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Global stability of stationary solutions for a class of semilinear stochastic functional differential equations with additive white noise ☆

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Abstract

This paper gives a criterion for the existence of a stationary solution for a class of semilinear stochastic functional differential equations with additive white noise and its global stability. Under the condition that the global Lipschitz constant of nonlinear term f is less than the absolute value of the top Lyapunov exponent for the linear flow Φ with f being monotone or anti-monotone, and the time delay is not very big, we show that the infinite-dimensional stochastic flow possesses a unique globally attracting random equilibrium in the state space of continuous functions, which produces the globally stable stationary solution. Compared to the result of Jiang and Lv (2016) [24], we remove the assumption of boundedness for f.

MSC: 34K50; 34K21; 34K20; 37L55

Keywords: Stochastic functional differential equations; Infinite-dimensional random dynamical systems; Stationary solutions; Stability

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1. Introduction

This paper is devoted to the global stability of stationary solutions for a class of semilinear stochastic functional differential equations (SFDEs) with additive white noise. These stationary solutions are produced by the globally attracting random equilibria for infinite-dimensional random dynamical systems (RDSs), which are generated by the solutions of SFDEs in the space of continuous functions. For the general theory on SFDEs and RDSs, we refer the reader to [1,11,31,34].

During the last three decades, the stability theory of stochastic differential equations (SDEs) and SFDEs has received a lot of attention in the areas of stochastic analysis and dynamical systems. Inspired by the pioneering works [26,27], all kinds of stability for SDEs and SFDEs have been extensively and intensively studied by many authors, see [7,9,30,32,35–37,46] and the references therein. Using the probability theory and the method of Lyapunov functionals, they have discussed the stability of solutions for SDEs and SFDEs, the existence of invariant measures and other properties.

In the meantime, more and more stochastic analysts and geometers want to use the theory of stochastic flows to investigate the finite or infinite dimensional stochastic systems. In fact, the finite-dimensional stochastic flows can arise from SDEs in the Euclidean space or other finite-dimensional manifolds, see [1,4–6,11,15,16,28]. Furthermore, there are fewer results on the infinite-dimensional stochastic semi-flows, which can arrive naturally from SFDEs and stochastic partial differential equations (SPDEs) with additive or multiplicative white noise. Here, we note that the problem on the existence of infinite-dimensional stochastic semi-flows (with general noise) is very hard. For more details, we refer the reader to [3,8,13,14,17–19,29,38–40]. Most of them dealt with the stochastic semi-flows generated by SPDEs, and [8,13,38,39] treated the stochastic semi-flows generated by SFDEs. To be specific, Mohammed and Scheutzow [38,39] proved the existence of stochastic semi-flows and stable manifolds on a Hilbert space. On the other hand, motivated by the theory of deterministic functional differential equations, it is natural to choose the space of continuous functions as the state space, see [21,42]. This is the main reason that Chueshov and Scheutzow [13] considered the invariance and monotonicity of RDSs generated by stochastic delay differential equations in the space of continuous functions. Moreover, Caraballo, Garrido-Atienza and Schmalfuss [8] showed the existence of exponentially attracting stationary solutions for stochastic delay evolution equations with multiplicative white noise, where they used a general random fixed point theorem and some sufficient conditions were given, see Theorem 6 in [8]. To the best of our knowledge, up to now, there are no results on the existence and the global stability of stationary solutions for infinite-dimensional RDSs, which are generated by semilinear SFDEs with additive white noise in the space of continuous functions.

In this paper, motivated by our recent works [24,25], we will do some efforts on this problem. We will show that if the nonlinear term f is monotone or anti-monotone and the global Lipschitz constant of f is less than the absolute value of the top Lyapunov exponent for the linear flow Φ, and the time delay is not very big, then the infinite-dimensional stochastic flow admits a unique globally stable random equilibrium in the space of continuous functions, which produces a stationary solution for semilinear SFDEs with additive white noise. The main contribution is that the conditions given in this paper are easy to verify and we remove the assumption of boundedness for f, which is a key point in [24,25]. During carrying out the ideas in [24,25], we will meet some difficulties coming from the infinite-dimensional space of continuous functions and the lack of boundedness for f. Note that if f is unbounded, in order to use the characteristic operator $\mathcal{K}$ defined in (2.14) for proving some inequalities in the sense of partial order, we need to give some energy estimates for the pullback trajectories. To be more precise, the pullback trajectories are bounded by some tempered and integrable random variables, see Proposition A.1 in the appendix. Furthermore, due to the unboundedness of f, the work space for the Banach fixed point theorem is also different from that in [24] and we need to show that the bounds for pullback trajectories are tempered, integrable and measurable with respect to the past σ-algebra generated by the Brownian motion, which are proved in Proposition A.1 and Proposition A.2 in the appendix.

Our theory on the global stability of stationary solutions for semilinear SFDEs with additive white noise can be applied to various stochastic delay systems, such as stochastic delay positive feedback systems (neural networks and Othmer-Tyson systems), stochastic delay negative feedback systems (Goodwin systems) and stochastic systems with distributed delay, see Examples 5.1, 5.2 and 5.3. Besides this, the method and the thought used in this paper may bring some new sights to the research on the stability of more general infinite-dimensional stochastic systems, such as SPDEs and so on.

2. Formulation and main results

In this section, we aim to describe precisely the existence of stochastic flows and some hypotheses used in the subsequent content. Based on some preliminaries, we will present the main results at the end of this section. First, we shall recall some notations related to RDSs. The reader is referred to [1,11] for more details.

Let X be a complete separable metric space (i.e., Polish space) equipped with the Borel σ-algebra $\mathscr{B}(X)$ and $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

Definition 2.1. A quadruple $\big(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{R}\}\big)$ is called a metric dynamical system if θ is a measurable flow:

$$\theta : \mathbb{R} \times \Omega \mapsto \Omega, \qquad \theta_0 = \mathrm{id}, \qquad \theta_{t_2} \circ \theta_{t_1} = \theta_{t_1+t_2}$$

for all $t_1, t_2 \in \mathbb{R}$, which is $\big(\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}, \mathscr{F}\big)$-measurable. In addition, we assume that $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 2.2. An RDS on the state space X induced by a metric dynamical system $\big(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{R}\}\big)$ is a mapping

$$\varphi : \mathbb{R}_+ \times \Omega \times X \mapsto X, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x),$$

which is $\big(\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F} \otimes \mathscr{B}(X), \mathscr{B}(X)\big)$-measurable such that for any $\omega \in \Omega$,

(i) $\varphi(0, \omega, \cdot)$ is the identity on X;
(ii) $\varphi(t_1 + t_2, \omega, x) = \varphi\big(t_2, \theta_{t_1}\omega, \varphi(t_1, \omega, x)\big)$ for all $t_1, t_2 \in \mathbb{R}_+$ and $x \in X$;
(iii) the mapping $\varphi(t, \omega, \cdot) : X \to X$ is continuous for all $t \in \mathbb{R}_+$.

Definition 2.3. A family $\big\{D(\omega), \omega \in \Omega\big\}$ of nonempty subsets of the state space X is said to be a random closed (resp. compact) set if for each $\omega \in \Omega$, it is closed (resp. compact) and $\omega \to d\big(x, D(\omega)\big)$ is measurable for each $x \in X$. Here, $d(x, B)$ is the distance in X between the point x and the set $B \subset X$.

Definition 2.4. Let X be a Banach space with a closed convex cone X_+, which gives a partial order relation on X via $x \le y$ if $y - x \in X_+$. An element $x \in X$ is called an upper bound for a subset $A \subset X$ if $y \le x$ for all $y \in A$. An upper bound $\bar{x}$ is called the least upper bound (or supremum), denoted by $\bar{x} = \sup A$, if $\bar{x} \le x$ for any other upper bound x. Moreover, the lower bound and the greatest lower bound (or infimum) can be defined similarly.

Definition 2.5. A cone X_+ is said to be minihedral if every finite set M in X which is order-bounded has a supremum. A cone X_+ is called strongly minihedral if every set M in X which is order-bounded has a supremum.

Throughout this paper, we define the Euclidean norm $|x| := \big(\sum_{i=1}^n |x_i|^2\big)^{1/2}$, $x \in \mathbb{R}^n$, where $\mathbb{R}^n$ is the n-dimensional Euclidean space. For any matrix $D = (D_{ij})_{n\times m} \in \mathbb{R}^{n\times m}$, set $\|D\| := \big(\sum_{i=1}^n \sum_{j=1}^m |D_{ij}|^2\big)^{1/2}$, where $\mathbb{R}^{n\times m}$ denotes the set of all $n \times m$-dimensional real matrices. Let $\tau > 0$ and denote by $C_\tau := C\big([-\tau, 0], \mathbb{R}^n\big)$ the Banach space of continuous functions $\xi : [-\tau, 0] \to \mathbb{R}^n$ equipped with the supremum norm $\|\xi\|_{C_\tau} = \sup_{-\tau\le s\le 0}\big|\xi(s)\big|$ and by C_τ^+ all the nonnegative continuous functions in C_τ. For any given $x_0, y_0 \in C_\tau$, $x_0 \le_{C_\tau^+} y_0$ means that $y_0 - x_0 \in C_\tau^+$. Similarly, set $\mathbb{R}_+^n = \big\{x = (x_1, \dots, x_n) : x_i \ge 0, i = 1, \dots, n\big\}$. For any given $x, y \in \mathbb{R}^n$, $x \le_{\mathbb{R}_+^n} y$ means that $y - x \in \mathbb{R}_+^n$.

Now, we will show that an RDS can be generated by the following SFDEs with additive white noise

$$dx(t) = \big[Ax(t) + f(x_t)\big]dt + \sigma dW_t, \tag{2.1}$$

with the initial value

$$x_0 = \xi \in C_\tau, \tag{2.2}$$

where $x_t \in C_\tau$ is defined by $x_t(s) = x(t + s)$ for $-\tau \le s \le 0$, $A = (a_{ij})_{n\times n}$ is an $n \times n$-dimensional matrix, $f : C_\tau \to \mathbb{R}^n$ and $\sigma = (\sigma_{ij})_{n\times m}$ is an $n \times m$-dimensional matrix, $W_t = (W_t^1, \dots, W_t^m)^{\mathrm{T}}$ is an m-dimensional two-sided Wiener process on the canonical probability space $(\Omega, \mathscr{F}, \mathbb{P})$. For our purposes, we will give some conditions on the drift term:

(H1) A is cooperative in the sense that $a_{ij} \ge 0$ for any $i, j \in \{1, \dots, n\}$ with $i \ne j$. Moreover, we assume that all real parts of eigenvalues of A are negative, i.e., there exist constants $\lambda > 0$ and $C_A > 0$ such that

$$\|\Phi(t)\| \triangleq \left(\sum_{i=1}^n \sum_{j=1}^n \big|\Phi_{ij}(t)\big|^2\right)^{\frac{1}{2}} \le C_A \mathrm{e}^{-\lambda t} \tag{2.3}$$

for all $t \ge 0$. Here, $\Phi(t)$ is the fundamental matrix of the linear ordinary differential equations:

$$dx(t) = Ax(t)dt. \tag{2.4}$$

(H2) $f: C_\tau \to \mathbb{R}^n$ satisfies the global Lipschitz condition

$$\left|f(x_0) - f(y_0)\right| \le L\|x_0 - y_0\|_{C_\tau} \tag{2.5}$$

for all $x_0, y_0 \in C_\tau$, where $L > 0$ is the Lipschitz constant such that $\frac{C_A L e^{\lambda\tau}}{\lambda} < 1$. Furthermore, we assume that f is monotone, i.e.,

$$x_0 \le_{C_\tau^+} y_0 \quad \Rightarrow \quad f(x_0) \le_{\mathbb{R}_+^n} f(y_0) \qquad \text{for all } x_0, y_0 \in C_\tau,$$

or anti-monotone, i.e.,

$$x_0 \le_{C_\tau^+} y_0 \quad \Rightarrow \quad f(x_0) \ge_{\mathbb{R}_+^n} f(y_0) \qquad \text{for all } x_0, y_0 \in C_\tau.$$

Note that the background for the monotonicity and anti-monotonicity comes from neural networks and biochemical reactions, which are presented in Section 5.

In what follows, we will consider a very important metric dynamical system driven by the Brownian motion. Let $W_t = (W_t^1, \dots, W_t^m)^{\mathrm{T}}$ be an m-dimensional two-sided Brownian motion on the canonical probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Here, $\mathscr{F}$ is the Borel σ-algebra of $\Omega = C_0(\mathbb{R}, \mathbb{R}^m) = \left\{\omega = (\omega_1, \omega_2, \dots, \omega_m) \in C(\mathbb{R}, \mathbb{R}^m),\ \omega(0) = 0\right\}$, which is equipped with the following metric

$$\varrho(\omega, \omega^*) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\varrho_k(\omega, \omega^*)}{1 + \varrho_k(\omega, \omega^*)}, \quad \varrho_k(\omega, \omega^*) = \max_{t \in [-k,k]} \left|\omega(t) - \omega^*(t)\right|,$$

and $\mathbb{P}$ is the corresponding Wiener measure. On this set we take the shift operator $\theta = \{\theta_t, t \in \mathbb{R}\}$, defined by $\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t)$ for $t \in \mathbb{R}$, which is an ergodic metric dynamical system.

