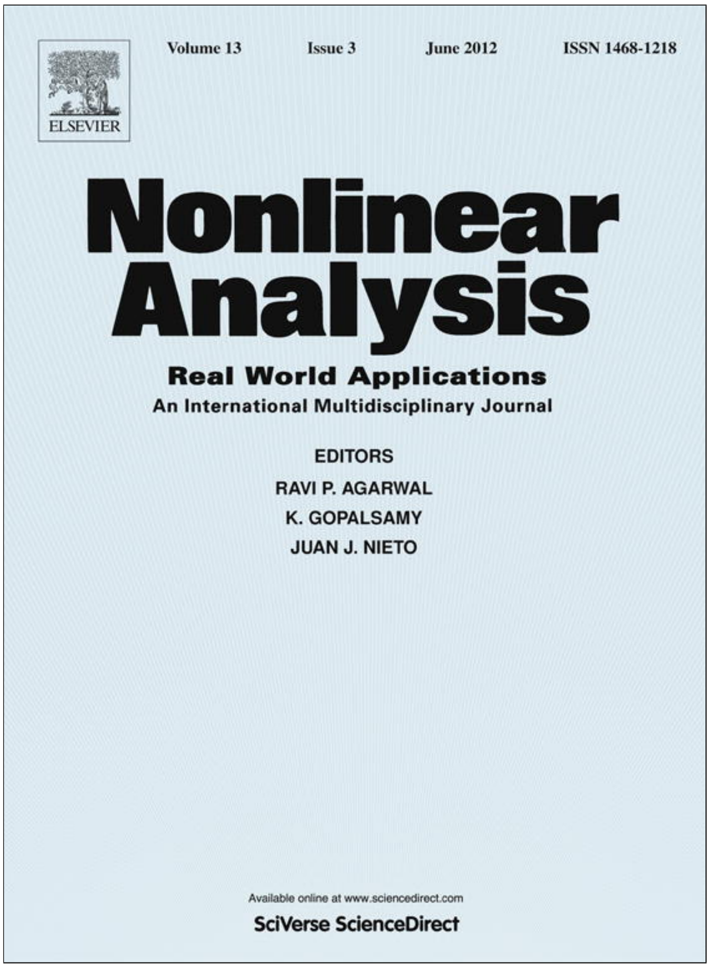
Volume 13
Issue 3
June 2012
ISSN 1468-1218
ELSEVIER
Nonlinear Analysis
Real World Applications
An International Multidisciplinary Journal
EDITORS
RAVI P. AGARWAL
K. GOPALSAMY
JUAN J. NIETO
Available online at www.sciencedirect.com
SciVerse ScienceDirect

Nonlinear Analysis: Real World Applications 13 (2012) 1152–1158

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Existence of homoclinic solutions for a class of second-order Hamiltonian systems with general potentials☆

Xiang Lv*, Jifa Jiang

Department of Mathematics, Shanghai Normal University, Shanghai 200234, PR China

ARTICLE INFO

Article history:
Received 15 February 2011
Accepted 16 September 2011

Keywords:
Homoclinic solutions
Hamiltonian systems
Variational methods

ABSTRACT

Under some local conditions on $W(t,u)$, the existence of homoclinic solutions is obtained for the nonperiodic second-order Hamiltonian systems $\ddot{u}(t) - L(t)u(t) + \nabla W(t,u(t)) = f(t)$ as a limit of periodic solutions of a certain sequence of boundary-value problems which are obtained by a new critical point theorem.

1. Introduction and main results

In this paper, we consider the following second-order nonautonomous Hamiltonian systems

$$\ddot{u}(t) - L(t)u(t) + \nabla W\big(t, u(t)\big) = f(t) \tag{1.1}$$

where $t \in R,\ u \in R^n,\ L \in C(R, R^{n\times n})$ is a symmetric matrix valued function, $W : R \times R^n \to R$ and $f : R \to R^n$. As usual, we say that a solution $u(t)$ of (1.1) is nontrivial homoclinic (to 0) if $u \neq 0$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $t \to \pm\infty$. Consequently, $\nabla W(t,x)$ denotes the gradient with respect to x, $(\cdot,\cdot) : R^n \times R^n \to R$ denotes the standard inner product in R^n and $|\cdot|$ is the induced norm.

