, 3$.

2. Main results

For the sake and convenience, we introduce the definition of cone and the celebrated Krasnoselskii's fixed point theorem.

Definition 2.1. Let X be a Banach space, and let P be a closed, nonempty subset of X. P is a cone if

(i) $\alpha x + \beta y \in P$ for all $x, y \in P$ and all $\alpha, \beta \geq 0$;

(ii) $x, -x \in P$ imply $x = 0$.

Lemma 2.2 (Krasnoselskii, [16]). *Let X be a Banach space, and let* $P \subset X$ *be a cone in* X. *Assume that* Ω_1, Ω_2 *are open bounded subsets of* X *with* $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, *and let*

$$\varphi : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \longrightarrow P$$

be a completely continuous operator such that either

(i) $\|\varphi x\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1$ *and* $\|\varphi x\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2$;

or

(ii) $\|\varphi x\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1$ *and* $\|\varphi x\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$.

Then φ has a fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Let

$$X = \{x(t) = (x_1, x_2, x_3)^{\mathrm{T}} \in C(R, R^3) : x(t+w) = x(t)\}, \tag{2.1}$$

$$\|x\| = \sum_{j=1}^{3} |x_j|_0,\ |x_j|_0 = \max_{t\in[0,w]} |x_j(t)|, \quad j = 1, 2, 3. \tag{2.2}$$

Then X is a Banach space endowed with the above norm $\|\cdot\|$. If $x(t) = (x_1, x_2, x_3)^{\mathrm{T}} \in X$ is a solution of Eq. (1.5), then

$$\begin{cases} \left[x_1(t)\exp\left(-\int_0^t r_1(s)\mathrm{d}s\right)\right]' \\ \quad = -\exp\left(-\int_0^t r_1(s)\mathrm{d}s\right) x_1(t)\,(a_{11}(t)x_1(t-\tau_{11}(t)) + a_{12}(t)x_2(t-\tau_{12}(t)) + a_{13}(t)x_3(t-\tau_{13}(t))) \\ \left[x_2(t)\exp\left(\int_0^t r_2(s)\mathrm{d}s\right)\right]' \\ \quad = \exp\left(\int_0^t r_2(s)\mathrm{d}s\right) x_2(t)\,(a_{21}(t)x_1(t-\tau_{21}(t)) - a_{22}(t)x_2(t-\tau_{22}(t)) - a_{23}(t)x_3(t-\tau_{23}(t))) \\ \left[x_3(t)\exp\left(\int_0^t r_3(s)\mathrm{d}s\right)\right]' \\ \quad = \exp\left(\int_0^t r_3(s)\mathrm{d}s\right) x_3(t)\,(a_{31}(t)x_1(t-\tau_{31}(t)) - a_{32}(t)x_2(t-\tau_{32}(t)) - a_{33}(t)x_3(t-\tau_{33}(t))). \end{cases} \tag{2.3}$$

Integrating both sides of (2.3) over $[t, t+w]$, we obtain

$$\begin{cases} x_1(t) = \int_t^{t+w} G_1(t,s)x_1(s)\,(a_{11}(s)x_1(s-\tau_{11}(s)) + a_{12}(s)x_2(s-\tau_{12}(s)) + a_{13}(s)x_3(s-\tau_{13}(s)))\,\mathrm{d}s \\ x_2(t) = \int_t^{t+w} G_2(t,s)x_2(s)\,(a_{21}(s)x_1(s-\tau_{21}(s)) - a_{22}(s)x_2(s-\tau_{22}(s)) - a_{23}(s)x_3(s-\tau_{23}(s)))\,\mathrm{d}s \\ x_3(t) = \int_t^{t+w} G_3(t,s)x_3(s)\,(a_{31}(s)x_1(s-\tau_{31}(s)) - a_{32}(s)x_2(s-\tau_{32}(s)) - a_{33}(s)x_3(s-\tau_{33}(s)))\,\mathrm{d}s, \end{cases} \tag{2.4}$$

where

$$\begin{cases} G_1(t,s) = \dfrac{1}{1-\mathrm{e}^{-\bar{r}_1 w}}\exp\left(-\int_t^s r_1(\xi)\mathrm{d}\xi\right) \\ G_2(t,s) = \dfrac{1}{\mathrm{e}^{\bar{r}_2 w}-1}\exp\left(\int_t^s r_2(\xi)\mathrm{d}\xi\right) \\ G_3(t,s) = \dfrac{1}{\mathrm{e}^{\bar{r}_3 w}-1}\exp\left(\int_t^s r_3(\xi)\mathrm{d}\xi\right). \end{cases} \tag{2.5}$$