In order to apply the theory of RDSs, we first need to transform SFDEs with additive white noise (2.1) into deterministic equations with random coefficients. To this end, we consider the auxiliary n-dimensional Ornstein-Uhlenbeck equations

$$dz(t) = Az(t)dt + \sigma dW_t. \tag{2.6}$$

Direct computation shows that one stationary solution of (2.6) called the Ornstein-Uhlenbeck process is given by

$$z(t, \omega) \equiv z(\theta_t\omega) = \int_{-\infty}^{t} \exp\left\{A(t-u)\right\}\sigma dW_u = \int_{-\infty}^{t} \Phi(t-u)\sigma dW_u \tag{2.7}$$

for all $t \in \mathbb{R}$ and $\omega \in \Omega$. In fact, using the integration by parts formula, we can rewrite the form of $z(t, \omega)$ as the following

$$z(t, \omega) = \int_{-\infty}^{0} A\Phi(-u)\sigma W_{u+t}(\omega)du + \sigma W_t(\omega) \tag{2.8}$$

for all $t \in \mathbb{R}$ and $\omega \in \Omega$. Therefore, by (H1), it is easily seen that the random variable $z(\omega)$ is tempered with respect to θ and $z(\theta_t\omega)$ is continuous on $\mathbb{R}$ for any $\omega \in \Omega$, see Lemma 2.5.1 in [11] or Proposition 3.1 in [12].

The existence and uniqueness of solutions for SFDEs (2.1) with the initial value condition (2.2) can be followed by [31,34]. To generate an RDS, we need to define $y(t) = x(t) - z(\theta_t\omega)$ for all $\omega \in \Omega$, where $x(t)$ is the solution of (2.1) and (2.2). Then, using Itô's formula, it follows that y satisfies

$$\frac{dy}{dt} = Ay + f\big(y_t + z_0(\theta_t\omega)\big), \tag{2.9}$$

with the initial value

$$y_0 = x_0 - z_0(\omega), \tag{2.10}$$

where $y_t \in C_\tau$ is defined by $y_t(s) = y(t+s)$ for $-\tau \le s \le 0$ and $z_0(\omega) \in C_\tau$ is defined by $z_0(\omega)(s) := z(\theta_s\omega)$ for $-\tau \le s \le 0$ and $\omega \in \Omega$.

Under the local Lipschitz condition, it was proved in [2, Theorem 2.9] that the solutions of random functional differential equations can generate an RDS if and only if all its solutions can be extended to $[0, +\infty)$, which implies that (2.9) and (2.10) under the global Lipschitz condition can generate an RDS $\psi(t, \omega, y_0) := y_t(\omega, y_0)$ which is continuous with respect to (t, y_0) for each fixed ω. Therefore, the same conclusion holds for the solution $x_t(\omega, x_0)$ of SFDEs (2.1) and (2.2).

Define $\varphi(t, \omega, x_0) : \mathbb{R}_+ \times \Omega \times C_\tau \to C_\tau$ by

$$\varphi(t, \omega, x_0) := x_t(\omega, x_0) = \psi\big(t, \omega, x_0 - z_0(\omega)\big) + z_0(\theta_t\omega) \tag{2.11}$$

for all $t \ge 0$, $\omega \in \Omega$ and $x_0 \in C_\tau$. Then $\varphi(t, \omega, x_0) : \mathbb{R}_+ \times \Omega \times C_\tau \to C_\tau$ is an RDS.

Next, we shall rewrite the form of solutions for SFDEs (2.1) and (2.2). Let $t \ge 0$ and $-\tau \le s \le 0$. Using the variation of constants formula [31, Theorem 3.1], one can have that for all $t + s \ge 0$,

$$\begin{aligned}
\varphi(t, \omega, x_0)(s) &= \Phi(t+s)x_0(0) + \Phi(t+s)\int_0^{t+s} \Phi^{-1}(u) f\big(\varphi(u, \omega, x_0)\big)du \\
&\quad + \Phi(t+s)\int_0^{t+s} \Phi^{-1}(u)\sigma dW_u \\
&= \Phi(t+s)x_0(0) + \int_0^{t+s} \Phi(t+s-u) f\big(\varphi(u, \omega, x_0)\big)du \\
&\quad + \int_0^{t+s} \Phi(t+s-u)\sigma dW_u.
\end{aligned} \tag{2.12}$$

If $t+s \leq 0$, it is obvious that $\varphi(t,\omega,x_0)(s) = x_0(t+s)$ for all $\omega \in \Omega$. Combining the definition of θ and (2.12), it is clear that $\varphi(t,\theta_{-t}\omega,x_0)(s) = x_0(t+s)$ for all $t+s \leq 0$ and $\omega \in \Omega$, and

$$\begin{aligned}\varphi(t,\theta_{-t}\omega,x_0)(s) = \Phi(t+s)x_0(0) &+ \int_0^{t+s} \Phi(t+s-u) f\big(\varphi(u,\theta_{-t}\omega,x_0)\big)du \\ &+ \int_0^{t+s} \Phi(t+s-u)\sigma dW_u(\theta_{-t}\omega) \\ &= \Phi(t+s)x_0(0) + \int_{-t}^{s} \Phi(s-u) f\big(\varphi(u+t,\theta_{-t}\omega,x_0)\big)du \\ &+ \int_{-t}^{s} \Phi(s-u)\sigma dW_u, \quad t+s \geq 0,\ \omega \in \Omega. \end{aligned} \tag{2.13}$$

At the end of this section, motivated by our recent work [24], we will introduce an important characteristic operator associated with (2.13), which is given by

$$\big[\mathcal{K}(r)\big](s,\omega) = \int_{-\infty}^{s} \Phi(s-u) r(\theta_u\omega)du + \int_{-\infty}^{s} \Phi(s-u)\sigma dW_u \tag{2.14}$$

for all $-\tau \leq s \leq 0$ and $\omega \in \Omega$. Here, $r : \Omega \longmapsto \mathbb{R}^n$ is a tempered random variable with respect to the ergodic metric dynamical system θ, i.e., $\sup_{t\in\mathbb{R}}\left\{e^{-\delta|t|}\big|r(\theta_t\omega)\big|\right\} < \infty$ for all $\delta > 0$ and $\omega \in \Omega$.

Remark 2.1. Following the same procedure in [24], by (H1) and (H2), it is easy to check that the characteristic operator $\mathcal{K}$ is well defined for all $-\tau \leq s \leq 0$ and $\omega \in \Omega$. Moreover, for any fixed $\omega \in \Omega$ and tempered random variable r, $\big[\mathcal{K}(r)\big](s,\omega)$ is continuous with respect to $s \in [-\tau, 0]$. We remind that $\big[\mathcal{K}(r)\big](\bullet,\omega) \in C_\tau$ for each ω, which induces a mapping, still denoted by $\big[\mathcal{K}(r)\big] : \Omega \to C_\tau$.

Let $L^1_{\mathscr{F}_-} = L^1(\Omega, \mathscr{F}_-, \mathbb{P}; \mathbb{R}^n)$ denote the space of all $\mathscr{F}_-$-measurable and integrable functions $r : \Omega \to \mathbb{R}^n$, where $\mathscr{F}_- = \sigma\{\omega \mapsto W_t(\omega) : t \leq 0\}$ is the past σ-algebra. In addition, define the operator $\mathcal{K}^f$ to be $f \circ \mathcal{K}$, which means that $\big[\mathcal{K}^f(r)\big](\omega) = f\Big(\big[\mathcal{K}(r)\big](\bullet,\omega)\Big)$ for any random variable $r : \Omega \longmapsto \mathbb{R}^n$. With the help of the characteristic operator $\mathcal{K}$, we can now state our main results.

Theorem 2.1. *Suppose that* (H1) *and* (H2) *hold. Then there exists a unique fixed point* $r \in L^1_{\mathscr{F}_-}$ *for the operator* $\mathcal{K}^f : L^1_{\mathscr{F}_-} \to L^1_{\mathscr{F}_-}$*, which gives that for any* $x_0 \in C_\tau$

$$\lim_{t\to\infty} \varphi(t,\theta_{-t}\omega,x_0)(\bullet) = \big[\mathcal{K}(r)\big](\bullet,\omega) \quad \mathbb{P}\text{-}a.s. \tag{2.15}$$

in C_τ. *Accordingly,* $\varphi\Big(t, \omega, \big[\mathcal{K}(r)\big](\bullet, \omega)\Big)(\bullet) = \big[\mathcal{K}(r)\big](\bullet, \theta_t\omega)$, $\mathbb{P}$*-a.s.,* $t \geq 0$. *To be more precise,* $\big[\mathcal{K}(r)\big](\bullet, \omega)$ *is an* $\mathscr{F}_-$*-measurable random equilibrium in* C_τ *for the stochastic flow* φ*, which yields that* $\big[\mathcal{K}(r)\big](\bullet, \theta_t\omega)$ *is a stationary solution for* (2.1) *and* (2.2).

3. Estimates and monotonicity of the stochastic flow φ generated by SFDEs

In this section, we will establish some useful inequalities in the sense of partial order, which play the key role in the presentation of the dynamical behavior of stochastic flow φ generated by (2.1) and (2.2). We start with a lemma, which can be found in [33].

Lemma 3.1 *(*[33, Lemma A.2]*). Let* $(x_t)_{t\in\Lambda}$ *is a net in a normed space* X *associated with a solid, normal cone* $X_+ \subseteq X$*. Assume that the net converges to a point* $x \in X$*, and that*

$$\underline{x}_t := \inf\{x_{\tilde{t}} : \tilde{t} \geq t\} \quad \text{and} \quad \overline{x}_t := \sup\{x_{\tilde{t}} : \tilde{t} \geq t\}$$

exist for all $t \in \Lambda$*. Then the nets* $(\underline{x}_t)_{t\in\Lambda}$ *and* $(\overline{x}_t)_{t\in\Lambda}$ *also converge to* x.

Lemma 3.2. *Suppose that* (H1) *and* (H2) *hold. For any* $t \geq \tau$*, define*

$$a_t^f(\omega) = \inf \overline{\Big\{ f\big(\varphi(u, \theta_{-u}\omega, x_0)\big) : u \geq t \Big\}} = \inf \Big\{ f\big(\varphi(u, \theta_{-u}\omega, x_0)\big) : u \geq t \Big\}$$

and

$$b_t^f(\omega) = \sup \overline{\Big\{ f\big(\varphi(u, \theta_{-u}\omega, x_0)\big) : u \geq t \Big\}} = \sup \Big\{ f\big(\varphi(u, \theta_{-u}\omega, x_0)\big) : u \geq t \Big\},$$

where $x_0 \in C_\tau$ *and* $\omega \in \Omega$*. Here,* inf *and* sup *represent the infimum and the supremum in* $\mathbb{R}^n$*, respectively. Then* $a_t^f(\omega)$ *and* $b_t^f(\omega)$ *are two tempered* $\mathscr{F}_-$*-measurable random variables for all* $t \geq \tau$.