When $f = 0$, (1.1) reduces to the following second-order Hamiltonian system

$$\ddot{u}(t) - L(t)u(t) + \nabla W\big(t, u(t)\big) = 0. \tag{1.2}$$

In applied sciences, as a special case of dynamical systems, Hamiltonian systems play a key role in practical problems concerning gas dynamics, fluid mechanics, relativistic mechanics and nuclear physics. The existence of homoclinic solutions is one of the most important problems in the theory of Hamiltonian systems. Inspired by the excellent monographs [1,2], the existence and multiplicity of periodic solutions and homoclinic orbits for Hamiltonian systems have been extensively and intensively studied in many papers via critical theory; see [3–25] and the references therein. Moreover, many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly; thus impulsive differential equations appear as a natural description of observed evolution phenomena of several real world problems. Due to their applications in many fields, second-order Hamiltonian systems with impulses via critical point theory have been recently considered in [26,27], and in papers [28–30], Nieto et al. studied some Dirichlet impulsive problems by variational approach.

☆ Supported by the NNSF of China (No. 11071166), the NSF of Shanghai (09ZR1423100), Research Fund for the Doctoral Program of Higher Education of China (RFDP) and Innovation Program of Shanghai Municipal Education Commission.

* Corresponding author.
E-mail address: lvxiang541@yahoo.com.cn (X. Lv).

1468-1218/$ – see front matter
doi:10.1016/j.nonrwa.2011.09.008

The case where $L(t)$ and $W(t,x)$ are either independent of t or periodic in t is studied by several authors; see [3,6,8,17,19,20]. More precisely, in paper [17], Rabinowitz has shown the existence of homoclinic orbits as a limit of $2kT$-periodic solutions of (1.2). Later, using the same method, several results for general Hamiltonian systems were obtained by Felmer et al. [6], Izydorek and Janczewska [8], and Tang and Xiao [19,20].

When $L(t)$ and $W(t,x)$ are not periodic with respect to t, the problem of existence of homoclinic orbits for (1.2) is quite different from the one just described, because of the lack of compactness of the Sobolev embedding. In [18], Rabinowitz and Tanaka studied system (1.2) without a periodicity assumption, both for L and W. More precisely, they assumed that the smallest eigenvalue of $L(t)$ tends to $+\infty$ as $|t| \to \infty$, using a variant of the Mountain Pass theorem without the Palais–Smale condition, and proved that system (1.2) possesses a homoclinic orbit.

Theorem A (*[18]*). *Assume that L and W satisfy the following conditions:*

(L) *$L(t)$ is positive definite symmetric matrix for all $t \in R$ and there exists an $l \in C(R, (0,\infty))$ such that $l(t) \to +\infty$ as $|t| \to \infty$ and*

$$\big(L(t)x, x\big) \geq l(t)|x|^2$$

for all $t \in R$ and $x \in R^n$;

(A1) *$W \in C^1(R \times R^n, R)$ and there is a constant $\mu > 2$ such that*

$$0 < \mu W(t,x) \leq \big(x, \nabla W(t,x)\big)$$

for all $t \in R$ and $x \in R^n \setminus \{0\}$;

(A2) *$|\nabla W(t,x)| = o(|x|)$ as $|x| \to 0$ uniformly with respect to $t \in R$;*

(A3) *there is a $\overline{W} \in C(R^n, R)$ such that*

$$|W(t,x)| + |\nabla W(t,x)| \leq |\overline{W}(x)|$$

for all $t \in R$ and $x \in R^n$.

Then system (1.2) possesses a nontrivial homoclinic solution.

In paper [21], Lv et al. further investigated the existence of homoclinic orbits as a limit of solutions of a certain sequence of boundary-value problems, which are obtained by using the Mountain Pass theorem. In detail, they obtained the following theorem.