Let $\sigma = \min\{\mathrm{e}^{-2\bar{r}_i w} : i = 1, 2, 3\}$. Now, we choose the cone defined by

$$P = \{x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}} \in X : x_i(t) \geq \sigma |x_i|_0, i = 1, 2, 3\} \tag{2.6}$$

and define an operator $\Phi : X \to X$ by

$$(\Phi x)(t) = ((\Phi x)_1(t), (\Phi x)_2(t), (\Phi x)_3(t))^{\mathrm{T}}, \tag{2.7}$$

where

$$\begin{cases} (\Phi x)_1(t) = \int_t^{t+w} G_1(t,s)x_1(s)\,(a_{11}(s)x_1(s-\tau_{11}(s)) + a_{12}(s)x_2(s-\tau_{12}(s)) + a_{13}(s)x_3(s-\tau_{13}(s)))\,\mathrm{d}s \\ (\Phi x)_2(t) = \int_t^{t+w} G_2(t,s)x_2(s)\,(a_{21}(s)x_1(s-\tau_{21}(s)) - a_{22}(s)x_2(s-\tau_{22}(s)) - a_{23}(s)x_3(s-\tau_{23}(s)))\,\mathrm{d}s \\ (\Phi x)_3(t) = \int_t^{t+w} G_3(t,s)x_3(s)\,(a_{31}(s)x_1(s-\tau_{31}(s)) - a_{32}(s)x_2(s-\tau_{32}(s)) - a_{33}(s)x_3(s-\tau_{33}(s)))\,\mathrm{d}s. \end{cases} \tag{2.8}$$

By (2.4) and (2.8), we can easily verify that $x^*(t) \equiv (x_1^*(t), x_2^*(t), x_3^*(t))^{\mathrm{T}} \in X$ is a positive w-periodic solution of system (1.5) provided that $x^*(t)$ is a fixed point of Φ and $(\Phi x^*)_i(t) > 0$, $i = 1, 2, 3$ for any $t \in R$.

Set

$$\begin{cases} A_1 = \dfrac{e^{-\bar{r}_1 w}}{1-e^{-\bar{r}_1 w}} \le G_1(t,s) \le \dfrac{1}{1-e^{-\bar{r}_1 w}} = B_1 \\ A_2 = \dfrac{1}{e^{\bar{r}_2 w}-1} \le G_2(t,s) \le \dfrac{e^{\bar{r}_2 w}}{e^{\bar{r}_2 w}-1} = B_2 \\ A_3 = \dfrac{1}{e^{\bar{r}_3 w}-1} \le G_3(t,s) \le \dfrac{e^{\bar{r}_3 w}}{e^{\bar{r}_3 w}-1} = B_3, \end{cases} \tag{2.9}$$

and

$$A = \min\{\sigma^2 \bar{r}_1 A_1, \bar{r}_2 A_2, \bar{r}_3 A_3\}, \qquad B = \max\{\bar{r}_1 B_1, \bar{r}_2 B_2, \bar{r}_3 B_3\}. \tag{2.10}$$

In the following discussion, we assume that

(H2) *the solution of* (1.3) $(n = 3)$ *satisfies*

$$\begin{cases} \left(1+\dfrac{B^3}{\sigma^2 A^3}\right)(\bar{a}_{22}x_2^* + \bar{a}_{23}x_3^*) < \bar{r}_2 \\ \left(1+\dfrac{B^3}{\sigma^2 A^3}\right)(\bar{a}_{32}x_2^* + \bar{a}_{33}x_3^*) < \bar{r}_3, \end{cases} \tag{2.11}$$

where we denote $h_2 = \frac{B^3}{\sigma^2 A^3}$,

(H3) *the solution of* (1.3) $(n = 3)$ *satisfies*

$$\begin{cases} \left(\dfrac{A\sigma^2\left(\sigma^{\frac{1}{2}}-\sigma\right)}{B\left(1-\sigma^{\frac{3}{2}}\right)}+1\right)\bar{a}_{21}x_1^* > \bar{r}_2 \\ \left(\dfrac{A\sigma^2\left(\sigma^{\frac{1}{2}}-\sigma\right)}{B\left(1-\sigma^{\frac{3}{2}}\right)}+1\right)\bar{a}_{31}x_1^* > \bar{r}_3, \end{cases} \tag{2.12}$$

where we denote $h_3 = \frac{A\sigma^2(\sigma^{\frac{1}{2}}-\sigma)}{B(1-\sigma^{\frac{3}{2}})}$.

