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GLOBAL STABILITY OF FEEDBACK SYSTEMS WITH MULTIPLICATIVE NOISE ON THE NONNEGATIVE ORTHANT*

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Abstract. We investigate the dynamical behavior of pull-back trajectories for feedback systems with multiplicative noise and prove that there exists a globally stable positive random equilibrium in the nonnegative orthant $\mathbb{R}^d_+$, where the global stability means that all pull-back trajectories originating from nonnegative orthant converge to this positive random equilibrium almost surely. The output functions (feedback functions) are assumed to either possess bounded derivatives or be uniformly bounded away from zero. In the first case, we first prove the joint measurability of the metric dynamical system θ with respect to the σ-algebra $\mathscr{B}(\mathbb{R}_-) \otimes \mathscr{F}_-$, where $\mathscr{F}_- = \sigma\{\omega \mapsto W_t(\omega) : t \leq 0\}$ is the past σ-algebra and $W_t(\omega)$ is an $\mathbb{R}^d$-valued two-sided Wiener process, and then combine the $\mathcal{L}^1$-integrability of the tempered random variable coming from the definition of the top Lyapunov exponent and the independence between the past σ-algebra and the future σ-algebra $\mathscr{F}_+ = \sigma\{\omega \mapsto W_t(\omega) : t \geq 0\}$ to obtain a globally stable random equilibrium by constructing the contraction mapping on an $\mathscr{F}_-$-measurable, $\mathcal{L}^1$-integrable, and complete metric input space; in the second case, the sublinearity of output functions (feedback functions) and the part metric play the main roles in the existence and uniqueness of globally attracting positive fixed point in the part of a normal, solid cone. Our results can be applied to a well-known stochastic Goodwin negative feedback system, Othmer–Tyson positive feedback system, and Griffith positive feedback system as well as other stochastic cooperative, competitive, and predator-prey systems.

Key words. stochastic feedback systems, random dynamical systems, random equilibrium, global stability

AMS subject classifications. 93E03, 93E15, 60H10, 37H10

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1. Introduction.

Feedback (positive and negative) loops play distinctively diverse roles in many biochemical control systems, which often occur in the study of the reaction process in cellular signalling, such as [4, 5, 29, 30]. It is well known that positive feedback can promote multistability, which is a necessary dynamic feature of control systems having multiple stable steady states; see [37]. Negative feedback control systems frequently describe biological systems whose synthesis rates cannot keep growing and which have a level of saturation. For instance, the operon in the lactose system [15, 16, 29] or in the bacterium *E. coli* [30] is under negative control (e.g., Goodwin system), i.e., some specific proteins are devoted to inhibiting the translation of the DNA to RNA. Stability analysis of nonlinear deterministic feedback systems has been extensively and intensively studied in [15, 18, 19, 20, 27, 36, 38]. However, real world systems are rife with stochastic fluctuation. At the cellular level, it has been recognized that biochemical reactions inside a cell are discrete and exhibit inherent

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randomness. For example, noise may perform as a trigger for phenotypic variability and different types of dynamics; see [11, 26]. Rather than considering deterministic ordinary differential equations, noise-perturbed systems for continuous-time are usually described by stochastic differential equations. For practical purposes, one of the main goals in the analysis of stochastic feedback systems is to consider the long-term behavior of solutions and establish various types of stochastic stability, including exponential stability in mean square, globally asymptotical stability in probability, and almost sure stability; see [9, 10, 24, 25, 28, 39, 40, 42], where the main method used is to construct Lyapunov functions.

This paper will make use of the theory of random dynamical systems (RDS) established by Ludwig Arnold and others, [1, 6]. If the stochastic system admits the stochastic comparison principle, i.e., the system is cooperative or monotone, the powerful theory of monotone random dynamical systems can be applied to investigate the global stability of stochastic flows; see [1, 6]. However, if the stochastic flow is not monotone, the global stability analysis still is a challenging work. Recently, the theory of random dynamical systems has been developed to investigate the stability of feedback systems involving *real noise* perturbation by Marcondes de Freitas and Sontag [12, 13, 14], providing new insights to consider nonmonotone systems. Specifically, they proved the existence of a globally attractive random equilibrium for random systems with inputs and outputs by iterating the gain operator $\mathcal{K}^h(u)$; this random equilibrium is usually nontrivial and only qualitatively exists. However, their methods and results cannot directly be applied to investigate stochastic feedback systems driven by Brownian motion. The biggest difficulty is the perfection of the crude cocycle $\varphi(t, \omega, x, u)$ for nonautonomous stochastic systems with input u, as analyzed in Remark 3 of [21]. Motivated by [14], the present authors [21] considered the global stability of nonlinear stochastic feedback systems driven by additive white noise. Suppose that the output function is either order-preserving or anti-order-preserving in the usual vector order and the global Lipschitz constant of the output function is less than the absolute of the negative principal eigenvalue of linear matrix; we directly consider the existence and uniqueness of globally attracting fixed points of stochastic feedback systems via the Banach fixed point theorem.