Theorem B (*[21]*). *Assume that L and W satisfy assumption* (L) *and the following conditions:*

(B1) *$W(t,0) \equiv 0$, $W \in C^1(R \times R^n, R)$ and $|\nabla W(t,x)| = o(|x|)$ as $|x| \to 0$ uniformly in $t \in R$;*

(B2) *there are two constants $\mu > 2$ and $\nu \in \big[0, \frac{\mu}{2} - 1\big)$ and $\beta \in L^1(R, R^+)$ such that*

$$\big(\nabla W(t,x), x\big) - \mu W(t,x) \geq -\nu\big(L(t)x, x\big) - \beta(t)$$

for all $t \in R$ and $x \in R^n \setminus \{0\}$;

(B3) *there exists $T_0 > 0$ such that*

$$\liminf_{|x| \to \infty} \frac{W(t,x)}{|x|^2} > \frac{\pi^2}{2T_0^2} + \frac{l_1}{2}$$

uniformly in $t \in [-T_0, T_0]$, where l_1 is the biggest eigenvalue of $L(t)$ on $[-T_0, T_0]$.

Then system (1.2) possesses a nontrivial homoclinic solution.

Motivated by papers [13,18,21], in this paper, we will study the existence of Homoclinic solutions of (1.1) under more general local conditions. Our main results are the following theorems.

Theorem 1.1. *Assume that L and W satisfy the following conditions:*

(L′) *$L(t)$ is a positive definite symmetric matrix for all $t \in R$ and there exists an $l \in C(R, (0,\infty))$ such that*

$$\big(L(t)x, x\big) \geq l(t)|x|^2, \quad \text{for all } t \in R \text{ and } x \in R^n$$

and $\sup_{t \in R} |L_{ij}(t)| < \infty$, where $L(t) = (L_{ij}(t))_{n \times n}$;

(W1) *$W(t,0) = \nabla W(t,0) \equiv 0$ and $W \in C^1(R \times R^n, R)$ uniformly in $t \in R$;*

(W2) *there exist a constant $\rho > 0$ and an $a \in C(R, (0,\infty))$ such that*

$$W(t,x) \leq a(t)|x|^2, \quad \text{for all } t \in R \text{ and } |x| \leq \rho,$$

where $0 < a(t) < \frac{1}{2}l(t)$, for all $t \in R$ and $m_1 = \inf_{t \in R}\left\{\frac{1}{2}l(t) - a(t)\right\} > 0$;

(F) *$f \not\equiv 0$ is a continuous and bounded function such that $\int_R |f(t)|^2 dt < \infty$ and*

$$\min\left\{\frac{1}{2}, m_1\right\}\rho^2 - \sqrt{2}M\rho > 0 \tag{1.3}$$

where $M = \left(\int_R |f(t)|^2 dt\right)^{\frac{1}{2}}$.

Then system (1.1) possesses a nontrivial homoclinic solution.

Remark 1.1. When $\rho \le 1$, $W(t,x) = a(t)|x|^{\mu}$, where $\mu > 2$, we can see that (W2) holds, i.e. $W(t,x)$ can be superquadratic.

Remark 1.2. There are many functions satisfying (F). For example, if $\rho = m_1 = 1$, we can choose $f(t) = (f_1(t), \ldots, f_n(t))^T$, where $f_i(t) = \left(\frac{1}{16n\pi(1+t^2)}\right)^{\frac{1}{2}}$, $i = 1, \ldots, n$. Then $M = \frac{1}{4}$; we can easily see that (F) holds.

Theorem 1.2. *Assume that L and W satisfy assumptions* (L′), (W1) *and the following conditions:*

(W2′) *there exist constants $\rho > 0$, $1 < \gamma < 2$ and an integrable function $b \in L^{\frac{2}{2-\gamma}}(R, (0,\infty))$ such that*

$$W(t,x) \le b(t)|x|^{\gamma}, \quad \text{for all } t \in R \text{ and } |x| \le \sqrt{2}\rho,$$

where $m_2 = \left(\int_R |b(t)|^{\frac{2}{2-\gamma}} dt\right)^{\frac{2-\gamma}{2}}$;

(F′) *$f \not\equiv 0$ is a continuous and bounded function such that $\int_R |f(t)|^2 dt < \infty$ and*

$$\min\left\{\frac{1}{2}, l_*\right\}\rho^2 - m_2\rho^{\gamma} - M\rho > 0 \tag{1.4}$$

where $l_* = \inf_{t\in R} l(t) > 0$ and M is determined by (1.3).