From (2.11) *and* (2.12), *we can easily see that when* $h_2h_3 \ge 1$, (H2) *implies* (H3), *when* $h_2h_3 \le 1$, (H3) *implies* (H2). *Actually, we can choose a little more large B such that* $h_2h_3 \ge 1$, *so for the system* (1.5), *the condition* (H2) *is enough.*

For the sake and convenience, we only assume that (H2) *and* (H3) *hold together.*

From (2.11), *we can choose a constant* $k \in N$ *such that*

$$\begin{cases} \left(1-\dfrac{B^2\sigma^{k-2}}{A^2}\right)\bar{r}_2 - \left(1+\dfrac{B^3}{\sigma^2 A^3}\right)(\bar{a}_{22}x_2^* + \bar{a}_{23}x_3^*) > 0 \\ \left(1-\dfrac{B^2\sigma^{k-2}}{A^2}\right)\bar{r}_3 - \left(1+\dfrac{B^3}{\sigma^2 A^3}\right)(\bar{a}_{32}x_2^* + \bar{a}_{33}x_3^*) > 0. \end{cases} \tag{2.13}$$

Let

$$\begin{cases} y_1^* = (\bar{a}_{21})^{-1}\left(\bar{r}_2 + \dfrac{B\bar{a}_{22}}{A\sigma^k}x_2^* + \dfrac{B\bar{a}_{23}}{A\sigma^k}x_3^*\right) \\ z_1^* = (\bar{a}_{31})^{-1}\left(\bar{r}_3 + \dfrac{B\bar{a}_{32}}{A\sigma^k}x_2^* + \dfrac{B\bar{a}_{33}}{A\sigma^k}x_3^*\right), \end{cases} \tag{2.14}$$

and $y^* = \max\{y_1^*, z_1^*\}$. *Obviously, we have* $y^* > x_1^*$.

Next, we define

$$\Omega_1 = \left\{x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}} \in X : |x_i|_0 < \frac{x_i^*}{Bw}, i = 1, 2, 3\right\}, \tag{2.15}$$

$$\Omega_2 = \left\{x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}} \in X : |x_1|_0 < \frac{y^*}{\sigma^2 Aw}, |x_2|_0 < \frac{x_2^*}{Aw}, |x_3|_0 < \frac{x_3^*}{Aw}\right\}. \tag{2.16}$$

Obviously, Ω_1 *and* Ω_2 *are open bounded subsets of X with* $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$.

Lemma 2.3. *If* $x(t) \in P \cap (\bar{\Omega}_2 \setminus \Omega_1)$, (H1) $(n = 3)$ *and* (H2) *hold, then* $((\Phi x)_i(t) > 0, i = 1, 2, 3$ *for any* $t \in R$.

Proof. First, from (2.13) and (2.14), we see that

$$\begin{aligned}\frac{A_2\sigma^2}{Bw}\bar{a}_{21}x_1^* - \frac{B_2\sigma^k\bar{a}_{21}y_1^*}{Aw} &= \frac{A_2}{Bw}\left(\sigma^2(\bar{r}_2 - \bar{a}_{22}x_2^* - \bar{a}_{23}x_3^*) - \frac{B_2B\sigma^k}{A_2A}\left(\bar{r}_2 + \frac{B\bar{a}_{22}}{A\sigma^k}x_2^* + \frac{B\bar{a}_{23}}{A\sigma^k}x_3^*\right)\right)\\ &\geq \frac{A_2}{Bw}\left(\sigma^2(\bar{r}_2 - \bar{a}_{22}x_2^* - \bar{a}_{23}x_3^*) - \frac{B^2}{A^2}\left(\sigma^k\bar{r}_2 + \frac{B\bar{a}_{22}}{A}x_2^* + \frac{B\bar{a}_{23}}{A}x_3^*\right)\right)\\ &\geq \frac{A_2\sigma^2}{Bw}\left(\left(1 - \frac{B^2\sigma^{k-2}}{A^2}\right)\bar{r}_2 - \left(1 + \frac{B^3}{\sigma^2A^3}\right)(\bar{a}_{22}x_2^* + \bar{a}_{23}x_3^*)\right) > 0. \end{aligned} \tag{2.17}$$

Since $x(t) \in P \cap (\bar{\Omega}_2 \setminus \Omega_1)$, then from (2.14) and (2.17), we have

$$\begin{aligned}&\int_t^{t+w} G_2(t,s)x_2(s)\,(a_{21}(s)x_1(s-\tau_{21}(s)) - a_{22}(s)x_2(s-\tau_{22}(s)) - a_{23}(s)x_3(s-\tau_{23}(s)))\,\mathrm{d}s\\ &\geq A_2\sigma^2|x_2|_0 w\bar{a}_{21}|x_1|_0 - B_2|x_2|_0 w\bar{a}_{22}|x_2|_0 - B_2|x_2|_0 w\bar{a}_{23}|x_3|_0\\ &\geq |x_2|_0\left(\frac{A_2\sigma^2}{B}\bar{a}_{21}x_1^* - \frac{B_2}{A}\bar{a}_{22}x_2^* - \frac{B_2}{A}\bar{a}_{23}x_3^*\right)\\ &\geq |x_2|_0\left(\frac{B_2\sigma^k\bar{a}_{21}y_1^*}{A} - \frac{B_2}{A}\bar{a}_{22}x_2^* - \frac{B_2}{A}\bar{a}_{23}x_3^*\right)\\ &= \frac{B_2}{A}|x_2|_0(\sigma^k\bar{a}_{21}y_1^* - \bar{a}_{22}x_2^* - \bar{a}_{23}x_3^*)\\ &\geq \frac{\sigma^kB_2}{A}\bar{r}_2|x_2|_0 > 0. \end{aligned} \tag{2.18}$$