When we investigate the feedback problems originating from biology, ecology, and biochemistry, etc., it is more realistic to restrain the state space on the nonnegative orthant $\mathbb{R}^d_+$. For these problems, we have to consider stochastic feedback systems driven by multiplicative white noise, which usually preserve the invariance for solutions on the nonnegative orthant $\mathbb{R}^d_+$. These types of SDEs often appear in some applications including diffusion models in population dynamics [3]. In the study of such SDEs, one of the most interesting things is to consider the existence, uniqueness, and domain of attraction of stationary solutions (or quasi-stationary distributions), which represent some absorbing states; see [3]. This paper is a continuation of the paper [21]. We will assume that the output functions (feedback functions) possess bounded derivatives or are uniformly bounded away from zero. In both cases, we will prove that there exists a globally stable positive random equilibrium (stationary solution) in the nonnegative orthant $\mathbb{R}^d_+$ for the corresponding random dynamical systems, where the global stability means that all pull-back trajectories originating from nonnegative orthant converge to this positive random equilibrium (stationary solution) almost surely. Our results can be successfully applied to a well-known stochastic Goodwin negative feedback system, Othmer–Tyson positive feedback system, and Griffith positive feedback system as well as other stochastic cooperative, competitive, and predator-prey systems.

The remainder of this paper is organized as follows. Section 2 contains the considered problem, some preliminary definitions, notation, and the definition for the input-to-state characteristic map. Section 3 specifies the measurability of pull-back trajectories and the metric dynamical system θ and the dynamical behavior of stochastic flows, and shows some relative order-preserving results. Section 4 presents two stochastic stability theorems, which are applied to a series of examples. Section 5 ends this paper with some concluding remarks and discussions.

2. Problem and preliminaries. In this section, we will investigate a stochastic biochemical model consisting of d interacting components, which may be more realistic for describing and simulating the dynamical behavior of biochemical networks under fluctuations of intrinsic and extrinsic noise. Let X_i represent the ith variable of biochemical reactions (protein concentrations or levels of gene expression), which can be modeled by the following nonlinear stochastic system with multiplicative white noise

$$dX_t = [AX_t + h(X_t)]dt + \sum_{k=1}^{d} \sigma_k X_t dW_t^k, \tag{2.1}$$

where $A = (a_{ij})_{d\times d}$ and $\sigma_k = (\sigma_k^{ij})_{d\times d}$ are $(d \times d)$-dimensional matrices, $k = 1, \dots, d$, $h : \mathbb{R}_+^d \to \mathbb{R}_+^d$, and $W_t(\omega) = \big(W_t^1(\omega), \dots, W_t^d(\omega)\big)$ is an $\mathbb{R}^d$-valued two-sided Wiener process on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, where $\mathscr{F}$ is the Borel σ-algebra of $\Omega = C_0(\mathbb{R}, \mathbb{R}^d) = \{\omega = (\omega_1, \omega_2, \dots, \omega_d) \in C(\mathbb{R}, \mathbb{R}^d),\ \omega(0) = 0\}$ induced by the compact-open topology, which is generated by the following metric:

$$\varrho(\omega, \omega^*) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\varrho_n(\omega, \omega^*)}{1 + \varrho_n(\omega, \omega^*)}, \quad \varrho_n(\omega, \omega^*) = \max_{t\in[-n,n]} |\omega(t) - \omega^*(t)|,$$

and $\mathbb{P}$ is the corresponding Wiener measure. Furthermore, we will be concerned with the dynamical behavior of stochastic differential equations on the nonnegative orthant $\mathbb{R}_+^d$. For this purpose, we assume that σ_k, $k = 1, \dots, d$, has the following form throughout this paper:

$$\sigma_k = \begin{bmatrix} \sigma_k^1 & & \\ & \ddots & \\ & & \sigma_k^d \end{bmatrix}, \qquad \sigma_k^i \in \mathbb{R},\ k, i = 1, \dots, d. \tag{2.2}$$

For the convenience of readers, we will give some definitions and notation of random dynamical systems for later use; see [1, 6] for more details.