Then system (1.1) possesses a nontrivial homoclinic solution.

Remark 1.3. There are many functions satisfying (F′). For example, if $\rho = l_* = 1$, $\gamma = \frac{3}{2}$, we can let $f(t)$ be the same as the example given in Remark 1.2 and $b(t) = \frac{1}{8}\left(\frac{1}{\pi(1+t^2)}\right)^{\frac{1}{4}}$. Then $m_2 = \frac{1}{8}$, $M = \frac{1}{4}$, we can obtain that (F′) holds.

Remark 1.4. For system (1.1), Theorems 1.1 and 1.2 give some new criteria for the existence of homoclinic solutions by relaxing conditions, which are essentially new.

2. Proof of theorems

Motivated by papers [7,11,21], we will prove the existence of the homoclinic solutions for (1.1) as the limit of periodic solutions of the following boundary-value problem

$$\begin{cases} \ddot{u}(t) - L(t)u(t) + \nabla W\big(t, u(t)\big) = f(t), & \text{for } t \in [-T, T] \\ u(-T) - u(T) = \dot{u}(-T) - \dot{u}(T) = 0, \end{cases} \tag{2.1}$$

for $\forall T \in R^+$.

For $\forall T \in R^+$, let

$$E_T := W^{1,2}\big([-T,T], R^n\big) = \big\{u : [-T,T] \longrightarrow R^n | u \text{ is absolutely continuous}, u(-T) = u(T) \text{ and } \dot{u} \in L^2([-T,T], R^n)\big\}$$

and for $u \in E_T$, let

$$\|u\|_{E_T} = \left\{\int_{-T}^{T} \left[|\dot{u}(t)|^2 + |u(t)|^2\right] dt\right\}^{\frac{1}{2}};$$

then E_T is a Hilbert space on the above norm.

We consider a functional $I_T : E_T \to R$, defined by

$$I_T(u) = \int_{-T}^{T}\left[\frac{1}{2}|\dot{u}(t)|^2 + \frac{1}{2}\big(L(t)u(t), u(t)\big) - W\big(t, u(t)\big) + \big(f(t), u(t)\big)\right] dt. \tag{2.2}$$

Then we can easily check that $I_T \in C^1(E_T, R)$ is weakly lower semi-continuous as the sum of a convex continuous function and of a weakly continuous one and

$$\langle I_T'(u), v\rangle = \int_{-T}^{T} \left[\big(\dot{u}(t), \dot{v}(t)\big) + \big(L(t)u(t), v(t)\big) - \big(\nabla W\big(t, u(t)\big), v(t)\big) + \big(f(t), v(t)\big)\right] dt \tag{2.3}$$

for all $u, v \in E_T$. Furthermore, it is well known that the critical points of I_T in E_T are classical solutions of (2.1) (see [1,2]).

We will obtain a critical point of I_T by using a new critical point theorem, which is important for what follows. Therefore, we state this theorem precisely.

Lemma 2.1 (*See [13]*). *Let X be a real reflexive Banach space and $\Omega \subset X$ be a closed bounded convex subset of X. Suppose that $\varphi : X \to R$ is a weakly lower semi-continuous functional. If there exists a point $x_0 \in \Omega \setminus \partial\Omega$ such that*

$$\varphi(x) > \varphi(x_0), \quad \forall x \in \partial\Omega. \tag{2.4}$$

Then there must be a $x^* \in \Omega \setminus \partial\Omega$ such that

$$\varphi(x^*) = \inf_{u\in\Omega} \varphi(u).$$

Lemma 2.2 (*See [8]*). *Let $u : R \to R^n$ be a continuous mapping such that $\dot{u} \in L^2_{\text{loc}}(R, R^n)$. Then for every $t \in R$ the following inequality holds*

$$|u(t)| \le \sqrt{2}\left[\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \left(|\dot{u}(s)|^2 + |u(s)|^2\right) ds\right]^{\frac{1}{2}}. \tag{2.5}$$

Lemma 2.3. *Let $u \in E_T$. Then the following inequality holds*

$$\|u\|_{L^\infty_{[-T,T]}} \le \left(\int_{-T}^{T} |u(t)|^2 dt\right)^{\frac{1}{2}} + \left(\int_{-T}^{T} |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}. \tag{2.6}$$

This lemma is a special case of Corollary 2.2 in [20].