Proof. Given any $t \geq \tau$, by (H2) and (A.2) in Proposition A.1 (See the appendix), it is easy to see that

$$\Big| f\big(\varphi(t, \theta_{-t}\omega, x_0)\big) \Big| \leq L\big|\widetilde{R}(\omega)\big| + \big|f(0)\big| < \infty \tag{3.1}$$

for all $\omega \in \Omega$ and $x_0 \in C_\tau$, where $\widetilde{R}(\omega)$ is a tempered random variable defined in Proposition A.1. This implies that $\overline{\Big\{ f\big(\varphi(u, \theta_{-u}\omega, x_0)\big) : u \geq t \Big\}}$ is a compact set in $\mathbb{R}^n$, and then $a_t^f(\omega)$ and $b_t^f(\omega)$ are both well defined for all $t \geq \tau$, $x_0 \in C_\tau$ and $\omega \in \Omega$. Here, we use the fact that $\mathbb{R}^n_+$ is a strongly minihedral cone, $\inf \overline{A} = \inf A$ and $\sup \overline{A} = \sup A$, where A is a bounded set in $\mathbb{R}^n$, see Lemma A.1 in [33]. In what follows, we will show that the mapping

$$t \longmapsto \varphi(t, \theta_{-t}\omega, x_0)$$

is continuous from $[\tau, \infty)$ into C_τ for all $\omega \in \Omega$ and $x_0 \in C_\tau$. By (2.11), it is obvious that

$$\varphi(t, \theta_{-t}\omega, x_0) = \psi(t, \theta_{-t}\omega, x_0 - z_0(\theta_{-t}\omega)) + z_0(\omega) = y_t\big(\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big) + z_0(\omega),$$

which implies that we only need to prove the continuity of $y_t\big(\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big)$ for any fixed $\omega \in \Omega$ and $x_0 \in C_\tau$. Since Ω is a metric space and $\theta(\bullet, \omega) : (\mathbb{R}, |\bullet|) \mapsto (\Omega, \varrho)$ is continuous for all $\omega \in \Omega$, see Proposition 3.2 in [25], using the quite same method as in the proof of the continuous dependence of solutions for functional differential equations with respect to the parameter "ω" and the initial value "y_0", it is sufficient to prove that $z_0(\theta_t\omega)$ is continuous with respect to the time t for all $\omega \in \Omega$. By (2.7), it is easy to see that $z(\theta_t\omega)$ is continuous on $\mathbb{R}$, which together with Lemma 2.1 in [21, Chapter 2] yields that this property is true. Therefore, by (H2), it is clear that

$$t \mapsto f\big(\varphi(t, \theta_{-t}\omega, x_0)\big) \text{ is continuous}$$

from $[\tau, \infty)$ into $\mathbb{R}^n$ for all $\omega \in \Omega$ and $x_0 \in C_\tau$. Applying the same way as Proposition 3.5 in [25], we conclude that $a_t^f(\omega)$ and $b_t^f(\omega)$ are two $\mathscr{F}_-$-measurable random variables for all $t \geq \tau$. Finally, by (3.1), it is evident that

$$-R_f(\omega) \leq_{\mathbb{R}^n_+} a_t^f(\omega) \leq_{\mathbb{R}^n_+} b_t^f(\omega) \leq_{\mathbb{R}^n_+} R_f(\omega) \tag{3.2}$$

for all $t \geq \tau$ and $\omega \in \Omega$, where $R_f(\omega) = \Big[L\big|\widetilde{R}(\omega)\big| + \big|f(0)\big|\Big] \cdot (1, \dots, 1)^{\mathrm{T}}$. This shows that

$$\big|a_t^f(\omega)\big| \leq \sqrt{n}\Big[L\big|\widetilde{R}(\omega)\big| + \big|f(0)\big|\Big] \text{ and } \big|b_t^f(\omega)\big| \leq \sqrt{n}\Big[L\big|\widetilde{R}(\omega)\big| + \big|f(0)\big|\Big] \tag{3.3}$$

for all $t \geq \tau$ and $\omega \in \Omega$, which together with Proposition A.1 implies that $a_t^f(\omega)$ and $b_t^f(\omega)$ are both tempered for all $t \geq \tau$. The proof is complete. □

Lemma 3.3. *Suppose that* (H1) *and* (H2) *hold. For any $t \geq \tau$, set*

$$a_t^{\varphi}(\omega) = \inf \overline{\big\{\varphi(u, \theta_{-u}\omega, x_0)(\bullet) : u \geq t\big\}} = \inf\big\{\varphi(u, \theta_{-u}\omega, x_0)(\bullet) : u \geq t\big\}$$

and

$$b_t^{\varphi}(\omega) = \sup \overline{\big\{\varphi(u, \theta_{-u}\omega, x_0)(\bullet) : u \geq t\big\}} = \sup\big\{\varphi(u, \theta_{-u}\omega, x_0)(\bullet) : u \geq t\big\},$$

where $x_0 \in C_\tau$ and $\omega \in \Omega$. Here, inf *and* sup *represent the infimum and the supremum in C_τ, respectively. Then $a_t^{\varphi}(\omega) : \Omega \mapsto C_\tau$ is a well-posed $\mathscr{F}_-$-measurable function, and so also is $b_t^{\varphi}(\omega)$ for all $t \geq \tau$.*

Proof. By Lemma 3.2 and Proposition A.1 in the appendix, we have that for any $t \geq \tau$, $\overline{\gamma_{x_0}^t(\omega)} := \overline{\big\{\varphi(u, \theta_{-u}\omega, x_0) : u \geq t\big\}}$ is a compact set in C_τ and for all $\omega \in \Omega$ and $x_0 \in C_\tau$,

$$u \longmapsto \varphi(u, \theta_{-u}\omega, x_0) \text{ is a continuous mapping}$$

from $[t, \infty)$ into C_τ. Furthermore, it is easy to check that for all $u \geq t \geq \tau$ and $x_0 \in C_\tau$,

$$\omega \longmapsto \varphi(u, \theta_{-u}\omega, x_0) \text{is } \mathscr{F}_-\text{-measurable.}$$

Since C_τ^+ is a solid normal minihedral cone, using Proposition 1.5.3 and Theorem 3.2.1 in [11], we see at once that $a_t^\varphi(\omega) : \Omega \mapsto C_\tau$ and $b_t^\varphi(\omega) : \Omega \mapsto C_\tau$ are two well-posed $\mathscr{F}_-$-measurable functions. Here, we use the fact that $\inf \overline{A} = \inf A$ and $\sup \overline{A} = \sup A$, where A is a relatively compact set in C_τ, see Theorem 3.1.2 in [11] and Lemma A.1 in [33]. The proof is complete. □

Lemma 3.4. *Suppose that* (H1) *and* (H2) *hold. For any $t \geq \tau$, define*

$$C_{\tilde{t}}^\varphi(\omega) \triangleq \int_{t-\tilde{t}}^{\bullet} \Phi(\bullet - u) \inf\Big\{ f\big(\varphi(v, \theta_{-v+u}\omega, x_0)\big) + R_f(\theta_u\omega) : v \geq t \Big\} du$$
$$- \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u) R_f(\theta_u\omega) du + \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u)\sigma dW_u + \Phi(\tilde{t} + \bullet)x_0(0)$$

and

$$D_{\tilde{t}}^\varphi(\omega) \triangleq \int_{t-\tilde{t}}^{\bullet} \Phi(\bullet - u) \sup\Big\{ f\big(\varphi(v, \theta_{-v+u}\omega, x_0)\big) - R_f(\theta_u\omega) : v \geq t \Big\} du$$
$$+ \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u) R_f(\theta_u\omega) du + \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u)\sigma dW_u + \Phi(\tilde{t} + \bullet)x_0(0),$$

where $\tilde{t} \geq t + \tau$, $\omega \in \Omega$ and $x_0 \in C_\tau$. Then,

$$\big[\mathcal{K}(a_t^f)\big](\bullet, \omega) = \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\geq t+\tau}} C_{\tilde{t}}^\varphi(\omega) = \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\geq t+\tau}} c_{\tilde{t}}^\varphi(\omega) \qquad \text{in} \quad C_\tau, \tag{3.4}$$

$$\big[\mathcal{K}(b_t^f)\big](\bullet, \omega) = \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\geq t+\tau}} D_{\tilde{t}}^\varphi(\omega) = \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\geq t+\tau}} d_{\tilde{t}}^\varphi(\omega) \qquad \text{in} \quad C_\tau, \tag{3.5}$$

$$C_{\tilde{t}}^\varphi(\omega) \leq \varphi(\tilde{t}, \theta_{-\tilde{t}}\omega, x_0) \leq D_{\tilde{t}}^\varphi(\omega) \qquad \text{in} \quad C_\tau \tag{3.6}$$

and

$$c_{\tilde{t}}^\varphi(\omega) \leq a_{\tilde{t}}^\varphi(\omega) \leq b_{\tilde{t}}^\varphi(\omega) \leq d_{\tilde{t}}^\varphi(\omega) \qquad \text{in} \quad C_\tau. \tag{3.7}$$

Here, $\leq$ means $\leq_{C_\tau^+}$,

$$c_{\tilde{t}}^\varphi(\omega) = \inf\big\{ C_u^\varphi(\omega) : u \geq \tilde{t} \big\} \tag{3.8}$$

and

$$d_{\tilde{t}}^\varphi(\omega) = \sup\big\{ D_u^\varphi(\omega) : u \geq \tilde{t} \big\}. \tag{3.9}$$

Proof. By Lemma 3.3, it is evident that $a^{\varphi}_{\tilde{t}}(\omega):\Omega\mapsto C_\tau$ and $b^{\varphi}_{\tilde{t}}(\omega):\Omega\mapsto C_\tau$ are both well-defined for all $\tilde{t}\geq t+\tau$, $\omega\in\Omega$ and $x_0\in C_\tau$, where $t\geq\tau$. Moreover, we can observe that

$$\lim_{\substack{\tilde{t}\to\infty\\ \tilde{t}\geq t+\tau}}\int_{-\tilde{t}}^{t-\tilde{t}}\Phi(\bullet-u)R_f(\theta_u\omega)du=0 \tag{3.10}$$

in C_τ. In fact, by (H1), it is a simple matter to check that

$$\begin{aligned}&\sup_{-\tau\leq s\leq 0}\left|\int_{-\tilde{t}}^{t-\tilde{t}}\Phi(s-u)R_f(\theta_u\omega)du\right|\\ &\leq C_A\sup_{-\tau\leq s\leq 0}\left\|\Phi(s)\right\|\int_{-\tilde{t}}^{t-\tilde{t}}e^{\lambda u}\left|R_f(\theta_u\omega)\right|du\\ &\leq C_A\sup_{-\tau\leq s\leq 0}\left\|\Phi(s)\right\|\cdot\sup_{u\in\mathbb{R}}\left\{e^{-\frac{\lambda}{2}|u|}\left|R_f(\theta_u\omega)\right|\right\}\cdot\int_{-\tilde{t}}^{t-\tilde{t}}e^{\frac{\lambda}{2}u}du\\ &\longrightarrow 0\end{aligned}$$

as $\tilde{t}\to\infty$, which together with Lemma 3.2 and Lebesgue's dominated convergence theorem shows that

$$\begin{aligned}&\left[\mathcal{K}(a^f_t)\right](\bullet,\omega)\\ &=\int_{-\infty}^{\bullet}\Phi(\bullet-u)\inf\Big\{f\big(\varphi(v,\theta_{-v}\diamond,x_0)\big):v\geq t\Big\}(\theta_u\omega)du+\int_{-\infty}^{\bullet}\Phi(\bullet-u)\sigma dW_u\\ &=\lim_{\substack{\tilde{t}\to\infty\\ \tilde{t}\geq t+\tau}}\left\{\int_{t-\tilde{t}}^{\bullet}\Phi(\bullet-u)\inf\Big\{f\big(\varphi(v,\theta_{-v+u}\omega,x_0)\big)+R_f(\theta_u\omega):v\geq t\Big\}du\right.\\ &\left.\quad-\int_{-\tilde{t}}^{\bullet}\Phi(\bullet-u)R_f(\theta_u\omega)du+\int_{-\tilde{t}}^{\bullet}\Phi(\bullet-u)\sigma dW_u+\Phi(\tilde{t}+\bullet)x_0(0)\right\}\\ &=\lim_{\substack{\tilde{t}\to\infty\\ \tilde{t}\geq t+\tau}}C^{\varphi}_{\tilde{t}}(\omega)\end{aligned} \tag{3.11}$$

and

$$\left[\mathcal{K}(b^f_t)\right](\bullet,\omega)$$

$$= \int_{-\infty}^{\bullet} \Phi(\bullet - u) \sup\Big\{ f\big(\varphi(v, \theta_{-v}\diamond, x_0)\big) : v \ge t \Big\}(\theta_u \omega) du + \int_{-\infty}^{\bullet} \Phi(\bullet - u)\sigma dW_u$$

$$= \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\ge t+\tau}} \Bigg\{ \int_{t-\tilde{t}}^{\bullet} \Phi(\bullet - u) \sup\Big\{ f\big(\varphi(v, \theta_{-v+u}\omega, x_0)\big) - R_f(\theta_u\omega) : v \ge t \Big\} du$$

$$+ \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u) R_f(\theta_u\omega) du + \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u)\sigma dW_u + \Phi(\tilde{t} + \bullet)x_0(0) \Bigg\}$$

$$= \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\ge t+\tau}} D^{\varphi}_{\tilde{t}}(\omega). \tag{3.12}$$

Applying the same argument in Proposition A.1, it follows that $\big\{C^{\varphi}_u(\omega) : u \ge t+\tau\big\}$ and $\big\{D^{\varphi}_u(\omega) : u \ge t+\tau\big\}$ are two relatively compact sets in C_τ for all $t \ge \tau$, $\omega \in \Omega$ and $x_0 \in C_\tau$. In addition, this can also be obtained from (3.11), (3.12) and the fact that $C^{\varphi}_{\tilde{t}}(\omega)$ and $D^{\varphi}_{\tilde{t}}(\omega)$ are continuous with respect to $\tilde{t}$ in C_τ. Consequently, by Lemma 3.1 and Theorem 3.1.2 in [11], we have

$$\big[\mathcal{K}(a^f_t)\big](\bullet, \omega) = \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\ge t+\tau}} \inf\big\{C^{\varphi}_u(\omega) : u \ge \tilde{t}\big\} = \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\ge t+\tau}} \inf \overline{\big\{C^{\varphi}_u(\omega) : u \ge \tilde{t}\big\}}$$

and

$$\big[\mathcal{K}(b^f_t)\big](\bullet, \omega) = \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\ge t+\tau}} \sup\big\{D^{\varphi}_u(\omega) : u \ge \tilde{t}\big\} = \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\ge t+\tau}} \sup \overline{\big\{D^{\varphi}_u(\omega) : u \ge \tilde{t}\big\}}.$$

This yields that (3.4) and (3.5) are true.