Similarly, we can prove

$$\int_t^{t+w} G_3(t,s)x_3(s)(a_{31}(s)x_1(s-\tau_{31}(s)) - a_{32}(s)x_2(s-\tau_{32}(s)) - a_{33}(s)x_3(s-\tau_{33}(s)))\mathrm{d}s > 0. \tag{2.19}$$

The proof is complete. □

Lemma 2.4. *Assume that* (H1) $(n = 3)$, (H2) *and* (H3) *hold, the mapping* $\Phi : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \longrightarrow P$.

Proof. Since $\sigma^{\frac{1}{2}} \leq \frac{A_i}{B_i} \leq 1$, $i = 1, 2, 3$, from (2.8) and (2.9), we can easily have that for $t \leq s \leq t+w$, $x(t) \in P \cap (\bar{\Omega}_2 \setminus \Omega_1)$,

$$|(\Phi x)_1|_0 \leq B_1 \int_0^w x_1(s)\,(a_{11}(s)x_1(s-\tau_{11}(s)) + a_{12}(s)x_2(s-\tau_{12}(s)) + a_{13}(s)x_3(s-\tau_{13}(s)))\,\mathrm{d}s$$

and

$$\begin{aligned}(\Phi x)_1 &\geq A_1 \int_0^w x_1(s)\,(a_{11}(s)x_1(s-\tau_{11}(s)) + a_{12}(s)x_2(s-\tau_{12}(s)) + a_{13}(s)x_3(s-\tau_{13}(s)))\,\mathrm{d}s\\ &\geq \frac{A_1}{B_1}|(\Phi x)_1|_0 \geq \sigma|(\Phi x)_1|_0.\end{aligned}$$

And then,

$$\begin{aligned}|(\Phi x)_2|_0 &\leq B_2\int_0^w x_2(s)a_{21}(s)x_1(s-\tau_{21}(s))\mathrm{d}s - A_2\int_0^w x_2(s)a_{22}(s)x_2(s-\tau_{22}(s))\mathrm{d}s\\ &\quad - A_2\int_0^w x_2(s)a_{23}(s)x_3(s-\tau_{23}(s))\mathrm{d}s,\\ (\Phi x)_2 &\geq A_2\int_0^w x_2(s)a_{21}(s)x_1(s-\tau_{21}(s))\mathrm{d}s - B_2\int_0^w x_2(s)a_{22}(s)x_2(s-\tau_{22}(s))\mathrm{d}s\\ &\quad - B_2\int_0^w x_2(s)a_{23}(s)x_3(s-\tau_{23}(s))\mathrm{d}s.\end{aligned}$$

Next, we have

$$\begin{aligned}&(A_2 - \sigma B_2)\int_0^w x_2(s)a_{21}(s)x_1(s-\tau_{21}(s))\mathrm{d}s - (B_2 - \sigma A_2)\int_0^w (x_2(s)a_{22}(s)x_2(s-\tau_{22}(s))\\ &\quad + x_2(s)a_{23}(s)x_3(s-\tau_{23}(s)))\mathrm{d}s\end{aligned}$$

$$\begin{aligned}
&\geq (A_2-\sigma B_2)\sigma^2|x_2|_0\frac{w\bar{a}_{21}x_1^*}{Bw}-(B_2-\sigma A_2)|x_2|_0\frac{w\bar{a}_{22}x_2^*+w\bar{a}_{23}x_3^*}{Aw}\\
&\geq \frac{(A_2-\sigma B_2)\sigma^2}{B}|x_2|_0\bar{a}_{21}x_1^*-\frac{B_2-\sigma A_2}{A}|x_2|_0(\bar{r}_2-\bar{a}_{21}x_1^*)\\
&=\frac{B_2-\sigma A_2}{A}|x_2|_0\left(\left(\frac{(A_2-\sigma B_2)\sigma^2}{B}\frac{A}{B_2-\sigma A_2}+1\right)\bar{a}_{21}x_1^*-\bar{r}_2\right).
\end{aligned} \tag{2.20}$$

For $\sigma^{\frac{1}{2}}\leq\frac{A_i}{B_i}\leq 1$, then we easily have

$$\frac{(A_2-\sigma B_2)\sigma^2}{B}\frac{A}{B_2-\sigma A_2}\geq\frac{A\sigma^2\left(\sigma^{\frac{1}{2}}-\sigma\right)}{B\left(1-\sigma^{\frac{3}{2}}\right)},$$

and so from (2.12), we can obtain that (2.20) >0 *i.e.* $(\Phi x)_2\geq\sigma|(\Phi x)_2|_0$.