Corollary 2.1. *Let $u \in E_T$. Then the following inequality holds*

$$\|u\|_{L^\infty_{[-T,T]}} \le \sqrt{2}\|u\|_{E_T} = \sqrt{2}\left\{\int_{-T}^{T} \left[|\dot{u}(t)|^2 + |u(t)|^2\right] dt\right\}^{\frac{1}{2}}. \tag{2.7}$$

Proof. Combining (2.6) and the inequality $\sqrt{a} + \sqrt{b} \le \sqrt{2}(a + b)^{\frac{1}{2}}$, we can easily see that (2.7) holds. □

Lemma 2.4. *Under the conditions of Theorem 1.1, problem (2.1) possesses a solution $u_T \in E_T$ such that*

$$\int_{-T}^{T} \left[|\dot{u}_T(t)|^2 + |u_T(t)|^2\right] dt < \frac{1}{2}\rho^2, \quad \forall T \in R^+. \tag{2.8}$$

Proof. Obviously, $I_T(0) = 0$ by (W1). In order to use Lemma 2.1, the first step in the proof is to construct a closed bounded convex subset of E_T. For $\forall\, T \in R^+$, let $\Omega_T := \left\{u \in E_T : \int_{-T}^{T} \left[|\dot{u}(t)|^2 dt + |u(t)|^2\right] dt \le \frac{1}{2}\rho^2\right\}$, where ρ is a constant given by (1.3). We can easily see that Ω_T is a closed bounded convex subset of E_T.

Next, we will show that (2.8) holds. If $u \in \partial\Omega_T$, then $\int_T^T \left[|\dot{u}(t)|^2 dt + |u(t)|^2\right] dt = \frac{1}{2}\rho^2$. By Corollary 2.1, it is easy to verify that for all $u \in \partial\Omega_T$, $\|u\|_{L^\infty_{[-T,T]}} \le \rho$, that is $|u(t)| \le \rho$ for all $t \in [-T, T]$, which together with (L′) and (W2) implies that

$$\begin{aligned}
I_T(u) &= \int_{-T}^{T} \left[\frac{1}{2}|\dot{u}(t)|^2 + \frac{1}{2}\big(L(t)u(t), u(t)\big) - W\big(t, u(t)\big) + \big(f(t), u(t)\big)\right] dt \\
&\ge \frac{1}{2}\int_{-T}^{T} |\dot{u}(t)|^2\, dt + \frac{1}{2}\int_{-T}^{T} l(t)|u(t)|^2 dt - \int_{-T}^{T} a(t)|u(t)|^2 dt + \int_{-T}^{T} \big(f(t), u(t)\big) dt \\
&\ge \frac{1}{2}\int_{-T}^{T} |\dot{u}(t)|^2\, dt + m_1\int_{-T}^{T} |u(t)|^2 dt - \left(\int_{-T}^{T} |f(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_{-T}^{T} |u(t)|^2 dt\right)^{\frac{1}{2}} \\
&\ge \min\left\{\frac{1}{4}, \frac{1}{2}m_1\right\}\rho^2 - \frac{1}{\sqrt{2}}M\rho
\end{aligned}$$

for all $u \in \partial\Omega_T$, which together with (1.3) yields

$$I_T(u) > I_T(0) = 0, \quad \forall u \in \partial\Omega_T.$$

Then by using Lemma 2.1, we can obtain that for $\forall T \in R^+$, there exists a point

$$u_T \in \widetilde{\Omega}_T := \left\{u \in E_T : \int_{-T}^{T} \left[|\dot{u}(t)|^2 + |u(t)|^2\right] dt < \frac{1}{2}\rho^2\right\}$$

such that

$$I_T(u_T) = \inf_{u \in \Omega_T} I_T(u).$$

Now, by Theorem 1.3 in [1] and the fact that $\widetilde{\Omega}_T$ is an open subset of E_T, we conclude that

$$I'_T(u_T) = 0.$$

Since $u_T \in \widetilde{\Omega}_T$, we get

$$\int_{-T}^{T} \left[|\dot{u}_T(t)|^2 + |u_T(t)|^2\right] dt < \frac{1}{2}\rho^2,$$

which shows that (2.8) holds. The proof is complete. □

Lemma 2.5. *Under the conditions of Theorem 1.2, problem (2.1) possesses a solution $u_T \in E_T$ such that*

$$\int_{-T}^{T} \left[|\dot{u}_T(t)|^2 + |u_T(t)|^2\right] dt < \rho^2, \quad \forall\, T \in R^+, \tag{2.9}$$

where ρ is determined by (W2′).

