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Existence of Homoclinic Solutions for a Class of Second-Order Hamiltonian Systems with Locally Subquadratic Potentials

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Abstract
In this paper, we mainly consider the existence of homoclinic orbits for the following second-order Hamiltonian systems

$$\ddot{u}(t) - L(t)u(t) + \nabla W\big(t, u(t)\big) = f(t),$$

where $L(t)$ is a positive definite and symmetric matrix for all $t \in \mathbb{R}$ and the potential function $W(t, u)$ is locally subquadratic. Here, the coefficient of the upper bound for W is a positive constant, whereas in the previous literature the corresponding coefficient need to be some integrable functions $a(t)$ on $\mathbb{R}$.

Keywords Homoclinic solutions · Hamiltonian systems · Variational methods

Mathematics Subject Classification Primary 34C37 · 70H05 · 58E05

1 Introduction and Main Results

The main purpose of this paper is to consider the following second-order Hamiltonian systems

$$\ddot{u}(t) - L(t)u(t) + \nabla W\big(t, u(t)\big) = f(t) \tag{1.1}$$

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where $L \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ is a symmetric and positive definite matrix valued function, $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $\nabla W(t, x)$ is the gradient with respect to x and $f : \mathbb{R} \to \mathbb{R}^n$. Usually, we say that a solution $u(t)$ of (1.1) is nontrivial homoclinic (to 0) if $u \not\equiv 0$ and $u(t) \to 0$ as $t \to \pm\infty$.

It is well known that homoclinic solutions of Hamiltonian systems play the key role in the study of gas dynamics and fluid mechanics. Recently, there has been an extensive theoretical works on the existence and multiplicity of homoclinic solutions for Hamiltonian systems via critical point theory and variational method, see [1–29] and the references therein.

In the case that $L(t)$ and $W(t, u)$ are either independent of t or periodic in t, the existence of homoclinic orbits can be obtained as a limit of $2kT$-periodic orbits of (1.1), see Rabinowitz [21], Izydorek and Janczewska [8,9] and so on. In the case that $L(t)$ and $W(t, u)$ are not periodic with respect to t, the problem will become more difficult because of the lack of compactness of the Sobolev embedding. Here, it is worth pointing out that most of the results in the literature were based on some globally conditions on W, such as the Ambrosetti-Rabinowitz condition and so on. In [23], Rabinowitz and Tanaka considered (1.1) without a periodicity assumption, both for L and W. By a variant of the Mountain Pass Theorem, they showed that (1.1) admits a nontrivial homoclinic orbit.

Theorem 1.1 [23] *Assume that L and W satisfy the following assumptions:*
(A1) $L(t)$ *is positive definite symmetric matrix for all* $t \in \mathbb{R}$ *and there exists an* $l \in C(\mathbb{R}, (0, \infty))$ *such that* $l(t) \to +\infty$ *as* $|t| \to \infty$ *and*

$$\big(L(t)x, x\big) \geq l(t)|x|^2 \qquad for\ all\ t \in \mathbb{R}\ and\ x \in \mathbb{R}^n;$$

(A2) $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ *and there is a constant* $\mu > 2$ *such that*

$$0 < \mu W(t, x) \leq \big(x, \nabla W(t, x)\big) \qquad for\ all\ t \in \mathbb{R}\ and\ x \in \mathbb{R}^n \backslash \{0\};$$

(A3) $|\nabla W(t, x)| = o(|x|)$ *as* $|x| \to 0$ *uniformly with respect to* $t \in \mathbb{R}$*;*
(A4) *There is a* $\overline{W} \in C(\mathbb{R}^n, \mathbb{R})$ *such that*

$$|W(t, x)| + |\nabla W(t, x)| \leq |\overline{W}(x)| \qquad for\ all\ t \in \mathbb{R}\ and\ x \in \mathbb{R}^n.$$

Then

$$\ddot{u}(t) - L(t)u(t) + \nabla W\big(t, u(t)\big) = 0 \qquad \text{(HS)}$$

possesses a nontrivial homoclinic solution.