Furthermore, by (3.2), we get that $a^f_t(\omega) + R_f(\omega) \ge_{\mathbb{R}^n_+} 0$ and $b^f_t(\omega) - R_f(\omega) \le_{\mathbb{R}^n_+} 0$ for all $t \ge \tau$ and $\omega \in \Omega$. Since $\Phi(t)x \ge_{\mathbb{R}^n_+} 0$ for all $t \ge 0$ and $x \in \mathbb{R}^n_+$, it follows immediately that

$$C^{\varphi}_{\tilde{t}}(\omega) \le \int_{t-\tilde{t}}^{\bullet} \Phi(\bullet - u)\Big[f\big(\varphi(\tilde{t} + u, \theta_{-\tilde{t}}\omega, x_0)\big) + R_f(\theta_u\omega)\Big] du$$

$$- \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u) R_f(\theta_u\omega) du + \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u)\sigma dW_u + \Phi(\tilde{t} + \bullet)x_0(0)$$

$$\le \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u)\Big[f\big(\varphi(\tilde{t} + u, \theta_{-\tilde{t}}\omega, x_0)\big) + R_f(\theta_u\omega)\Big] du$$

$$- \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u) R_f(\theta_u\omega) du + \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u)\sigma dW_u + \Phi(\tilde{t} + \bullet)x_0(0)$$

$$
\begin{aligned}
&= \varphi(\tilde{t}, \theta_{-\tilde{t}}\omega, x_0)(\bullet) \\
&= \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u)\Big[f\big(\varphi(\tilde{t}+u, \theta_{-\tilde{t}}\omega, x_0)\big) - R_f(\theta_u\omega)\Big] du \\
&\quad + \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u) R_f(\theta_u\omega) du + \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u)\sigma dW_u + \Phi(\tilde{t}+\bullet)x_0(0) \\
&\le \int_{t-\tilde{t}}^{\bullet} \Phi(\bullet - u)\Big[f\big(\varphi(\tilde{t}+u, \theta_{-\tilde{t}}\omega, x_0)\big) - R_f(\theta_u\omega)\Big] du \\
&\quad + \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u) R_f(\theta_u\omega) du + \int_{-\tilde{t}}^{\bullet} \Phi(\bullet - u)\sigma dW_u + \Phi(\tilde{t}+\bullet)x_0(0) \\
&\le D^{\varphi}_{\tilde{t}}(\omega)
\end{aligned} \tag{3.13}
$$

for all $\tilde{t} \ge t + \tau$, $\omega \in \Omega$ and $x_0 \in C_\tau$, which together with definitions of the infimum and supremum implies that

$$
c^{\varphi}_{\tilde{t}}(\omega) \le a^{\varphi}_{\tilde{t}}(\omega) \le b^{\varphi}_{\tilde{t}}(\omega) \le d^{\varphi}_{\tilde{t}}(\omega) \quad \text{in} \quad C_\tau .
$$

The proof is complete. □

Remark 3.1. Note that the positive cone C^+_τ is not strongly minihedral, it is necessary to verify some compactness of the pullback trajectories in C_τ, see Proposition A.1. Moreover, since C^+_τ is not regular, which yields that the monotonicity and boundedness of a sequence can not imply its convergence. That is, the limit of $a^{\varphi}_{\tilde{t}}(\omega)$ and $b^{\varphi}_{\tilde{t}}(\omega)$ may not exist in C_τ as $\tilde{t} \to \infty$. Therefore, the conclusion presented in Lemma 3.4 is different from that in [24].

Lemma 3.5. *Suppose that* (H1) *and* (H2) *hold. Set*

$$
\big[\underline{\lim}_\theta f(\varphi)\big](\omega) \triangleq \lim_{t\to\infty} a^f_t(\omega) = \lim_{t\to\infty} \inf\Big\{ f\big(\varphi(u, \theta_{-u}\omega, x_0)\big) : u \ge t \Big\} \tag{3.14}
$$

and

$$
\big[\overline{\lim}_\theta f(\varphi)\big](\omega) \triangleq \lim_{t\to\infty} b^f_t(\omega) = \lim_{t\to\infty} \sup\Big\{ f\big(\varphi(u, \theta_{-u}\omega, x_0)\big) : u \ge t \Big\}. \tag{3.15}
$$

Thus

(i) *If f is monotone, we deduce that for all $t \ge \tau$, $\omega \in \Omega$, $x_0 \in C_\tau$ and $k \in \mathbb{N}$,*

$$
\big[(\mathcal{K}^f)^k(a^f_t)\big](\omega) \le \big[\underline{\lim}_\theta f(\varphi)\big](\omega) \le \big[\overline{\lim}_\theta f(\varphi)\big](\omega) \le \big[(\mathcal{K}^f)^k(b^f_t)\big](\omega). \tag{3.16}
$$

(ii) *If f is anti-monotone, we deduce that for all $t \geq \tau$, $\omega \in \Omega$, $x_0 \in C_\tau$ and $k \in \mathbb{N}$,*

$$\left[(\mathcal{K}^f)^{2k}(a_t^f)\right](\omega) \leq \left[\underline{\lim}_\theta f(\varphi)\right](\omega) \leq \left[\overline{\lim}_\theta f(\varphi)\right](\omega) \leq \left[(\mathcal{K}^f)^{2k}(b_t^f)\right](\omega). \tag{3.17}$$

Proof. First, combining (3.2) and the monotone convergence theorem, this induces that $\left[\underline{\lim}_\theta f(\varphi)\right](\omega)$ and $\left[\overline{\lim}_\theta f(\varphi)\right](\omega)$ are both well defined $\mathscr{F}_-$-measurable random variables. Next, we will show (3.16) and (3.17). For simplicity, we only show the case that f is monotone. For any $t \geq \tau$, $\omega \in \Omega$ and $x_0 \in C_\tau$, it is immediate that

$$\begin{aligned}
\left[\mathcal{K}^f(a_t^f)\right](\omega) &= f\Big(\left[\mathcal{K}(a_t^f)\right](\bullet,\omega)\Big) \\
&= f\Big(\lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\geq t+\tau}} C_{\tilde{t}}^{\varphi}(\omega)\Big) && \text{by (3.4)} \\
&= \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\geq t+\tau}} f\left(C_{\tilde{t}}^{\varphi}(\omega)\right) && \text{by (H2)} \\
&= \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\geq t+\tau}} \inf\left\{ f\left(C_u^{\varphi}(\omega)\right) : u \geq \tilde{t}\right\} && \text{by Lemma 3.1} \\
&\leq \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\geq t+\tau}} \inf\left\{ f\left(\varphi(u,\theta_{-u}\omega,x_0)\right) : u \geq \tilde{t}\right\} && \text{by (3.6)} \\
&= \left[\underline{\lim}_\theta f(\varphi)\right](\omega)
\end{aligned} \tag{3.18}$$

and an argument similar to (3.18) gives that

$$\left[\mathcal{K}^f(b_t^f)\right](\omega) \geq \left[\overline{\lim}_\theta f(\varphi)\right](\omega). \tag{3.19}$$

Thus, (3.16) holds for $k = 1$. The rest proof can be obtained by mathematical induction. Suppose that (3.16) is true for some $k \in \mathbb{N}$, it follows from the monotonicity of $\mathcal{K}$ and f that

$$\begin{aligned}
\left[(\mathcal{K}^f)^{k+1}(a_t^f)\right](\omega) &\leq \left[\mathcal{K}^f\left(\underline{\lim}_\theta f(\varphi)\right)\right](\omega) \\
&= \left[\mathcal{K}^f\left(\lim_{u\to\infty} a_u^f\right)\right](\omega) \\
&= f\Big(\lim_{u\to\infty}\left[\mathcal{K}(a_u^f)\right](\bullet,\omega)\Big) && \text{by Lebesgue's DCT} \\
&= \lim_{u\to\infty} f\Big(\left[\mathcal{K}(a_u^f)\right](\bullet,\omega)\Big) && \text{by (H2)} \\
&\leq \left[\underline{\lim}_\theta f(\varphi)\right](\omega) && \text{by (3.18)}
\end{aligned}$$

and similarly

$$\left[(\mathcal{K}^f)^{k+1}(b_t^f)\right](\omega) \geq \left[\overline{\lim}_\theta f(\varphi)\right](\omega).$$

The proof is complete. □

Lemma 3.6. *Suppose that* (H1) *and* (H2) *hold. It follows that the space* $L^1_{\mathscr{F}_-} = L^1(\Omega, \mathscr{F}_-, \mathbb{P}; \mathbb{R}^n)$ *is complete under the metric* $\|r\|_{L^1} = \int_\Omega \big|r(\omega)\big|\mathbb{P}(\mathrm{d}\omega)$, $r \in L^1_{\mathscr{F}_-}$ *and the operator* $\mathcal{K}^f = f \circ \mathcal{K} : (L^1_{\mathscr{F}_-}, \|\cdot\|_{L^1}) \to (L^1_{\mathscr{F}_-}, \|\cdot\|_{L^1})$ *is a contraction mapping.*

Proof. By definition, it is clear that $L^1(\Omega, \mathscr{F}_-, \mathbb{P}; \mathbb{R}^n)$ is a Banach space with the norm

$$\|r\|_{L^1} = \int\limits_\Omega \big|r(\omega)\big|\mathbb{P}(\mathrm{d}\omega).$$

Next, we can assert that $\mathcal{K}^f : L^1_{\mathscr{F}_-} \to L^1_{\mathscr{F}_-}$ is well defined. Given any $r \in L^1_{\mathscr{F}_-}$, since $\theta : \mathbb{R}_- \times \Omega \mapsto \Omega$ is $\big(\mathscr{B}(\mathbb{R}_-) \otimes \mathscr{F}_-, \mathscr{F}_-\big)$-measurable, see Proposition 3.3 in [25], which together with Fubini's theorem induces that the mapping

$$\omega \longrightarrow \big[\mathcal{K}(r)\big](s,\omega) = \int\limits_{-\infty}^{s} \Phi(s-u) r(\theta_u\omega) du + \int\limits_{-\infty}^{s} \Phi(s-u)\sigma dW_u \tag{3.20}$$

is $\mathscr{F}_-$-measurable for all $-\tau \le s \le 0$. Here, from (3.20) and the fact that r is integrable, we can conclude that $\big[\mathcal{K}(r)\big](\bullet,\omega)$ exists almost surely due to (3.24). Combining this and Lemma II.2.1 in [34], it follows that the function $\omega \to \big[\mathcal{K}(r)\big](\bullet,\omega)$ is $\big(\mathscr{F}_-, \mathscr{B}(C_\tau)\big)$-measurable. Note that $f : C_\tau \to \mathbb{R}^n$ is continuous, and then $\omega \to \big[\mathcal{K}^f(r)\big](\omega)$ is also $\mathscr{F}_-$-measurable. In addition, by (3.20), it is evident that

$$\begin{aligned} &\mathbb{E}\left(\Big\|\big[\mathcal{K}(r)\big](\bullet,\omega)\Big\|_{C_\tau}\right) \\ &\le \mathbb{E}\left(\sup_{-\tau\le s\le 0}\left|\int\limits_{-\infty}^{s} \Phi(s-u)r(\theta_u\omega)du\right|\right) + \mathbb{E}\left(\sup_{-\tau\le s\le 0}\left|\int\limits_{-\infty}^{s} \Phi(s-u)\sigma dW_u\right|\right). \end{aligned} \tag{3.21}$$

Denote

$$I_1 = \mathbb{E}\left(\sup_{-\tau\le s\le 0}\left|\int\limits_{-\infty}^{s} \Phi(s-u)r(\theta_u\omega)du\right|\right)$$

and

$$I_2 = \mathbb{E}\left(\sup_{-\tau\le s\le 0}\left|\int\limits_{-\infty}^{s} \Phi(s-u)\sigma dW_u\right|\right).$$