Proof. Let $\Gamma_T := \left\{u \in E_T : \int_{-T}^{T} \left[|\dot{u}(t)|^2 dt + |u(t)|^2\right] dt \le \rho^2\right\}$. Clearly, Γ_T is a closed bounded convex subset of E_T. Similar to the proof of Lemma 2.4, it suffices to show that for $\forall\, T \in R^+$,

$$I_T(u) > I_T(0) = 0, \quad \forall u \in \partial\Gamma_T. \tag{2.10}$$

If $u \in \partial\Gamma_T$, from the proof of Lemma 2.4, we can obtain $|u(t)| \le \sqrt{2}\rho$ for all $t \in [-T, T]$, which together with (L′) and (W2′) implies that

$$\begin{aligned}
I_T(u) &= \int_{-T}^{T} \left[\frac{1}{2}|\dot{u}(t)|^2 + \frac{1}{2}\big(L(t)u(t), u(t)\big) - W\big(t, u(t)\big) + \big(f(t), u(t)\big)\right] dt \\
&\ge \frac{1}{2}\int_{-T}^{T} |\dot{u}(t)|^2\, dt + \frac{1}{2}\int_{-T}^{T} l(t)|u(t)|^2 dt - \int_{-T}^{T} b(t)|u(t)|^\gamma dt + \int_{-T}^{T} \big(f(t), u(t)\big) dt \\
&\ge \frac{1}{2}\int_{-T}^{T} |\dot{u}(t)|^2\, dt + l_* \int_{-T}^{T} |u(t)|^2 dt - \left(\int_{-T}^{T} |b(t)|^{\frac{2}{2-\gamma}} dt\right)^{\frac{2-\gamma}{2}} \left(\int_{-T}^{T} |u(t)|^2 dt\right)^{\frac{\gamma}{2}} \\
&\quad - \left(\int_{-T}^{T} |f(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_{-T}^{T} |u(t)|^2 dt\right)^{\frac{1}{2}} \\
&\ge \min\left\{\frac{1}{2}, l_*\right\}\rho^2 - m_2\rho^\gamma - M\rho
\end{aligned}$$

which together with (1.4) yields (2.10) holds. The proof is complete. □

Proof of Theorem 1.1. Take a sequence $T_n \to \infty$ and consider problem (2.1) on the interval $[-T_n, T_n]$. By the conclusions of Lemma 2.4, it has a solution u_n and $\|u_n\|_{E_{T_n}}$ is bounded uniformly in n.

Similar to the proof of Theorem 2.1 in [7], from the fact that

$$|u_n(t_1) - u_n(t_2)| \le \int_{t_1}^{t_2} |\dot{u}_n(t)|\, dt \le \sqrt{t_2 - t_1}\left(\int_{t_1}^{t_2} |\dot{u}_n(t)|^2\, dt\right)^{\frac{1}{2}},$$

we claim that the sequence $\{u_n\}$ is equicontinuous and uniformly bounded on every interval $[-T_n, T_n]$ and we can select a subsequence $\{u_{n_k}\}$ such that it converges uniformly on any bounded interval to a function u. Furthermore, since $\|u_n\|_{E_{T_n}}$ is bounded uniformly in n, we can conclude that $u \in W^{1,2}(R, R^n)$ and

$$\int_R \left[|\dot{u}(t)|^2 + |u(t)|^2\right] dt < +\infty. \tag{2.11}$$

Expressing $\ddot{u}_{n_k}$ using Eq. (2.1), we conclude that the sequence $\ddot{u}_{n_k}$, and then also $\dot{u}_{n_k}$ converges uniformly on bounded intervals. Writing