Motivated by [12,17,18,23], the main purpose of this paper is to consider the existence of Homoclinic solutions for (1.1), where we only give some locally subquadratic assumptions on $W(t, u)$. Our main results are the following theorems.

Theorem 1.2 *Assume that L and W satisfy the following assumptions:*

(L) *$L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists an $l \in C(\mathbb{R}, (0, \infty))$ such that $l(t) \to +\infty$ as $|t| \to \infty$ and*

$$\big(L(t)x, x\big) \geq l(t)|x|^2 \qquad for\ all\ t \in \mathbb{R}\ and\ x \in \mathbb{R}^n,$$

where $l \in L^{-\frac{\mu}{2-\mu}}(\mathbb{R}, (0, \infty))$ and $1 < \mu < 2$ is defined in the condition (W$_2$)*;*
(W$_1$) *$W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, $W(t, 0) = \nabla W(t, 0) \equiv 0$ for all $t \in \mathbb{R}$;*
(W$_2$) *there exist constants $\varrho > 0$, $a > 0$ and $1 < \mu < 2$ such that*

$$W(t, x) \leq a|x|^{\mu} \qquad for\ all\ t \in \mathbb{R}\ and\ |x| \leq \frac{1}{\sqrt{2\sqrt{l_*}}}\varrho, \tag{1.2}$$

where $l_* = \inf_{t\in\mathbb{R}} l(t)$;
(F) *$f \not\equiv 0$ is a continuous function such that $\int_{\mathbb{R}} |f(t)|^{\beta} dt < \infty$ and*

$$\frac{1}{2}\varrho^2 - aM_l^{\frac{\mu}{2}}\varrho^{\mu} - \left(\frac{1}{\sqrt{2\sqrt{l_*}}}\right)^{1-\frac{2}{\beta^*}} M_f l_*^{-\frac{1}{\beta^*}} \varrho > 0, \tag{1.3}$$

where $1 < \beta \leq 2$, $\frac{1}{\beta^*} + \frac{1}{\beta} = 1$,

$$M_l = \left(\int_{\mathbb{R}} l^{-\frac{\mu}{2-\mu}}(t)dt\right)^{\frac{2-\mu}{\mu}} \quad and \quad M_f = \left(\int_{\mathbb{R}} |f(t)|^{\beta} dt\right)^{\frac{1}{\beta}}.$$

Then (1.1) *admits a nontrivial homoclinic solution.*

Remark 1.3 The condition (W$_2$) is only defined in some local regions.

Remark 1.4 In the condition (W$_2$), the coefficient of the upper bound $a|x|^{\mu}$ is a positive constant, whereas in the previous literature the corresponding coefficient need to be some integrable functions $a(t)$ on $\mathbb{R}$, such as [16,17,24].

Example 1.5 If $l(t) = 1 + t^2$, $t \in \mathbb{R}$ and $\mu = \frac{3}{2}$, we can easily have that (L) holds, i.e., $\frac{1}{(1+t^2)^3}$ is integrable on $\mathbb{R}$.

2 Proof of Theorems

Motivated by [10,18], we will consider the existence of homoclinic solutions for (1.1), which can be seen as the limit of solutions for the following boundary-value problem

$$\begin{cases} \ddot{u}(t) - L(t)u(t) + \nabla W\big(t, u(t)\big) = f(t), & t \in [-T, T], \\ u(-T) = u(T) = 0, & T \in \mathbb{R}^+. \end{cases} \tag{2.1}$$

For all $T \in \mathbb{R}^+$, define

$$E_T = \left\{ u \in W^{1,2}\big([-T,T],\mathbb{R}^n\big) \Big| \int_{-T}^{T} \Big[|\dot{u}(t)|^2 + \big(L(t)u(t),u(t)\big) \Big] dt < +\infty \right\}$$

where

$$\begin{aligned} W^{1,2}\big([-T,T],\mathbb{R}^n\big) = \ & \{u : [-T,T] \longrightarrow \mathbb{R}^n \big| \text{u is absolutely continuous,} \\ & u(-T) = u(T) = 0 \text{ and } \dot{u} \in L^2([-T,T],\mathbb{R}^n)\} \end{aligned}$$

and for $u \in E_T$, introduce the inner product $(u,v)_{E_T} = \int_{-T}^{T} \Big[\big(\dot{u}(t),\dot{v}(t)\big) + \big(L(t)u(t),v(t)\big) \Big] dt$, then E_T is a Hilbert space equipped with the above inner product and the corresponding norm is $\|u\|_{E_T} = (u,u)_{E_T}^{\frac{1}{2}}$.