Therefore,

$$
\begin{aligned}
I_1 &\leq \sup_{-\tau\leq s\leq 0}\left\|\Phi(s)\right\| \cdot \mathbb{E}\left(\int_{-\infty}^{0}\left|\Phi(-u)r(\theta_u\omega)\right|du\right)\\
&\leq C_A \sup_{-\tau\leq s\leq 0}\left\|\Phi(s)\right\| \cdot \int_{-\infty}^{0} e^{\lambda u}\mathbb{E}\left|r(\theta_u\omega)\right|du\\
&= C_A \sup_{-\tau\leq s\leq 0}\left\|\Phi(s)\right\| \cdot \mathbb{E}|r| \cdot \int_{-\infty}^{0} e^{\lambda u}du\\
&= \frac{\mu C_A}{\lambda}\|r\|_{L^1},
\end{aligned}
\tag{3.22}
$$

where $\mu = \sup_{-\tau\leq s\leq 0}\left\|\Phi(s)\right\|$. Moreover, observe that

$$
M_t \triangleq \int_{-t}^{0}\Phi(-u)\sigma dW_u, \quad t\geq 0,
$$

is a continuous martingale with respect to the filtration $\mathscr{G}_t = \mathscr{F}_{-t}^0 = \sigma\{W_s : -t \leq s \leq 0\}$, $t \geq 0$. Then, using Doob's martingale inequality (see Theorem 3.8 in [31]) and Hölder's inequality, we have

$$
\begin{aligned}
I_2 &\leq \mathbb{E}\left(\sup_{-\tau\leq s\leq 0}\left|\int_{-\infty}^{0}\Phi(s-u)\sigma dW_u\right|\right) + \mathbb{E}\left(\sup_{-\tau\leq s\leq 0}\left|\int_{s}^{0}\Phi(s-u)\sigma dW_u\right|\right)\\
&\leq \mu\left[\mathbb{E}\left(\left|\int_{-\infty}^{0}\Phi(-u)\sigma dW_u\right|\right) + \mathbb{E}\left(\sup_{-\tau\leq s\leq 0}\left|\int_{s}^{0}\Phi(-u)\sigma dW_u\right|\right)\right]\\
&\leq \mu\left[\left(\mathbb{E}\left(\left|\int_{-\infty}^{0}\Phi(-u)\sigma dW_u\right|^2\right)\right)^{\frac{1}{2}} + \left(\mathbb{E}\left(\sup_{-\tau\leq s\leq 0}\left|\int_{s}^{0}\Phi(-u)\sigma dW_u\right|^2\right)\right)^{\frac{1}{2}}\right]\\
&= \mu\left[\left(\mathbb{E}\left(\left|\int_{-\infty}^{0}\Phi(-u)\sigma dW_u\right|^2\right)\right)^{\frac{1}{2}}\right.\\
&\quad \left. + \left(\mathbb{E}\left(\sup_{-\tau\leq s\leq 0}\sum_{i=1}^{n}\left(\sum_{j=1}^{m}\int_{s}^{0}\left[\Phi(-u)\sigma\right]_{ij}dW_u^j\right)^2\right)\right)^{\frac{1}{2}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le \mu \left[\left(\mathbb{E} \left(\left| \int_{-\infty}^{0} \Phi(-u)\sigma dW_u \right|^2 \right) \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \left(m \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{E} \left(\sup_{-\tau \le s \le 0} \left| \int_{s}^{0} \left[\Phi(-u)\sigma \right]_{ij} dW_u^j \right|^2 \right) \right)^{\frac{1}{2}} \right] \\
&\le \mu \left[\left(\mathbb{E} \left(\left| \int_{-\infty}^{0} \Phi(-u)\sigma dW_u \right|^2 \right) \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \left(4m \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{E} \left(\left| \int_{-\tau}^{0} \left[\Phi(-u)\sigma \right]_{ij} dW_u^j \right|^2 \right) \right)^{\frac{1}{2}} \right] \\
&= \mu \left[\left(\mathbb{E} \left(\left| \int_{-\infty}^{0} \Phi(-u)\sigma dW_u \right|^2 \right) \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \left(4m \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\int_{-\tau}^{0} \left| \left[\Phi(-u)\sigma \right]_{ij} \right|^2 du \right) \right)^{\frac{1}{2}} \right] \\
&\le \mu \left[\left(\int_{-\infty}^{0} \left\| \Phi(-u)\sigma \right\|^2 du \right)^{\frac{1}{2}} + \left(4m^2 n \left(\int_{-\infty}^{0} \left\| \Phi(-u)\sigma \right\|^2 du \right) \right)^{\frac{1}{2}} \right] \\
&\le \mu \left(1 + 2m\sqrt{n} \right) \left(\int_{-\infty}^{0} \left\| \Phi(-u)\sigma \right\|^2 du \right)^{\frac{1}{2}} \\
&\le \mu \left(1 + 2m\sqrt{n} \right) C_A \|\sigma\| \left(\int_{-\infty}^{0} e^{2\lambda u} du \right)^{\frac{1}{2}} \\
&= \frac{\mu \left(1 + 2m\sqrt{n} \right) C_A \|\sigma\|}{\sqrt{2\lambda}},
\end{aligned}
\tag{3.23}
$$

where we use the following inequality

$$
(x_1 + x_2 + \cdots + x_m)^2 \le m(x_1^2 + x_2^2 + \cdots + x_m^2), \quad x_i \ge 0,\ i = 1, \ldots, m.
$$

Consequently, from (3.21), (3.22) and (3.23), we can obtain

$$\mathbb{E}\left(\left\|\big[\mathcal{K}(r)\big](\bullet,\omega)\right\|_{C_\tau}\right) \le \frac{\mu C_A}{\lambda}\|r\|_{L^1} + \frac{\mu\big(1+2m\sqrt{n}\big)C_A\|\sigma\|}{\sqrt{2\lambda}}, \tag{3.24}$$

which together with (H2) implies that

$$\begin{aligned}\mathbb{E}\big|\mathcal{K}^f(r)\big| &\le L\mathbb{E}\left(\left\|\big[\mathcal{K}(r)\big](\bullet,\omega)\right\|_{C_\tau}\right) + \big|f(0)\big| \\ &\le L\left(\frac{\mu C_A}{\lambda}\|r\|_{L^1} + \frac{\mu\big(1+2m\sqrt{n}\big)C_A\|\sigma\|}{\sqrt{2\lambda}}\right) + \big|f(0)\big| \\ &< \infty.\end{aligned}$$

That is, $\mathcal{K}^f : L^1_{\mathscr{F}_-} \to L^1_{\mathscr{F}_-}$ is well defined.

Now, we turn to prove that $\mathcal{K}^f : L^1_{\mathscr{F}_-} \to L^1_{\mathscr{F}_-}$ is a contraction mapping. For any $r_1, r_2 \in L^1_{\mathscr{F}_-}$, by (H1) and (H2), it is easy to see that

$$\begin{aligned}\mathbb{E}\big|\mathcal{K}^f(r_1) - \mathcal{K}^f(r_2)\big| &\le L\mathbb{E}\left(\left\|\big[\mathcal{K}(r_1)\big](\bullet,\omega) - \big[\mathcal{K}(r_2)\big](\bullet,\omega)\right\|_{C_\tau}\right) \\ &= L\mathbb{E}\left(\sup_{-\tau\le s\le 0}\left|\int_{-\infty}^{s} \Phi(s-u)\big[r_1(\theta_u\omega) - r_2(\theta_u\omega)\big]du\right|\right) \\ &\le C_A L\mathbb{E}\left(\sup_{-\tau\le s\le 0}\int_{-\infty}^{s} e^{-\lambda(s-u)}\big|r_1(\theta_u\omega) - r_2(\theta_u\omega)\big|du\right) \\ &\le C_A L e^{\lambda\tau}\mathbb{E}\left(\int_{-\infty}^{0} e^{\lambda u}\big|r_1(\theta_u\omega) - r_2(\theta_u\omega)\big|du\right) \\ &= C_A L e^{\lambda\tau}\left(\int_{-\infty}^{0} e^{\lambda u}\mathbb{E}\big|r_1(\theta_u\omega) - r_2(\theta_u\omega)\big|du\right) \\ &= \frac{C_A L e^{\lambda\tau}}{\lambda}\|r_1 - r_2\|_{L^1},\end{aligned} \tag{3.25}$$

where $\frac{C_A L e^{\lambda\tau}}{\lambda} < 1$. The proof is complete. □

4. Proof of Theorem 2.1

Proof. Assume that f is monotone or anti-monotone, by Lemma 3.5, it is clear that,

$$\big[(\mathcal{K}^f)^{2k}(a_t^f)\big](\omega) \le \big[\underline{\lim}_\theta f(\varphi)\big](\omega) \le \big[\overline{\lim}_\theta f(\varphi)\big](\omega) \le \big[(\mathcal{K}^f)^{2k}(b_t^f)\big](\omega) \tag{4.1}$$

for all $t \geq \tau$, $\omega \in \Omega$ and $k \in \mathbb{N}$. Combining Lemma 3.2, Proposition A.1 and Proposition A.2, we can easily check that $a_t^f \in L^1(\Omega, \mathscr{F}_-, \mathbb{P}; \mathbb{R}^n)$ and $b_t^f \in L^1(\Omega, \mathscr{F}_-, \mathbb{P}; \mathbb{R}^n)$ for all $t \geq \tau$. This together with Lemma 3.6 and the Banach fixed point theorem shows that there is a unique random variable $r \in L^1(\Omega, \mathscr{F}_-, \mathbb{P}; \mathbb{R}^n)$ such that

$$\big[\mathcal{K}^f(r)\big](\omega) = r(\omega) \quad \mathbb{P}\text{-a.s.}$$

and

$$\lim_{k\to\infty} \mathbb{E}\left|(\mathcal{K}^f)^{2k}(a_t^f) - r\right| = \lim_{k\to\infty} \mathbb{E}\left|(\mathcal{K}^f)^{2k}(b_t^f) - r\right| = 0 \tag{4.2}$$

for all $t \geq \tau$. Therefore, for any $x_0 \in C_\tau$, we can choose a subsequence $\{k_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty}\big[(\mathcal{K}^f)^{2k_j}(a_t^f)\big](\omega) = r(\omega) = \lim_{j\to\infty}\big[(\mathcal{K}^f)^{2k_j}(b_t^f)\big](\omega) \quad \mathbb{P}\text{-a.s.} \tag{4.3}$$

for all $t \geq \tau$. From (4.1) and (4.3), we have that

$$\big[\underline{\lim}_\theta f(\varphi)\big](\omega) = \big[\overline{\lim}_\theta f(\varphi)\big](\omega) = r(\omega) \quad \mathbb{P}\text{-a.s.} \tag{4.4}$$

That is,

$$\lim_{t\to\infty} a_t^f(\omega) = \lim_{t\to\infty} b_t^f(\omega) = r(\omega) \quad \mathbb{P}\text{-a.s.},$$

which together with Lebesgue's dominated convergence theorem yields that

$$\lim_{t\to\infty}\big[\mathcal{K}(a_t^f)\big](\bullet, \omega) = \lim_{t\to\infty}\big[\mathcal{K}(b_t^f)\big](\bullet, \omega) = \big[\mathcal{K}(r)\big](\bullet, \omega) \quad \mathbb{P}\text{-a.s.} \tag{4.5}$$

in C_τ. Note that C_τ^+ is a normal cone, using (4.5) and Lemma 3.4, it follows easily that

$$\lim_{t\to\infty} \varphi(t, \theta_{-t}\omega, x_0)(\bullet) = \big[\mathcal{K}(r)\big](\bullet, \omega) \quad \mathbb{P}\text{-a.s.}$$

in C_τ. Finally, by the cocycle property, we get that $\big[\mathcal{K}(r)\big](\bullet, \omega)$ is an $\mathscr{F}_-$-measurable random equilibrium in C_τ. The proof is complete. □

5. Applications

In this section, we will give some examples to illustrate the effect of Theorem 2.1. For simplicity, we assume that $n = m = 3$.