Similarly, we can obtain that $(\Phi x)_3\geq\sigma|(\Phi x)_3|_0$. The proof is complete. □

Lemma 2.5. *$\Phi: P\cap(\bar{\Omega}_2\setminus\Omega_1)\longrightarrow P$ is a completely continuous.*

Proof. Let

$$\begin{cases}
f_1(t,x_t)=x_1(t)\left(a_{11}(t)x_1(t-\tau_{11}(t))+a_{12}(t)x_2(t-\tau_{12}(t))+a_{13}(t)x_3(t-\tau_{13}(t))\right)\\
f_2(t,x_t)=x_2(t)\left(a_{21}(t)x_1(t-\tau_{21}(t))-a_{22}(t)x_2(t-\tau_{22}(t))-a_{23}(t)x_3(t-\tau_{23}(t))\right)\\
f_3(t,x_t)=x_3(t)\left(a_{31}(t)x_1(t-\tau_{31}(t))-a_{32}(t)x_2(t-\tau_{32}(t))-a_{33}(t)x_3(t-\tau_{33}(t))\right).
\end{cases}$$

We first show that Φ is continuous. For any $L>0$ and $\varepsilon>0$, there exists a $\delta>0$ such that for any $\phi,\psi\in X$, $\|\phi\|\leq L$, $\|\psi\|\leq L$, and $\|\phi-\varphi\|<\delta$ imply

$$\max_{s\in[0,w]}|f_i(s,\phi_s)-f_i(s,\psi_s)|<\frac{\varepsilon}{3B^*w},\quad i=1,2,3 \tag{2.21}$$

where $B^*=\max_{1\leq i\leq 3}B_i$. If $x,y\in X$ with $\|x\|\leq L$, $\|y\|\leq L$, and $\|x-y\|\leq\delta$, then from (2.8), (2.9) and (2.21), we have

$$\begin{aligned}
|(\Phi x)_i-(\Phi y)_i|_0 &\leq \int_t^{t+w}|G_i(t,s)||f_i(s,x_s)-f_i(s,y_s)|\mathrm{d}s\\
&\leq B^*\int_0^w|f_i(s,x_s)-f_i(s,y_s)|\mathrm{d}s\\
&<\frac{\varepsilon}{3},\quad i=1,2,3.
\end{aligned}$$

Obviously,

$$\|(\Phi x)-(\Phi y)\|=\sum_{i=1}^3|(\Phi x)_i-(\Phi y)_i|_0<\varepsilon.$$

Then Φ is continuous.

Now, we prove that Φ is a compact operator. Set $a=\max_{1\leq i\leq 3}\sum_{j=1}^3\bar{a}_{ij}$. Let $M>0$ be any constant and let $S=\{x\in X:\|x\|\leq M\}$ be a bounded set. For any $x\in S$, from (2.8) and (2.9), we can have

$$|(\Phi x)_i|_0\leq B_i\int_0^w|x_i(s)|\sum_{j=1}^3a_{ij}(s)|x_j(s-\tau_{ij}(s))|\mathrm{d}s\leq wB^*M^2\sum_{j=1}^3\bar{a}_{ij}\leq awB^*M^2,$$

and so

$$\|(\Phi x)\|=\sum_{i=1}^3|(\Phi x)_i|_0\leq 3awB^*M^2,\quad\forall x\in S.$$

Furthermore, by the Lemma 2.3 and from (2.8), we have

$$\begin{cases}
[(\Phi x)_1(t)]'=r_1(t)(\Phi x)_1(t)-x_1(t)\left(a_{11}(t)x_1(t-\tau_{11}(t))+a_{12}(t)x_2(t-\tau_{12}(t))+a_{13}(t)x_3(t-\tau_{13}(t))\right)\\
[(\Phi x)_2(t)]'=x_2(t)\left(a_{21}(t)x_1(t-\tau_{21}(t))-a_{22}(t)x_2(t-\tau_{22}(t))-a_{23}(t)x_3(t-\tau_{23}(t))\right)-r_2(t)(\Phi x)_2(t)\\
[(\Phi x)_3(t)]'=x_3(t)\left(a_{31}(t)x_1(t-\tau_{31}(t))-a_{32}(t)x_2(t-\tau_{32}(t))-a_{33}(t)x_3(t-\tau_{33}(t))\right)-r_3(t)(\Phi x)_3(t).
\end{cases}$$

Then for $x \in S$,

$$\begin{aligned} |[(\Phi x)_i(t)]'| &\le r_i(t)|(\Phi x)_i(t)| + |x_i(t)| \sum_{j=1}^{3} a_{ij}(t)|x_j(t-\tau_{ij}(t))| \\ &\le r_i^u awB^* M^2 + M^2 \sum_{j=1}^{3} a_{ij}^u \\ &\le KM^2, \quad i = 1, 2, 3, \end{aligned}$$

where $K = \max_{1\le i\le 3}(r_i^u awB^* + \sum_{j=1}^{3} a_{ij}^u)$ and

$$r_i^u = \max_{t\in[0,w]} r_i(t), \qquad a_{ij}^u = \max_{t\in[0,w]} a_{ij}(t), \quad i, j = 1, 2, 3.$$

Hence, $\Phi S \subset X$ is a family of uniformly bounded and equi-continuous functions. By the Ascoli–Arzela Theorem (see, e.g., [17, p. 169]), Φ is a compact operator, thus it is completely continuous. Now the proof is completed. □

Our main result on the existence of a positive periodic solution of system (1.5) is stated in the following theorem.