Let $I_T : E_T \to \mathbb{R}$ defined by

$$I_T(u) = \frac{1}{2}\|u\|_{E_T}^2 - \int_{-T}^{T} W\big(t,u(t)\big)dt + \int_{-T}^{T} \big(f(t),u(t)\big)dt. \tag{2.2}$$

It is obvious that $I_T \in C^1(E_T,\mathbb{R})$ is weakly lower semi-continuous based on the fact that it is the sum of a convex continuous function and of a weakly continuous one. Direct computation shows that

$$< I_T'(u), v >= \int_{-T}^{T} \Big[\big(\dot{u}(t),\dot{v}(t)\big) + \big(L(t)u(t),v(t)\big) - \big(\nabla W\big(t,u(t)\big),v(t)\big) + \big(f(t),v(t)\big) \Big] dt \tag{2.3}$$

for all $u, v \in E_T$. Moreover, we can have that critical points of I_T in E_T are classical solutions of (2.1) (see [19]).

In order to prove our main result, we will give some lemmas as follows.

Lemma 2.1 (See [12]) *Let X be a real reflexive Banach space and $\Omega \subset X$ be a closed and bounded convex subset of X. Suppose that $\varphi : X \to \mathbb{R}$ is weakly lower semi-continuous. If there exists a point $x_0 \in \Omega \backslash \partial\Omega$ such that $\varphi(x) > \varphi(x_0)$ for all $x \in \partial\Omega$. Then there exists a $x^* \in \Omega \backslash \partial\Omega$ such that $\varphi(x^*) = \inf_{u \in \Omega} \varphi(u)$.*

Lemma 2.2 (See [18]) *Let $u \in E_T$. It follows that*

$$\|u\|_{L^\infty_{[-T,T]}} \le \frac{1}{\sqrt{2\sqrt{l_*}}}\|u\|_{E_T} = \frac{1}{\sqrt{2\sqrt{l_*}}} \left\{ \int_{-T}^{T} \Big[|\dot{u}(t)|^2 + \big(L(t)u(t),u(t)\big) \Big] dt \right\}^{\frac{1}{2}}, \tag{2.4}$$

where $l_* = \inf_{t \in \mathbb{R}} l(t)$.

Lemma 2.3 *Assume that all conditions of Theorem 1.2 hold, then there exists a solution $u_T \in E_T$ for the boundary-value problem (2.1) and*

$$\int_{-T}^{T} \Big[|\dot{u}(t)|^2 + \big(L(t)u(t),u(t)\big) \Big] dt < \varrho^2 \quad \text{for all } T \in \mathbb{R}_+. \tag{2.5}$$

Proof Evidently, $I_T(0) = 0$ by (W_1) for all $T \in \mathbb{R}_+$. In order to use Lemma 2.1, we shall construct a closed and bounded convex subset of E_T for all $T \in \mathbb{R}_+$. For any $T \in \mathbb{R}_+$, define

$$\Omega_T := \left\{ u \in E_T : \int_{-T}^{T} \left[|\dot{u}(t)|^2 + \big(L(t)u(t), u(t)\big) \right] dt \le \varrho^2 \right\},$$

where ϱ is defined in (1.2). It is clear that Ω_T is a closed and bounded convex subset of E_T for all $T \in \mathbb{R}_+$.