$$u_{n_k}(t) = \int_0^t (t-s)\ddot{u}_{n_k}(s)ds + t\dot{u}_{n_k}(0) + u_{n_k}(0),$$

we obtain that $u \in C^2(R, R^n)$ and $\ddot{u}_{n_k} \to \ddot{u}$ uniformly on bounded intervals. Consequently, at first, we consider Eq. (2.1) on interval $[-m, m]$, $m \in N$ and then by the diagonal process and let $m \to \infty$, we can conclude that u satisfies (1.1), i.e., u is a classical solution of (1.1). Note that, by Lemma 2.2, we can have

$$|u(t)| \le \sqrt{2}\left[\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \left(|\dot{u}(s)|^2 + |u(s)|^2\right) ds\right]^{\frac{1}{2}}, \quad \text{for every } t \in R.$$

From (2.11), we can easily see that the limits of $u(t)$ exist as $|t| \to \infty$, that is $u(\pm\infty) = 0$.

We now prove that $\dot{u}(t) \to 0$ as $|t| \to \infty$. By Lemma 2.2, we get that

$$|\dot{u}(t)|^2 \le 2\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \left(|u(s)|^2 + |\dot{u}(s)|^2\right) ds + 2\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{u}(s)|^2 ds, \quad \text{for every } t \in R.$$

By (2.11), we can have

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \left(|u(s)|^2 + |\dot{u}(s)|^2\right) ds \to 0, \quad \text{as } |t| \to \infty.$$

Hence we only need to prove that

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{u}(s)|^2 ds \to 0, \quad \text{as } |t| \to \infty. \tag{2.12}$$

For $\forall\, x \in R^n$, by (L′) and Cauchy inequality, we can easily have that

$$|L(t)x| \le nL|x|, \quad \text{where } L = \sup_{t\in R} |L_{ij}(t)|. \tag{2.13}$$

It follows from (1.1) and (2.13)

$$\begin{aligned}\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{u}(s)|^2 ds &= \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \left|L(s)u(s) - \nabla W\big(s, u(s)\big) + f(s)\right|^2 ds \\ &\le 3\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \left(|L(s)u(s)|^2 + \left|\nabla W\big(s, u(s)\big)\right|^2 + |f(s)|^2\right) ds \\ &\le 3\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \left(n^2L^2\,|u(s)|^2 + \left|\nabla W\big(s, u(s)\big)\right|^2 + |f(s)|^2\right) ds.\end{aligned}$$

By (W1), (F) and the fact that $u(t) \to 0$ as $|t| \to \infty$, we can have $\left|\nabla W\big(t, u(t)\big)\right| \to 0$ and $f(t) \to 0$ as $|t| \to \infty$. Consequently, from (2.11), we conclude that (2.12) holds.

In the end, since $\nabla W(t, 0) = 0$ and $f \not\equiv 0$, then u is a nontrivial homoclinic orbit of system (1.1). The proof is complete. □

Proof of Theorem 1.2. Since the proof of Theorem 1.2 is exactly similar to the proof of Theorem 1.1, we omit it here. □

Acknowledgment

The authors would like to thank the reviewer for the valuable comments and suggestions.

References

[1] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian Systems, in: Appl. Math. Sci, vol. 74, Springer-Verlag, New York, 1989.
[2] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, in: CBMS Regional Conference Series in Mathematics, vol. 65, American Mathematical Society, Providence, RI, 1986, Published for the Conference Board of the Mathematical Sciences, Washington, DC.
[3] Y. Ding, M. Girardi, Periodic and homoclinic solutions to a class of Hamiltonian systems with the potentials changing sign, Dynam. Systems Appl. 2 (1993) 131–145.