Theorem 2.6. *Assume that* (H1) ($n = 3$), (H2) *and* (H3) *hold. Then system* (1.5) *has at least one positive w-periodic solution.*

Proof. Let $x^* = (x_1^*, x_2^*, x_3^*)^{\mathrm{T}}$ with $x_i^* > 0$, $i = 1, 2, 3$, be a positive solution of (1.3).

From (2.9) and (2.10), we have $0 < A < B < \infty$.

By the (2.15) and (2.16), if $x = x(t) \in P \cap \partial\Omega_1$, then $\sigma|x_i|_0 \le x_i(t) \le |x_i|_0 = (Bw)^{-1}x_i^*$, $i = 1, 2, 3$, and

$$\begin{aligned} |(\Phi x)_i|_0 &\le B_i \int_0^w x_i(s) \sum_{j=1}^{3} a_{ij}(s) x_j(s-\tau_{ij}(s))\mathrm{d}s \\ &\le B_i w |x_i|_0 \sum_{j=1}^{3} \bar{a}_{ij} x_j^* \\ &= B_i \bar{r}_i w (Bw)^{-1} |x_i|_0 \\ &\le |x_i|_0, \quad i = 1, 2, 3, \end{aligned}$$

and so

$$\|\Phi x\| = \sum_{i=1}^{3} |(\Phi x)_i|_0 \le \sum_{i=1}^{3} |x_i|_0 = \|x\|, \quad \forall x = x(t) \in P \cap \partial\Omega_1. \tag{2.22}$$

Next, if $x = x(t) \in P \cap \partial\Omega_2$, then $\sigma|x_i|_0 \le x_i(t) \le |x_i|_0$, $i = 1, 2, 3$, and it follows that

$$\begin{aligned} (\Phi x)_1(t) &\ge A_1 \int_0^w x_1(s) \sum_{j=1}^{3} a_{1j}(s) x_j(s-\tau_{1j}(s))\mathrm{d}s \\ &\ge \sigma^2 A_1 w |x_1|_0 (\bar{a}_{11}|x_1|_0 + \bar{a}_{12}|x_2|_0 + \bar{a}_{13}|x_3|_0) \\ &= \sigma^2 A_1 w |x_1|_0 (Aw)^{-1} \left(\frac{\bar{a}_{11}}{\sigma^2} y^* + \bar{a}_{12} x_2^* + \bar{a}_{13} x_3^*\right) \\ &\ge \sigma^2 A_1 w |x_1|_0 (Aw)^{-1} (\bar{a}_{11} x_1^* + \bar{a}_{12} x_2^* + \bar{a}_{13} x_3^*) \\ &= \sigma^2 A_1 w \bar{r}_1 (Aw)^{-1} |x_1|_0 \\ &\ge |x_1|_0 \\ (\Phi x)_2(t) &\ge A_2 \int_0^w x_2(s) a_{21}(s) x_1(s-\tau_{21}(s))\mathrm{d}s - B_2 \int_0^w x_2(s)(a_{22}(s)x_2(s-\tau_{22}(s)) + a_{23}(s)x_3(s-\tau_{23}(s)))\mathrm{d}s \\ &\ge \sigma^2 A_2 w |x_2|_0 \bar{a}_{21} |x_1|_0 - B_2 w |x_2|_0 (\bar{a}_{22}|x_2|_0 + \bar{a}_{23}|x_3|_0) \\ &= |x_2|_0 \left(\frac{A_2}{A} \bar{a}_{21} y^* - \frac{B_2}{A} \bar{a}_{22} x_2^* - \frac{B_2}{A} \bar{a}_{23} x_3^*\right) \\ &\ge |x_2|_0 \left(\frac{1}{r_2} \bar{a}_{21} y_1^* - \frac{B_2}{A} \bar{a}_{22} x_2^* - \frac{B_2}{A} \bar{a}_{23} x_3^*\right) \\ &= |x_2|_0 \left(1 + \left(\frac{B}{\sigma^k \bar{r}_2 A} - \frac{B_2}{A}\right) \bar{a}_{22} x_2^* + \left(\frac{B}{\sigma^k \bar{r}_2 A} - \frac{B_2}{A}\right) \bar{a}_{23} x_3^*\right) \\ &\ge |x_2|_0 \left(1 + \left(\frac{1}{\sigma^k} - 1\right) \frac{B_2}{A} \bar{a}_{22} x_2^* + \left(\frac{1}{\sigma^k} - 1\right) \frac{B_2}{A} \bar{a}_{23} x_3^*\right) \\ &\ge |x_2|_0. \end{aligned}$$