In this work, let X be a Polish space endowed with the Borel σ-algebra $\mathscr{B}(X)$, i.e., a separable complete metric space, and $(\Omega, \mathscr{F}, \mathbb{P})$ a probability space.

DEFINITION 2.1. *$\theta \equiv (\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{R}\})$ is called a metric dynamical system if*
(i) *$\theta : \mathbb{R} \times \Omega \mapsto \Omega$ is $(\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}, \mathscr{F})$-measurable;*
(ii) *$\theta_0 = \mathrm{id}$ is the identity on Ω and $\theta_t \circ \theta_s = \theta_{t+s}$ for all $t, s \in \mathbb{R}$;*
(iii) *$\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$, i.e., θ_t preserves the probability measure $\mathbb{P}$ for all $t \in \mathbb{R}$.*

DEFINITION 2.2. *A random dynamical system on the Polish space X consists of two elements: a metric dynamical system $\theta \equiv (\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{R}\})$ and the mapping*

$$\varphi : \mathbb{R}_+ \times \Omega \times X \mapsto X, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x),$$

which is $(\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F} \otimes \mathscr{B}(X), \mathscr{B}(X))$-measurable and satisfies the following:

(i) $\varphi(t,\omega,\cdot): X \to X$ *is continuous for all* $t \in \mathbb{R}_+$ *and* $\omega \in \Omega$;

(ii) *the mappings* $\varphi(t,\omega) := \varphi(t,\omega,\cdot)$ *satisfy the cocycle (over* θ*) property:*

$$\varphi(0,\omega) = \mathrm{id}, \quad \varphi(t+s,\omega) = \varphi(t,\theta_s\omega) \circ \varphi(s,\omega)$$

for all $t, s \in \mathbb{R}_+$ *and* $\omega \in \Omega$.

DEFINITION 2.3. *The multifunction* $D: \Omega \to 2^X \backslash \{\varnothing\}$ *is said to be a random set if the mapping* $\omega \to \mathrm{dist}_X(x, D(\omega))$ *is measurable for any* $x \in X$, *where* $\mathrm{dist}_X(x, B)$ *means the distance in* X *between the point* x *and the set* $B \subset X$. *If* $D(\omega)$ *is closed (resp., compact) in* X *for each* $\omega \in \Omega$, *the mapping* $\omega \to D(\omega)$ *is called a random closed (resp., compact) set.*

Motivated by the work of [14, 21], the above stochastic model can be rewritten as a stochastic system with inputs

$$dX_t = [AX_t + u(t)]dt + \sum_{k=1}^{d} \sigma_k X_t dW_t^k \tag{2.3}$$

together with outputs

$$u(t) = h(X_t).$$

From this viewpoint, we can regard the nonlinear feedback function $u(t) = h(X_t)$ as a known stochastic process, which results in the fact that the stochastic system (2.1) will become a linear nonhomogeneous stochastic differential equation.

Let us first consider the corresponding linear homogeneous stochastic Itô type differential equation

$$dX_t = AX_t dt + \sum_{k=1}^{d} \sigma_k X_t dW_t^k, \tag{2.4}$$

which is equivalent to the following system of Stratonovich stochastic differential equations:

$$dX_t = \left(A - \frac{1}{2}C\right)X_t dt + \sum_{k=1}^{d} \sigma_k X_t \circ dW_t^k, \tag{2.5}$$

where we write C in the form

$$C = \begin{bmatrix} \sum_{k=1}^{d} (\sigma_k^1)^2 & & \\ & \ddots & \\ & & \sum_{k=1}^{d} (\sigma_k^d)^2 \end{bmatrix}. \tag{2.6}$$

In order to make use of the technique of monotone systems, it is necessary to make the following assumption on A:

(A) A is *cooperative*, i.e., $a_{ij} \geq 0$ for all $i, j \in \{1, \ldots, d\}$ and $i \neq j$.