[4] Y.H. Ding, Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems, Nonlinear Anal. 25 (1995) 1095–1113.
[5] G.H. Fei, The existence of homoclinic orbits for Hamiltonian systems with the potential changing sign, Chinese Ann. Math. Ser. A 17 (4) (1996) 651 (a Chinese summary), Chinese Ann. Math. Ser. B 4 (1996) 403–410.
[6] P.L. Felmer, E.A. De, B.E. Silva, Homoclinic and periodic orbits for Hamiltonian systems, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 26 (2) (1998) 285–301.
[7] P. Korman, A.C. Lazer, Homoclinic orbits for a class of symmetric Hamiltonian systems, Electron. J. Differential Equations 1 (1994) 1–10.
[8] I. Marek, J. Janczewska, Homoclinic solutions for a class of second order Hamiltonian systems, J. Differential Equations 219 (2005) 375–389.
[9] W. Omana, M. Willem, Homoclinic orbits for a class of Hamiltonian systems, Differential Integral Equations 5 (1992) 1115–1120.
[10] Z.Q. Qu, C.L. Tang, Existence of homoclinic orbits for the second order Hamiltonian systems, J. Math. Anal. Appl. 291 (2004) 203–213.
[11] Y. Lv, C.L. Tang, Existence of even homoclinic orbits for second-order Hamiltonian systems, Nonlinear Anal. 67 (2007) 2189–2198.
[12] X.H. Tang, X.Y. Xiao, Homoclinic solutions for a class of second-order Hamiltonian systems, J. Math. Anal. Appl. 354 (2009) 539–549.
[13] S. Lu, Homoclinic solutions for a nonlinear second order differential system with p-Laplacian operator, Nonlinear Anal. RWA 12 (2011) 525–534.
[14] J. Sun, H. Chen, J.J. Nieto, Homoclinic solutions for a class of subquadratic second-order Hamiltonian systems, J. Math. Anal. Appl. 373 (2011) 20–29.
[15] J. Sun, H. Chen, J.J. Nieto, Homoclinic orbits for a class of first-order nonperiodic asymptotically quadratic Hamiltonian systems with spectrum point zero, J. Math. Anal. Appl. 378 (2011) 117–127.
[16] C. Vidal, Dynamics aspects in a galactic type Hamiltonian, Nonlinear Anal. RWA 12 (2011) 918–930.
[17] P.H. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, Proc. Roy. Soc. Edinburgh Sect. A 114 (1990) 33–38.
[18] P.H. Rabinowitz, K. Tanaka, Some results on connecting orbits for a class of Hamiltonian systems, Math. Z. 206 (1991) 473–499.
[19] X.H. Tang, L. Xiao, Homoclinic solutions for a class of second-order Hamiltonian systems, Nonlinear Anal. 71 (2009) 1140–1152.
[20] X.H. Tang, L. Xiao, Homoclinic solutions for ordinary p-Laplacian systems with a coercive potential, Nonlinear Anal. 71 (2009) 1124–1132.
[21] X. Lv, S. Lu, P. Yan, Existence of homoclinic solutions for a class of second-order Hamiltonian systems, Nonlinear Anal. 72 (2010) 390–398.
[22] X. Lv, S. Lu, P. Yan, Homoclinic solutions for nonautonomous second-order Hamiltonian systems with a coercive potential, Nonlinear Anal. 72 (2010) 3484–3490.
[23] X. Lv, S. Lu, J. Jiang, Homoclinic solutions for a class of second-order Hamiltonian systems, Nonlinear Anal. RWA 13 (2012) 176–185.
[24] J. Wei, J. Wang, Infinitely many homoclinic orbits for the second order Hamiltonian systems with general potentials, J. Math. Anal. Appl. 366 (2010) 694–699.
[25] S.Q. Zhang, Symmetrically homoclinic orbits for symmetric Hamiltonian systems, J. Math. Anal. Appl. 247 (2000) 645–652.
[26] H. Zhang, Z.X. Li, Periodic and homoclinic solutions generated by impulses, Nonlinear Anal. RWA 12 (2011) 39–51.
[27] Z.X. Han, H. Zhang, Periodic and homoclinic solutions generated by impulses for asymptotically linear and sublinear Hamiltonian system, J. Comput. Appl. Math. 235 (2011) 1531–1541.
[28] J.J. Nieto, Variational formulation of a damped Dirichlet impulsive problem, Appl. Math. Lett. 23 (2010) 940–942.
[29] J.J. Nieto, D. O'Regan, Variational approach to impulsive differential equations, Nonlinear Anal. RWA 10 (2009) 680–690.
[30] J. Xiao, J.J. Nieto, Variational approach to some damped Dirichlet nonlinear impulsive differential equations, J. Franklin Inst. 348 (2011) 369–377.