Similarly, we can have

$$(\Phi x)_3(t) \geq |x_3|_0$$

and so

$$\|\Phi x\| = \sum_{i=1}^{3} |(\Phi x)_i|_0 \geq \sum_{i=1}^{3} |x_i|_0 = \|x\|, \quad \forall x = x(t) \in P \cap \partial \Omega_2. \tag{2.23}$$

Hence, $\Phi : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \longrightarrow P$ is a completely continuous operator and satisfies condition (i) in Lemma 2.2. By Lemma 2.2, there exists a point $x^*(t) = (x_1^*(t), x_2^*(t), x_3^*(t))^{\mathrm{T}} \in P \cap (\bar{\Omega}_2 \setminus \Omega_1)$ such that $x^*(t) = (\Phi x^*)(t)$, and by Lemma 2.3 $x^*(t)$ is a positive w-periodic solution of system (1.5). The proof is completed. □

Remark 2.7. This result is very concise and pretty in mathematics because of numerous parameters, while only there are a linear constant system and very simple inequalities.

Remark 2.8. The method and technique in this paper may be used to more general mathematical ecology models than system (1.1) and system (1.5).

3. Global attractivity of positive periodic solutions

In this section, we will always assume that the existence of positive periodic solutions of system (1.5) and study the global attractivity of positive periodic solutions of system (1.5).

Theorem 3.1. *In addition to the existence of positive periodic solutions, assume that* $\tau_{ii}(t) \equiv 0$, $\dot{\tau}_{ij}(t) < 1$, $i, j = 1, 2, 3$, *and that there exist* $\nu_i > 0$, $i = 1, 2, 3$, *such that*

$$\nu_i a_{ii}(t) > \sum_{j\neq i}^{3} \frac{\nu_j a_{ji}(\xi_{ji}^{-1}(t))}{1 - \dot{\tau}_{ji}(\xi_{ji}^{-1}(t))}, \quad i = 1, 2, 3, \tag{3.1}$$

where $\xi_{ij}^{-1}(t)$ *is the inverse function of* $\xi_{ij}(t) = t - \tau_{ij}(t)$, $i, j = 1, 2, 3$. *Then system* (1.5) *has a unique positive* w*-periodic solution* $x^*(t) = (x_1^*(t), x_2^*(t), x_3^*(t))^{\mathrm{T}}$ *which attracts all positive solutions* $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ *of system* (1.5), *that is,*

$$\lim_{t\to\infty} |x_i(t) - x_i^*(t)| = 0, \quad i = 1, 2, 3. \tag{3.2}$$

Proof. Set $\xi_{ij}(s) = t$. Then

$$\xi_{ij}(s + w) = s + w - \tau_{ij}(s + w) = \xi_{ij}(s) + w = t + w,$$

and so

$$\xi_{ij}^{-1}(t + w) = s + w = \xi_{ij}^{-1}(t) + w.$$

Thus, $a_{ij}(\xi_{ij}^{-1}(t))$ and $\dot{\tau}_{ij}(\xi_{ij}^{-1}(t))$ are still w-periodic functions for $i, j = 1, 2, 3$. Set

$$\theta_i = \max_{t\in[0,w]} \left\{ \frac{1}{\nu_i a_{ii}(t)} \sum_{j\neq i}^{3} \frac{\nu_j a_{ji}(\xi_{ji}^{-1}(t))}{1 - \dot{\tau}_{ji}(\xi_{ji}^{-1}(t))} \right\}, \quad i = 1, 2, 3.$$

From (3.1), we can have that $0 \leq \theta_i < 1$, $i = 1, 2, 3$, and

$$\theta_i \nu_i a_{ii}(t) \geq \sum_{j\neq i}^{3} \frac{\nu_j a_{ji}(\xi_{ji}^{-1}(t))}{1 - \dot{\tau}_{ji}(\xi_{ji}^{-1}(t))}, \quad i = 1, 2, 3. \tag{3.3}$$

Let $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ be any positive solution of system (1.5). Set

$$V(t) = \sum_{i=1}^{3} \nu_i \left[\left| \ln\left(\frac{x_i(t)}{x_i^*(t)} \right) \right| + \sum_{j\neq i}^{3} \int_{t-\tau_{ij}(t)}^{t} \frac{a_{ij}(\xi_{ij}^{-1}(s))}{1 - \dot{\tau}_{ij}(\xi_{ij}^{-1}(s))} |x_j(s) - x_j^*(s)| \mathrm{d}s \right], \quad t \geq 0. \tag{3.4}$$