Example 5.1. First, we consider the following delayed positive feedback system with additive white noise, including neural networks [23,44,45] and Othmer-Tyson systems [41,43] as special cases, which is given by

$$dx(t) = \Big[Ax(t) + f\big(x_1(t-\tau_1), x_2(t-\tau_2), x_3(t-\tau_3)\big)\Big]dt + \sigma dW_t, \tag{5.1}$$

with the initial value

$$x_0 = \xi \in C_\tau. \tag{5.2}$$

Here,

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 2 & -4 \end{bmatrix}, \tag{5.3}$$

$x = (x_1, x_2, x_3)^{\mathrm{T}} \in \mathbb{R}^3$, $\sigma = (\sigma_{ij})_{3\times 3}$ is a 3×3-dimensional matrix, $W_t = (W_t^1, W_t^2, W_t^3)^{\mathrm{T}}$ is a 3-dimensional Brownian motion. In addition, set τ_1, τ_2 and τ_3 are non-negative constants such that $\tau = \tau_1 \vee \tau_2 \vee \tau_3 \le \frac{1}{4}$ and $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a monotone function such that $L \le \frac{1}{3}$, where L is the global Lipschitz constant of f. Actually, f can be chosen such as the Arctan function, the Tanh function, the Rational function, the Linear function and so on. Direct computation shows that the eigenvalues of A are $\lambda_1 = -4$, $\lambda_{2,3} = \frac{-5\pm\sqrt{5}}{2}$ and the corresponding fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \frac{5+\sqrt{5}}{10} e^{\frac{-5+\sqrt{5}}{2}t} + \frac{5-\sqrt{5}}{10} e^{\frac{-5-\sqrt{5}}{2}t} & \frac{\sqrt{5}}{5} e^{\frac{-5+\sqrt{5}}{2}t} - \frac{\sqrt{5}}{5} e^{\frac{-5-\sqrt{5}}{2}t} & 0 \\ \frac{\sqrt{5}}{5} e^{\frac{-5+\sqrt{5}}{2}t} - \frac{\sqrt{5}}{5} e^{\frac{-5-\sqrt{5}}{2}t} & \frac{5-\sqrt{5}}{10} e^{\frac{-5+\sqrt{5}}{2}t} + \frac{5+\sqrt{5}}{10} e^{\frac{-5-\sqrt{5}}{2}t} & 0 \\ 2e^{-4t} + \frac{-5+3\sqrt{5}}{5} e^{\frac{-5+\sqrt{5}}{2}t} - \frac{5+3\sqrt{5}}{5} e^{\frac{-5-\sqrt{5}}{2}t} & -4e^{-4t} + \frac{10-4\sqrt{5}}{5} e^{\frac{-5+\sqrt{5}}{2}t} + \frac{10+4\sqrt{5}}{5} e^{\frac{-5-\sqrt{5}}{2}t} & e^{-4t} \end{bmatrix}.$$

In order to prove that

$$\|\Phi(t)\| \triangleq \left(\sum_{i=1}^{3} \sum_{j=1}^{3} \left|\Phi_{ij}(t)\right|^2 \right)^{\frac{1}{2}} \le C_A e^{\frac{-5+\sqrt{5}}{2}t} = \sqrt{7} e^{\lambda_2 t} \tag{5.4}$$

for all $t \ge 0$, we only need to verify that $\left|\Phi_{31}(t)\right| \le e^{\frac{-5+\sqrt{5}}{2}t}$ and $\left|\Phi_{32}(t)\right| \le e^{\frac{-5+\sqrt{5}}{2}t}$ for all $t \ge 0$. The proof of the remaining components of $\Phi(t)$ is trivial. Observe that the matrix A is cooperative, it follows that $\Phi_{ij}(t) \ge 0$ for all $i, j = 1, 2, 3$ and $t \ge 0$, see Proposition 3.1.1 in [42]. This yields that it is sufficient to check that

$$\Psi_{31}(t) \triangleq \frac{\Phi_{31}(t)}{e^{\frac{-5+\sqrt{5}}{2}t}} \le 1 \quad \text{and} \quad \Psi_{32}(t) \triangleq \frac{\Phi_{32}(t)}{e^{\frac{-5+\sqrt{5}}{2}t}} \le 1 \tag{5.5}$$

for all $t \ge 0$. By (5.5), we see at once that

$$\Psi_{31}(t) = 2e^{\frac{-3-\sqrt{5}}{2}t} + \frac{-5+3\sqrt{5}}{5} - \frac{5+3\sqrt{5}}{5} e^{-\sqrt{5}t}$$

and

$$\frac{\mathrm{d}\Psi_{31}(t)}{\mathrm{d}t}=-(3+\sqrt{5})e^{\frac{-3-\sqrt{5}}{2}t}+(3+\sqrt{5})e^{-\sqrt{5}t}\geq 0$$

for all $t\geq 0$. Since $\Psi_{31}(0)=0$ and $\lim_{t\to\infty}\Psi_{31}(t)=\frac{-5+3\sqrt{5}}{5}<1$, it is immediate that $\Psi_{31}(t)\leq 1$ for all $t\geq 0$. Similarly, we can get that

$$\Psi_{32}(t)=-4e^{\frac{-3-\sqrt{5}}{2}t}+\frac{10-4\sqrt{5}}{5}+\frac{10+4\sqrt{5}}{5}e^{-\sqrt{5}t}$$

and

$$\frac{\mathrm{d}\Psi_{32}(t)}{\mathrm{d}t}=2(3+\sqrt{5})e^{\frac{-3-\sqrt{5}}{2}t}-(4+2\sqrt{5})e^{-\sqrt{5}t}$$

for all $t\geq 0$. This together with $\Psi_{32}'(0)=2>0$ implies that there exists a unique local maximum point $t_0>0$ such that $\Psi_{32}(t_0)$ is the biggest value of $\Psi_{32}(t)$ for all $t\geq 0$. Furthermore, it is a simple matter to get that $\frac{2}{5}<t_0<\frac{3}{5}$, and then

$$\Psi_{32}(t_0)\leq -4e^{\frac{-3-\sqrt{5}}{2}\cdot\frac{3}{5}}+\frac{10-4\sqrt{5}}{5}+\frac{10+4\sqrt{5}}{5}e^{-\sqrt{5}\cdot\frac{2}{5}}\approx 0.928689<1,$$

which induces that (5.4) and (5.5) hold. Finally, let $L\leq\frac{1}{3}$, $\tau\leq\frac{1}{4}$, $C_A=\sqrt{7}$ and $\lambda=\frac{5-\sqrt{5}}{2}$, we have that

$$\frac{C_A L e^{\lambda\tau}}{\lambda}\leq\frac{2\sqrt{7}e^{\frac{5-\sqrt{5}}{8}}}{3(5-\sqrt{5})}\approx 0.901520<1.$$

Consequently, using Theorem 2.1, there is a unique globally stable stationary solution (random equilibrium) for (5.1) and (5.2), which attracts all the pullback trajectories in the space C_τ.

Example 5.2. Secondly, we study the following delayed negative feedback system with additive white noise, including the Goodwin system [20,22] as a special case, which is modeled by

$$\begin{cases} dx_1(t)=\left[-2x_1(t)+f\left(x_3(t-\tau)\right)\right]dt+\sigma_1 dW_t^1,\\ dx_2(t)=\left[x_1(t)-3x_2(t)\right]dt+\sigma_2 dW_t^2,\\ dx_3(t)=\left[x_2(t)-4x_3(t)\right]dt+\sigma_3 dW_t^3,\end{cases}\tag{5.6}$$

with the initial value

$$x_0=\xi\in C_\tau,\tag{5.7}$$

where the constants σ_1, σ_2 and σ_3 represent the noise strength, the delay $0\leq\tau\leq\frac{1}{6}$, $W_t=(W_t^1,W_t^2,W_t^3)^{\mathrm{T}}$ is a 3-dimensional Brownian motion and $f:\mathbb{R}\to\mathbb{R}$ is a decreasing function such that $L\leq\frac{1}{2}$, where L is the global Lipschitz constant of f. The form of f can be determined similarly as that in Example 5.1. By (5.6), we can easily obtain that the eigenvalues of A are $\lambda_1=-2$, $\lambda_2=-3$ and $\lambda_3=-4$. Moreover, the fundamental matrix is

$$\Phi(t)=\begin{bmatrix} e^{-2t} & 0 & 0 \\ e^{-2t}-e^{-3t} & e^{-3t} & 0 \\ \frac{e^{-2t}}{2}-e^{-3t}+\frac{e^{-4t}}{2} & e^{-3t}-e^{-4t} & e^{-4t} \end{bmatrix},$$

which implies that

$$\|\Phi(t)\| \triangleq \left(\sum_{i=1}^{3}\sum_{j=1}^{3}\left|\Phi_{ij}(t)\right|^2\right)^{\frac{1}{2}} \le C_A e^{-2t} = \sqrt{6}e^{\lambda_1 t} \tag{5.8}$$

for all $t \ge 0$. Set $L \le \frac{1}{2}$, $\tau \le \frac{1}{6}$, $C_A = \sqrt{6}$ and $\lambda = 2$, it follows that

$$\frac{C_A L e^{\lambda\tau}}{\lambda} \le \frac{\sqrt{6}e^{\frac{1}{3}}}{4} \approx 0.854635 < 1.$$

Applying Theorem 2.1, it is immediate that (5.6) and (5.7) possesses a unique globally attracting stationary solution (random equilibrium) in the space C_τ.

Example 5.3. Finally, we discuss the following 3-dimensional stochastic system with distributed delay and additive white noise, which is given by

$$dx(t) = \left[Ax(t) + h\left(\int_{-\tau}^{0} G(s)x(t-s)ds\right)\right]dt + \sigma dW_t, \tag{5.9}$$

with the initial value

$$x_0 = \xi \in C_\tau. \tag{5.10}$$

Here, A, σ and W_t are the same as that in Example 5.1, $G(s)$ is a continuous matrix defined for $s \in [-\tau, 0]$ such that $G_{ij}(s) \ge 0$ (or $G_{ij}(s) \le 0$) for all $s \in [-\tau, 0]$ and $i, j = 1, 2, 3$. In addition, $h : \mathbb{R}^3 \to \mathbb{R}^3$ is a monotone (or anti-monotone) function such that $L_h \le \frac{1}{2}$, where L_h is the global Lipschitz constant of h. Then, we can have that

$$L \le L_h \int_{-\tau}^{0} \|G(s)\| ds. \tag{5.11}$$

Choose $L_h \le \frac{1}{2}$, $\sup_{-\tau\le s\le 0}\|G(s)\| \le 1$, $\tau \le \frac{1}{2}$, $C_A = \sqrt{7}$ and $\lambda = \frac{5-\sqrt{5}}{2}$, it is evident that

$$\frac{C_A L e^{\lambda\tau}}{\lambda} \le \frac{\sqrt{7}L_h \tau e^{\lambda\tau}}{\lambda} \le \frac{\sqrt{7}e^{\frac{5-\sqrt{5}}{4}}}{2(5-\sqrt{5})} \approx 0.955172 < 1.$$

Thus, by Theorem 2.1, the stochastic delay system (5.9) and (5.10) admits a unique globally stable stationary solution (random equilibrium) in the space C_τ.

Data availability

No data was used for the research described in the article.

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Appendix A. Compactness of the pullback orbits and the integrability for their bounds

Proposition A.1. *Assume that* (H1) *and* (H2) *hold. For any $x_0 \in C_\tau$, there exists a tempered random variable $R(\omega) = R(\omega, x_0)$ such that*

$$\sup_{t\geq 0}\left\|\varphi(t,\theta_{-t}\omega,x_0)\right\|_{C_\tau} \leq R(\omega), \tag{A.1}$$

where $R(\omega) = \widetilde{R}(\omega)\bigvee \widehat{R}(\omega)$, $\widehat{R}(\omega) = \sup_{t\in[0,\tau]\bigcap\mathbb{Q}} \widetilde{R}(\theta_{\tau-t}\omega) + \|x_0\|_{C_\tau}$ and

$$\widetilde{R}(\omega) = C_A e^{\lambda\tau}\left|x_0(0)\right| + C_A e^{\lambda\tau}\sup_{t\geq 0}\left\{e^{-(\lambda - C_A L e^{\lambda\tau})t}\left|z(\theta_{-t}\omega)\right|\right\}$$
$$+ C_A e^{\lambda\tau}\int_{-\infty}^{0} e^{(\lambda - C_A L e^{\lambda\tau})u}\Big(L\left\|z_0(\theta_u\omega)\right\|_{C_\tau} + \left|f(0)\right|\Big)du + \left\|z_0(\omega)\right\|_{C_\tau}$$

for all $\omega \in \Omega$. Furthermore, $\gamma_{x_0}^\tau(\omega)$ is relatively compact in C_τ for all $\omega \in \Omega$ and $x_0 \in C_\tau$, where $\gamma_{x_0}^u(\omega) := \{\varphi(t,\theta_{-t}\omega,x_0) \mid t \geq u\}$ for $u \geq 0$.

Proof. We shall first show the boundedness of the pullback trajectories and then prove their compactness.