Given any $T \in \mathbb{R}_+$, we will show that (2.5) is correct. Choose $u \in \partial\Omega_T$, it is obvious that

$$\int_{-T}^{T} \left[|\dot{u}(t)|^2 + \big(L(t)u(t), u(t)\big) \right] dt = \varrho^2.$$

Using Lemma 2.2, we can easily get that $\|u\|_{L^\infty_{[-T,T]}} \le \frac{1}{\sqrt{2\sqrt{l_*}}}\varrho$ for all $u \in \partial\Omega_T$. That is $|u(t)| \le \frac{1}{\sqrt{2\sqrt{l_*}}}\varrho$ for all $t \in [-T, T]$. This inequality together with (L), (W_2) and (F) implies that

$$\begin{aligned}
I_T(u) &= \frac{1}{2}\|u\|_{E_T}^2 - \int_{-T}^{T} W\big(t, u(t)\big)dt + \int_{-T}^{T} \big(f(t), u(t)\big)dt \\
&\ge \frac{1}{2}\|u\|_{E_T}^2 - a\left(\int_{-T}^{T} l(t)|u(t)|^2 dt\right)^{\frac{\mu}{2}} \left(\int_{-T}^{T} l^{-\frac{\mu}{2-\mu}}(t)dt\right)^{\frac{2-\mu}{2}} \\
&\quad - \left(\int_{-T}^{T} |f(t)|^{\beta} dt\right)^{\frac{1}{\beta}} \left(\int_{-T}^{T} |u(t)|^{\beta^*} dt\right)^{\frac{1}{\beta^*}} \\
&\ge \frac{1}{2}\|u\|_{E_T}^2 - a\left(\int_{-T}^{T} l(t)|u(t)|^2 dt\right)^{\frac{\mu}{2}} \left(\int_{-T}^{T} l^{-\frac{\mu}{2-\mu}}(t)dt\right)^{\frac{2-\mu}{2}} \\
&\quad - \|u\|_{L^\infty_{[-T,T]}}^{1-\frac{2}{\beta^*}} \left(\int_{-T}^{T} |f(t)|^{\beta} dt\right)^{\frac{1}{\beta}} \left(\int_{-T}^{T} |u(t)|^{2} dt\right)^{\frac{1}{\beta^*}} \\
&\ge \frac{1}{2}\|u\|_{E_T}^2 - aM_l^{\frac{\mu}{2}}\|u\|_{E_T}^{\mu} - \left(\frac{1}{\sqrt{2\sqrt{l_*}}}\right)^{1-\frac{2}{\beta^*}} M_f l_*^{-\frac{1}{\beta^*}} \|u\|_{E_T} \\
&= \frac{1}{2}\varrho^2 - aM_l^{\frac{\mu}{2}}\varrho^{\mu} - \left(\frac{1}{\sqrt{2\sqrt{l_*}}}\right)^{1-\frac{2}{\beta^*}} M_f l_*^{-\frac{1}{\beta^*}} \varrho
\end{aligned}$$

for all $u \in \partial\Omega_T$. This together with (1.3) yields

$$I_T(u) > I_T(0) = 0, \qquad \forall\, u \in \partial\Omega_T.$$

Therefore, applying Lemma 2.1, we can obtain that for any $T \in \mathbb{R}_+$, there exists $u_T \in \text{int}\Omega_T$ such that $I_T(u_T) = \inf_{u \in \Omega_T} I_T(u)$, where

$$\text{int}\Omega_T = \left\{ u \in E_T : \int_{-T}^{T} \left[|\dot{u}(t)|^2 + \big(L(t)u(t), u(t)\big) \right] dt < \varrho^2 \right\}.$$

Moreover, we note that Theorem 1.3 in [19] implies that $I'_T(u_T) = 0$. This shows that u_T is the solution of problem (2.1) and

$$\int_{-T}^{T} \left[|\dot{u}(t)|^2 + \big(L(t)u(t), u(t)\big) \right] dt < \varrho^2.$$

The proof is complete. □

Proof of Theorem 1.2 We first choose a sequence $T_n \to \infty$ and consider the problem (2.1) on the interval $[-T_n, T_n]$. By Lemma 2.3, there exists a sequence of solution u_n and $\|u_n\|_{E_{T_n}}$ is bounded uniformly in n. The rest procedure is standard, see [18], we omit it here.

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