Calculating the upper right derivative of $V(t)$ at time t, we have

$$\begin{aligned} D^+V(t) &\leq \sum_{i=1}^{3} \nu_i \left[-a_{ii}(t)|x_i(t) - x_i^*(t)| + \sum_{j\neq i}^{3} \frac{a_{ij}(\xi_{ij}^{-1}(t))}{1 - \dot{\tau}_{ij}(\xi_{ij}^{-1}(t))} |x_j(t) - x_j^*(t)| \right] \\ &= \sum_{i=1}^{3} \left[-\nu_i a_{ii}(t) + \sum_{j\neq i}^{3} \frac{\nu_j a_{ji}(\xi_{ji}^{-1}(t))}{1 - \dot{\tau}_{ji}(\xi_{ji}^{-1}(t))} \right] |x_i(t) - x_i^*(t)| \\ &\leq -\sum_{i=1}^{3} \nu_i (1 - \theta_i) a_{ii}(t) |x_i(t) - x_i^*(t)|, \quad t \geq 0. \end{aligned} \tag{3.5}$$

This shows that $V(t)$ is decreasing in $[0, \infty)$ and so the limit $\upsilon = \lim_{t\to\infty} V(t)$ exists.

Consequently, integrating (3.5) from 0 to ∞, we obtain

$$\sum_{i=1}^{3} \nu_i (1 - \theta_i) \int_0^\infty a_{ii}(s)|x_i(s) - x_i^*(s)|\mathrm{d}s \leq V(0) - \upsilon \leq V(0) < \infty,$$

and so

$$\sum_{i=1}^{3} \nu_i \int_0^\infty a_{ii}(s)|x_i(s) - x_i^*(s)|\mathrm{d}s < \infty. \tag{3.6}$$

From (3.1), we can note that

$$\int_0^\infty a_{ii}(s)\mathrm{d}s = \infty, \quad i = 1, 2, 3.$$

It follows from (3.6) that

$$\liminf_{t\to\infty} \sum_{i=1}^{3} \nu_i |x_i(t) - x_i^*(t)| = 0. \tag{3.7}$$

Again from (3.3) and (3.6), we have

$$\begin{aligned} \sum_{j\neq i}^{3} \int_{t-\tau_{ij}(t)}^{t} \frac{a_{ij}(\xi_{ij}^{-1}(s))}{1 - \dot{\tau}_{ij}(\xi_{ij}^{-1}(s))} |x_j(s) - x_j^*(s)|\mathrm{d}s &\leq \int_{t-\tau_M}^{t} \sum_{j\neq i}^{3} \frac{a_{ij}(\xi_{ij}^{-1}(s))}{1 - \dot{\tau}_{ij}(\xi_{ij}^{-1}(s))} |x_j(s) - x_j^*(s)|\mathrm{d}s \\ &\leq \frac{1}{\nu} \int_{t-\tau_M}^{t} \sum_{j\neq i}^{3} \nu_j a_{jj}(s)|x_j(s) - x_j^*(s)|\mathrm{d}s \to 0, \quad t \to \infty, i = 1, 2, 3, \end{aligned} \tag{3.8}$$

where $\tau_M = \max\{\tau_{ij}(t) : t \in [0, w], i, j = 1, 2, 3\}$ and $\nu = \min\{\nu_i : i = 1, 2, 3\}$. Combining (3.4) and (3.8), we can have

$$\lim_{t\to\infty} \sum_{i=1}^{3} \nu_i \left| \ln\left(\frac{x_i(t)}{x_i^*(t)}\right) \right| = \upsilon, \tag{3.9}$$

which yields

$$x_i(t) \geq \mathrm{e}^{-(\upsilon+1)/\nu} x_i^*(t), \quad \text{for large } t, i = 1, 2, 3. \tag{3.10}$$

By (3.7), (3.9) and (3.10),

$$\begin{aligned} \upsilon &= \liminf_{t\to\infty} \sum_{i=1}^{3} \nu_i \left| \ln\left(\frac{x_i(t)}{x_i^*(t)}\right) \right| \leq \liminf_{t\to\infty} \sum_{i=1}^{3} \frac{\nu_i}{\min\{x_i(t), x_i^*(t)\}} |x_i(t) - x_i^*(t)| \\ &\leq \mathrm{e}^{(\upsilon+1)/\nu} \liminf_{t\to\infty} \sum_{i=1}^{3} \frac{\nu_i}{x_i^*(t)} |x_i(t) - x_i^*(t)| \leq \frac{\mathrm{e}^{(\upsilon+1)/\nu}}{m} \liminf_{t\to\infty} \sum_{i=1}^{3} \nu_i |x_i(t) - x_i^*(t)| \\ &= 0, \end{aligned} \tag{3.11}$$

where $m = \min\{x_i^*(t) : t \in [0, w], i = 1, 2, 3\}$. Hence, it follows from (3.9) that

$$\lim_{t\to\infty} |x_i(t) - x_i^*(t)| = 0, \quad i = 1, 2, 3. \tag{3.12}$$

The proof is complete. □

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