The first issue is to show that there exists a tempered random variable $\widetilde{R}(\omega) = \widetilde{R}(\omega, x_0)$ such that

$$\sup_{t\geq\tau}\left\|\varphi(t,\theta_{-t}\omega,x_0)\right\|_{C_\tau} \leq \widetilde{R}(\omega) \tag{A.2}$$

for all $\omega \in \Omega$. Observe that $\varphi(t,\theta_{-t}\omega,x_0) = y_t\big(\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big) + z_0(\omega)$. In order to prove (A.2), we only need to show that for any $x_0 \in C_\tau$, there exists a tempered random variable $R_1(\omega) = R_1(\omega, x_0)$ such that

$$\sup_{t\geq\tau}\left\|y_t\big(\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big)\right\|_{C_\tau} \leq R_1(\omega) \tag{A.3}$$

for all $\omega \in \Omega$. Using the variation of constants formula [31, Theorem 3.1], (2.9) and (2.10), it follows that for all $-\tau \leq s \leq 0$, $t \geq \tau$ and $\omega \in \Omega$, we have

$$\begin{aligned} & y\big(t+s,\omega,y_0(\omega)\big) \\ & = \Phi(t+s)\big[y_0(\omega)\big](0) + \int_0^{t+s} \Phi(t+s-u) f\Big(y_u\big(\omega,y_0(\omega)\big) + z_0(\theta_u\omega)\Big)du. \end{aligned} \tag{A.4}$$

Combining (A.4), (H1) and (H2), it is clear that

$$\begin{aligned} & \sup_{-\tau\le s\le 0}\Big|y\big(t+s,\omega,y_0(\omega)\big)\Big| \\ & \le C_A e^{\lambda\tau}e^{-\lambda t}\Big|\big[y_0(\omega)\big](0)\Big| + C_A L e^{\lambda\tau}e^{-\lambda t}\int_0^t e^{\lambda u}\big(\|y_u\|_{C_\tau} + \|z_0(\theta_u\omega)\|_{C_\tau}\big)du \\ & \quad + C_A e^{\lambda\tau}e^{-\lambda t}\int_0^t e^{\lambda u}\big|f(0)\big|du \\ & = C_A e^{\lambda\tau}e^{-\lambda t}\Big|\big[y_0(\omega)\big](0)\Big| + C_A e^{\lambda\tau}e^{-\lambda t}\int_0^t e^{\lambda u}\Big(L\|z_0(\theta_u\omega)\|_{C_\tau} + \big|f(0)\big|\Big)du \\ & \quad + C_A L e^{\lambda\tau}e^{-\lambda t}\int_0^t e^{\lambda u}\|y_u\|_{C_\tau}du. \end{aligned} \tag{A.5}$$

That is,

$$\begin{aligned} & e^{\lambda t}\Big\|y_t\big(\omega,y_0(\omega)\big)\Big\|_{C_\tau} \\ & \le C_A e^{\lambda\tau}\Big|\big[y_0(\omega)\big](0)\Big| + C_A e^{\lambda\tau}\int_0^t e^{\lambda u}\Big(L\|z_0(\theta_u\omega)\|_{C_\tau} + \big|f(0)\big|\Big)du \\ & \quad + C_A L e^{\lambda\tau}\int_0^t e^{\lambda u}\|y_u\|_{C_\tau}du \end{aligned} \tag{A.6}$$

for all $t \ge \tau$ and $\omega \in \Omega$. Therefore, by the Gronwall inequality, we can easily see that

$$\begin{aligned} e^{\lambda t}\Big\|y_t\big(\omega,y_0(\omega)\big)\Big\|_{C_\tau} \le\ & C_A e^{\lambda\tau}\Big|\big[y_0(\omega)\big](0)\Big| e^{C_A L e^{\lambda\tau} t} \\ & + C_A e^{\lambda\tau}\int_0^t e^{\lambda u} e^{C_A L e^{\lambda\tau}(t-u)}\Big(L\|z_0(\theta_u\omega)\|_{C_\tau} + \big|f(0)\big|\Big)du, \end{aligned} \tag{A.7}$$

and then

$$\begin{aligned}\left\|y_t\big(\omega, y_0(\omega)\big)\right\|_{C_\tau} \le C_A e^{\lambda\tau}\left|\big[y_0(\omega)\big](0)\right| e^{-(\lambda - C_A L e^{\lambda\tau})t} \\ + C_A e^{\lambda\tau}\int_0^t e^{-(\lambda - C_A L e^{\lambda\tau})(t-u)}\Big(L\|z_0(\theta_u\omega)\|_{C_\tau} + \big|f(0)\big|\Big)du. \end{aligned} \tag{A.8}$$

Note that for all $t \ge 0$ and $\omega \in \Omega$, $\big[y_0(\theta_{-t}\omega)\big](0) = x_0(0) - z(-t,\omega) = x_0(0) - z(\theta_{-t}\omega)$. Since the random variable $z(\omega)$ is tempered with respect to θ and $\frac{C_A L e^{\lambda\tau}}{\lambda} < 1$, it follows that

$$\begin{aligned}&\left\|y_t\big(\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big)\right\|_{C_\tau} \\ &\le C_A e^{\lambda\tau}\left|\big[y_0(\theta_{-t}\omega)\big](0)\right| e^{-(\lambda - C_A L e^{\lambda\tau})t} \\ &\quad + C_A e^{\lambda\tau}\int_0^t e^{-(\lambda - C_A L e^{\lambda\tau})(t-u)}\Big(L\big\|z_0(\theta_{u-t}\omega)\big\|_{C_\tau} + \big|f(0)\big|\Big)du \\ &\le C_A e^{\lambda\tau}\Big(\big|x_0(0)\big| + \big|z(\theta_{-t}\omega)\big|\Big)e^{-(\lambda - C_A L e^{\lambda\tau})t} \\ &\quad + C_A e^{\lambda\tau}\int_0^t e^{-(\lambda - C_A L e^{\lambda\tau})(t-u)}\Big(L\big\|z_0(\theta_{u-t}\omega)\big\|_{C_\tau} + \big|f(0)\big|\Big)du \\ &\le C_A e^{\lambda\tau}\big|x_0(0)\big| + C_A e^{\lambda\tau}\sup_{t\ge 0}\Big\{e^{-(\lambda - C_A L e^{\lambda\tau})t}\big|z(\theta_{-t}\omega)\big|\Big\} \\ &\quad + C_A e^{\lambda\tau}\int_{-t}^0 e^{(\lambda - C_A L e^{\lambda\tau})u}\Big(L\big\|z_0(\theta_u\omega)\big\|_{C_\tau} + \big|f(0)\big|\Big)du\end{aligned}$$

for all $t \ge 0$ and $\omega \in \Omega$. Define

$$\begin{aligned}R_1(\omega) = C_A e^{\lambda\tau}\big|x_0(0)\big| + C_A e^{\lambda\tau}\sup_{t\ge 0}\Big\{e^{-(\lambda - C_A L e^{\lambda\tau})t}\big|z(\theta_{-t}\omega)\big|\Big\} \\ + C_A e^{\lambda\tau}\int_{-\infty}^0 e^{(\lambda - C_A L e^{\lambda\tau})u}\Big(L\big\|z_0(\theta_u\omega)\big\|_{C_\tau} + \big|f(0)\big|\Big)du,\ \omega \in \Omega,\end{aligned} \tag{A.9}$$

it is immediate that (A.3) holds. Here,

$$\begin{aligned}&\int_{-\infty}^0 e^{(\lambda - C_A L e^{\lambda\tau})u}\big\|z_0(\theta_u\omega)\big\|_{C_\tau}du \\ &\triangleq \int_{-\infty}^0 e^{\eta u}\big\|z_0(\theta_u\omega)\big\|_{C_\tau}du \qquad (\eta = \lambda - C_A L e^{\lambda\tau} > 0)\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{0} e^{\eta u} \sup_{-\tau \le s \le 0} \left| z(\theta_{u+s}\omega) \right| du \\
&\le \int_{-\infty}^{0} e^{\eta u} \sup_{-\tau \le s \le 0} \left\{ e^{-\frac{\eta}{2}(u+s)} \right\} \sup_{-\tau \le s \le 0} \left\{ e^{\frac{\eta}{2}(u+s)} \left| z(\theta_{u+s}\omega) \right| \right\} du \\
&\le e^{\frac{\eta}{2}\tau} \sup_{t \le 0} \left\{ e^{\frac{\eta}{2}t} \left| z(\theta_t \omega) \right| \right\} \int_{-\infty}^{0} e^{\frac{\eta}{2}u} du \\
&< \infty,
\end{aligned} \tag{A.10}
$$

which implies that $R_1(\omega)$ is well defined for all $\omega \in \Omega$.

Next, we will consider the temperedness of R_1. For this purpose, by (A.9), we only need to prove that $\sup_{t\ge 0}\left\{e^{-\eta t}\left|z(\theta_{-t}\omega)\right|\right\}$ and $\int_{-\infty}^{0} e^{\eta u}\left\|z_0(\theta_u\omega)\right\|_{C_\tau} du$ are both tempered. For any $\delta > 0$, we see that

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}} \left\{ e^{-\delta|t|} \sup_{u \ge 0} \left\{ e^{-\eta u} \left| z(\theta_{-u} \circ \theta_t \omega) \right| \right\} \right\} \\
&\le \sup_{t\in\mathbb{R}} \left\{ e^{-(\delta\wedge\eta)|t|} \sup_{u \ge 0} \left\{ e^{-(\delta\wedge\eta)u} \left| z(\theta_{-u+t}\omega) \right| \right\} \right\} \\
&\le \sup_{t\in\mathbb{R}} \left\{ e^{-(\delta\wedge\eta)|t|} \sup_{u \ge 0} \left\{ e^{-(\delta\wedge\eta)|-u+t|+(\delta\wedge\eta)|t|} \left| z(\theta_{-u+t}\omega) \right| \right\} \right\} \\
&\le \sup_{t\in\mathbb{R}} \left\{ e^{-(\delta\wedge\eta)|t|} \left| z(\theta_t\omega) \right| \right\} \\
&< \infty,
\end{aligned} \tag{A.11}
$$

which is due to the fact that $z(\omega)$ is tempered, where $\delta \wedge \eta = \min\{\delta, \eta\} > 0$. Moreover, in order to examine that $\int_{-\infty}^{0} e^{\eta u}\left\|z_0(\theta_u\omega)\right\|_{C_\tau} du$ is tempered, by (A.10), it is sufficient to show that $\sup_{t\le 0}\left\{e^{\frac{\eta}{2}t}\left|z(\theta_t\omega)\right|\right\}$ is tempered, which can be done by the same method in (A.11).

To show (A.2), it remains to prove that $\left\|z_0(\omega)\right\|_{C_\tau}$ is tempered. For any $\delta > 0$, it is obvious that

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}} \left\{ e^{-\delta|t|} \left\| z_0(\theta_t\omega) \right\|_{C_\tau} \right\} \\
&= \sup_{t\in\mathbb{R}} \left\{ e^{-\delta|t|} \sup_{-\tau \le s \le 0} \left| z(\theta_{s+t}\omega) \right| \right\} \\
&\le \sup_{t\in\mathbb{R}} \left\{ e^{-\delta|t|} \sup_{-\tau \le s \le 0} \left\{ e^{\delta|s+t|} \right\} \sup_{-\tau \le s \le 0} \left\{ e^{-\delta|s+t|} \left| z(\theta_{s+t}\omega) \right| \right\} \right\}
\end{aligned}
$$

$$\leq e^{\delta\tau} \sup_{t\in\mathbb{R}} \left\{ e^{-\delta|t|} \left| z(\theta_t \omega) \right| \right\}$$
$$< \infty.$$

Set $\widetilde{R}(\omega) = R_1(\omega) + \left\| z_0(\omega) \right\|_{C_\tau}$ for all $\omega \in \Omega$, which leads to (A.2).

Secondly, we need to find a tempered random variable $\widehat{R}(\omega) = \widehat{R}(\omega, x_0)$ such that

$$\sup_{0\leq t\leq\tau} \left\| \varphi(t, \theta_{-t}\omega, x_0) \right\|_{C_\tau} \leq \widehat{R}(\omega). \tag{A.12}$$

For any $0 \leq t \leq \tau$ and $\omega \in \Omega$, it is clear that

$$\begin{aligned}
&\left\| \varphi(t, \theta_{-t}\omega, x_0) \right\|_{C_\tau} \\
&= \sup_{-\tau\leq s\leq 0} \left| x(t+s, \theta_{-t}\omega, x_0) \right| \\
&\leq \sup_{-t\leq s\leq 0} \left| x(t+s, \theta_{-t}\omega, x_0) \right| + \sup_{-\tau\leq s\leq -t} \left| x(t+s, \theta_{-t}\omega, x_0) \right| \\
&\leq \left\| \varphi(\tau, \theta_{-t}\omega, x_0) \right\|_{C_\tau} + \sup_{-\tau\leq s\leq -t} \left| x(t+s, \theta_{-t}\omega, x_0) \right| \\
&\leq \left\| \varphi(\tau, \theta_{-t}\omega, x_0) \right\|_{C_\tau} + \|x_0\|_{C_\tau}.
\end{aligned} \tag{A.13}$$

Furthermore, by (2.7) and (2.11), we have that

$$\varphi(\tau, \theta_{-t}\omega, x_0) = y_\tau\left(\theta_{-t}\omega, y_0(\theta_{-t}\omega)\right) + z_0(\theta_{\tau-t}\omega),$$

which together with Lemma 3.2 gives that $\varphi(\tau, \theta_{-t}\omega, x_0)$ is continuous with respect to $t \in [0, \tau]$ for all $\omega \in \Omega$. Consequently, from (A.13) and (A.2), it is easy to see that

$$\begin{aligned}
&\sup_{0\leq t\leq\tau} \left\| \varphi(t, \theta_{-t}\omega, x_0) \right\|_{C_\tau} \\
&\leq \sup_{0\leq t\leq\tau} \left\| \varphi(\tau, \theta_{-t}\omega, x_0) \right\|_{C_\tau} + \|x_0\|_{C_\tau} \\
&= \sup_{t\in[0,\tau]\bigcap\mathbb{Q}} \left\| \varphi(\tau, \theta_{-t}\omega, x_0) \right\|_{C_\tau} + \|x_0\|_{C_\tau} \\
&= \sup_{t\in[0,\tau]\bigcap\mathbb{Q}} \left\| \varphi(\tau, \theta_{-\tau} \circ \theta_{\tau-t}\omega, x_0) \right\|_{C_\tau} + \|x_0\|_{C_\tau} \\
&\leq \sup_{t\in[0,\tau]\bigcap\mathbb{Q}} \widetilde{R}(\theta_{\tau-t}\omega) + \|x_0\|_{C_\tau} \\
&\triangleq \widehat{R}(\omega).
\end{aligned}$$

Here, $\sup_{t\in[0,\tau]\bigcap\mathbb{Q}} \widetilde{R}(\theta_{\tau-t}\omega)$ guarantees the measurability of $\widehat{R}$. In fact, we can also conclude that $\widehat{R}(\omega)$ is tempered. For any $\delta > 0$, it is evident that

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}}\left\{e^{-\delta|t|}\widehat{R}(\theta_t\omega)\right\}\\
&=\sup_{t\in\mathbb{R}}\left\{e^{-\delta|t|}\left(\sup_{u\in[0,\tau]\bigcap\mathbb{Q}}\widetilde{R}(\theta_{\tau-u+t}\omega)+\|x_0\|_{C_\tau}\right)\right\}\\
&\le\sup_{t\in\mathbb{R}}\left\{e^{-\delta|t|}\left(\sup_{u\in[0,\tau]\bigcap\mathbb{Q}}e^{\delta|\tau-u+t|}\sup_{u\in[0,\tau]\bigcap\mathbb{Q}}\left\{e^{-\delta|\tau-u+t|}\widetilde{R}(\theta_{\tau-u+t}\omega)\right\}\right)\right\}+\|x_0\|_{C_\tau}\\
&\le\sup_{u\in[0,\tau]\bigcap\mathbb{Q}}e^{\delta|\tau-u|}\sup_{t\in\mathbb{R}}\left\{e^{-\delta|t|}\widetilde{R}(\theta_t\omega)\right\}+\|x_0\|_{C_\tau}\\
&\le e^{\delta\tau}\sup_{t\in\mathbb{R}}\left\{e^{-\delta|t|}\widetilde{R}(\theta_t\omega)\right\}+\|x_0\|_{C_\tau}\\
&<\infty.
\end{aligned}
$$

Therefore, let $R(\omega)=\max\left\{\widetilde{R}(\omega),\widehat{R}(\omega)\right\}=\widetilde{R}(\omega)\bigvee\widehat{R}(\omega)$ for all $\omega\in\Omega$, it follows that (A.1) holds.

Finally, we will verify the relative compactness of $\gamma_{x_0}^{\tau}(\omega)$ in C_τ for all $\omega\in\Omega$ and $x_0\in C_\tau$. From (2.11) and the cocycle property, it follows that

$$
\gamma_{x_0}^{\tau}(\omega)=\psi\Big(\tau,\theta_{-\tau}\omega,\gamma_{x_0}^{0}(\theta_{-\tau}\omega)-z_0(\theta_{-\tau}\omega)\Big)+z_0(\omega). \tag{A.14}
$$

Thus, by the boundedness of $\gamma_{x_0}^{0}(\theta_{-\tau}\omega)$, it suffices to prove the mapping

$$
\psi\,(\tau,\theta_{-\tau}\omega,\bullet):C_\tau\to C_\tau
$$

is compact. Based on the Arzela-Ascoli theorem, we only need to check that for any bounded subset $B\subset C_\tau$ and $\omega\in\Omega$,

(i) $\bigcup_{t=0}^{\tau}\psi\,(t,\omega,B)$ is bounded; and

(ii) all functions in $\psi\,(\tau,\omega,B)$ are equicontinuous.

Both can be obtained by the global Lipschitz condition and standard priori estimations, which are omitted here. The proof is complete. □

Proposition A.2. *Let $\widetilde{R}$ be defined in Proposition* A.1*, i.e.,*

$$
\begin{aligned}
\widetilde{R}(\omega)=&\,C_Ae^{\lambda\tau}\left|x_0(0)\right|+C_Ae^{\lambda\tau}\sup_{t\ge0}\left\{e^{-(\lambda-C_ALe^{\lambda\tau})t}\left|z(\theta_{-t}\omega)\right|\right\}\\
&+C_Ae^{\lambda\tau}\int_{-\infty}^{0}e^{(\lambda-C_ALe^{\lambda\tau})u}\Big(L\left\|z_0(\theta_u\omega)\right\|_{C_\tau}+\left|f(0)\right|\Big)du+\left\|z_0(\omega)\right\|_{C_\tau}.
\end{aligned}
$$

Then $\widetilde{R}\in L^1(\Omega,\mathscr{F}_-,\mathbb{P};\mathbb{R}^n)$.

Proof. Since

$$z(t,\omega) \equiv z(\theta_t\omega) = \int_{-\infty}^{t} \Phi(t-u)\sigma dW_u,$$

which shows that $z(\bullet,\omega) : \mathbb{R} \mapsto \mathbb{R}^n$ is continuous for all $\omega \in \Omega$ and $z(t,\bullet) : \Omega \mapsto \mathbb{R}^n$ is $\mathscr{F}_-$-measurable for all $t \le 0$. From this and Lemma II.2.1 in [34], we have that $z_0(\theta_\bullet\omega) : \mathbb{R} \mapsto C_\tau$ is continuous for all $\omega \in \Omega$ and $z_0(\theta_t\bullet) : \Omega \mapsto C_\tau$ is $\mathscr{F}_-$-measurable for all $t \le 0$. Using Lemma 3.14 in [10], it is obvious that $z_0(\theta_t\omega) : \mathbb{R}_- \times \Omega \mapsto C_\tau$ is $\mathscr{B}(\mathbb{R}_-) \otimes \mathscr{F}_-$-measurable, which together with Fubini's theorem yields that $\widetilde{R}$ is $\mathscr{F}_-$-measurable.

Next, we will prove that $\mathbb{E}\left|\widetilde{R}\right| < \infty$. In fact, for any $\delta > 0$, we can assert that

$$\mathbb{E}\left(\sup_{t\le 0} e^{\delta t}|W_t|\right) < \infty.$$

Note that

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\le 0} e^{\delta t}|W_t|\right) &\le \sum_{i=1}^{m} \mathbb{E}\left(\sup_{t\le 0} e^{\delta t}|W_t^i|\right) \\
&= \sum_{i=1}^{m} \mathbb{E}\left(\sup_{n\in\mathbb{N}} \sup_{-n-1\le t\le -n} e^{\delta t}|W_t^i|\right) \\
&\le \sum_{i=1}^{m}\sum_{n=0}^{\infty} \mathbb{E}\left(\sup_{-n-1\le t\le -n} e^{\delta t}|W_t^i|\right) \\
&\le \sum_{i=1}^{m}\sum_{n=0}^{\infty} e^{-\delta n}\left(\mathbb{E}\left(\sup_{-n-1\le t\le -n} |W_t^i|^2\right)\right)^{\frac{1}{2}} \\
&\le \sum_{i=1}^{m}\sum_{n=0}^{\infty} e^{-\delta n}\left(4\mathbb{E}\left(|W_{-n-1}^i|^2\right)\right)^{\frac{1}{2}} \\
&= 2m\sum_{n=0}^{\infty} e^{-\delta n}\sqrt{n+1} \\
&< \infty,
\end{aligned}$$

where we use Hölder inequality and Doob's martingale inequality. In what follows, define $C_\delta = \mathbb{E}\left(\sup_{t\le 0} e^{\delta t}|W_t|\right)$ for any $\delta > 0$. By (2.8), it is immediate that

$$z(t,\omega) = \int_{-\infty}^{0} A\Phi(-u)\sigma W_{u+t}(\omega)du + \sigma W_t(\omega). \tag{A.15}$$

Write $\eta = \lambda - C_A L e^{\lambda\tau} > 0$, we have

$$
\begin{aligned}
&\sup_{t\geq 0}\Big\{e^{-\eta t}\big|z(\theta_{-t}\omega)\big|\Big\}\\
&\leq \|A\|\cdot\|\sigma\|\sup_{t\geq 0}\left\{e^{-\eta t}\int_{-\infty}^{0}\big\|\Phi(-u)\big\|\cdot\big|W_{u-t}(\omega)\big|du\right\}+\|\sigma\|\sup_{t\geq 0}\Big\{e^{-\eta t}\big|W_{-t}(\omega)\big|\Big\}\\
&\leq C_A\|A\|\cdot\|\sigma\|\sup_{t\geq 0}\left\{e^{-\eta t}\int_{-\infty}^{0}e^{\lambda u}\big|W_{u-t}(\omega)\big|du\right\}+\|\sigma\|\sup_{t\geq 0}\Big\{e^{-\eta t}\big|W_{-t}(\omega)\big|\Big\}\\
&\leq C_A\|A\|\cdot\|\sigma\|\sup_{t\geq 0}\left\{e^{-\frac{\eta\wedge\lambda}{2}t}\big|W_{-t}(\omega)\big|\right\}\int_{-\infty}^{0}e^{\frac{\lambda}{2}u}du+\|\sigma\|\sup_{t\geq 0}\Big\{e^{-\eta t}\big|W_{-t}(\omega)\big|\Big\}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{t\geq 0}\Big\{e^{-\eta t}\big|z(\theta_{-t}\omega)\big|\Big\}\right)\\
&\leq \frac{2}{\lambda}C_A\|A\|\cdot\|\sigma\|\mathbb{E}\left(\sup_{t\geq 0}\left\{e^{-\frac{\eta\wedge\lambda}{2}t}\big|W_{-t}(\omega)\big|\right\}\right)+\|\sigma\|\mathbb{E}\left(\sup_{t\geq 0}\Big\{e^{-\eta t}\big|W_{-t}(\omega)\big|\Big\}\right)\\
&\leq \frac{2}{\lambda}C_A C_{\frac{\eta\wedge\lambda}{2}}\|A\|\cdot\|\sigma\|+C_\eta\|\sigma\|\\
&<\infty.
\end{aligned}
\tag{A.16}
$$

In order to check $\widetilde{R}\in L^1(\Omega,\mathscr{F}_-,\mathbb{P};\mathbb{R}^n)$, it remains to show that

$$
\mathbb{E}\left(\int_{-\infty}^{0}e^{\eta u}\big\|z_0(\theta_u\omega)\big\|_{C_\tau}du\right)<\infty
$$

and

$$
\mathbb{E}\big\|z_0(\omega)\big\|_{C_\tau}<\infty.
$$

By definition of $\|\cdot\|_{C_\tau}$ and (A.16), we have

$$
\begin{aligned}
&\mathbb{E}\left(\int_{-\infty}^{0}e^{\eta u}\big\|z_0(\theta_u\omega)\big\|_{C_\tau}du\right)\\
&=\mathbb{E}\left(\int_{-\infty}^{0}e^{\eta u}\sup_{-\tau\leq s\leq 0}\big|z(u+s,\omega)\big|du\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}\left(\int_{-\infty}^{0} e^{\frac{\eta}{2}u} \sup_{-\tau\le s\le 0} e^{\frac{\eta}{2}(u+s)} e^{-\frac{\eta}{2}s}\left|z(\theta_{u+s}\omega)\right| du\right) \\
&\le \frac{2}{\eta} e^{\frac{\eta}{2}\tau}\mathbb{E}\left(\sup_{t\ge 0}\left\{e^{-\frac{\eta}{2}t}\left|z(\theta_{-t}\omega)\right|\right\}\right) \\
&< \infty
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\left\|z_0(\omega)\right\|_{C_\tau} \\
&= \mathbb{E}\left(\sup_{-\tau\le s\le 0}\left|z(s,\omega)\right|\right) \\
&\le e^{\eta\tau}\mathbb{E}\left(\sup_{-\tau\le s\le 0}\left\{e^{\eta s}\left|z(\theta_s\omega)\right|\right\}\right) \\
&\le e^{\eta\tau}\mathbb{E}\left(\sup_{t\ge 0}\left\{e^{-\eta t}\left|z(\theta_{-t}\omega)\right|\right\}\right) \\
&< \infty.
\end{aligned}
$$

The proof is complete. □

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