We will denote by $\Phi(t) = (\Phi_1(t), \ldots, \Phi_d(t)) = (\Phi_{ij}(t))_{d \times d}$ the fundamental matrix of (2.4), where $\Phi_j(t) = (\Phi_{1j}(t), \ldots, \Phi_{dj}(t))^T$ is the solution of (2.4) with initial value $x(0) = e_j$, $j = 1, \ldots, d$. By the classical existence and uniqueness of solutions for stochastic differential equations and the theory of monotone random dynamical systems, it is clear that (2.4), i.e., (2.5) generates a linear order-preserving random dynamical system (θ, Φ) in $\mathbb{R}^d_+$ (see [6, Proposition 6.2.2, p. 186]), where θ is the time shift on Ω, i.e.,

$$\theta_t\omega(\cdot) := \omega(t+\cdot) - \omega(t), \quad t \in \mathbb{R}.$$

That is, Φ satisfies the cocycle property: $\Phi(t+s,\omega) = \Phi(t,\theta_s\omega)\circ\Phi(s,\omega)$ for all $t,s \in \mathbb{R}_+$, $\omega \in \Omega$, and $\Phi(t,\omega)x \geq_{\mathbb{R}^d_+} \Phi(t,\omega)y$ for all $x, y \in \mathbb{R}^d_+$ such that $x \geq_{\mathbb{R}^d_+} y$, where $x \geq_{\mathbb{R}^d_+} y$ means that $x - y \in \mathbb{R}^d_+$. Furthermore, the following assumption on (θ, Φ) will be needed in what follows:

(L) The top Lyapunov exponent for the linear random dynamical system (θ, Φ) is a negative real number, i.e., there exist a constant $\lambda > 0$ and a tempered random variable $R(\omega) > 0$ such that

$$\|\Phi(t,\omega)\| := \max\{|\Phi_{ij}(t,\omega)| : i,j = 1,\dots,d\} \leq R(\omega)e^{-\lambda t} \tag{2.7}$$

holds for all $t \geq 0$, $\omega \in \Omega$.

Here, a random variable $R(\omega) > 0$ is called tempered if

$$\sup_{t\in\mathbb{R}} \left\{ e^{-\gamma|t|} \left|R(\theta_t\omega)\right|_2 \right\} < \infty \quad \text{for all } \omega \in \Omega \text{ and } \gamma > 0,$$

where $|x|_2 := (\sum_{i=1}^d |x_i|^2)^{\frac{1}{2}}$, $x \in \mathbb{R}^d$. Throughout this paper, we will use the norm $|x| := \max\{|x_i| : i = 1,\dots,d\}$, $x \in \mathbb{R}^d$ and $\|\Phi\|_2 := (\sum_{i,j=1}^d |\Phi_{ij}|^2)^{\frac{1}{2}}$, $\Phi \in \mathbb{R}^{d\times d}$.

In the remainder of this section, we are concerned with the existence and uniqueness of solutions for (2.1) and its pull-back trajectories; we shall make the following assumption on h, which is abstracted from the Othmer–Tyson positive feedback model [38] and the Goodwin negative feedback model [15]:

(H_1) $h \in C^1(\mathbb{R}^d_+, \mathbb{R}^d_+ \setminus \{0\})$ and is bounded in $\mathbb{R}^d_+$. Moreover, we assume that h is monotone, i.e.,

$$x_1 \leq_{\mathbb{R}^d_+} x_2 \quad \Rightarrow \quad h(x_1) \leq_{\mathbb{R}^d_+} h(x_2) \qquad \text{for all } x_1, x_2 \in \mathbb{R}^d_+$$

or antimonotone, i.e.,

$$x_1 \leq_{\mathbb{R}^d_+} x_2 \quad \Rightarrow \quad h(x_1) \geq_{\mathbb{R}^d_+} h(x_2) \qquad \text{for all } x_1, x_2 \in \mathbb{R}^d_+.$$

By (H_1), it is easy to check that (2.1) satisfies the conditions of local Lipschitz and linear growth (since h is bounded in $\mathbb{R}^d_+$) in $\mathbb{R}^d_+$. Motivated by the proof of Proposition 6.2.1 in [6], let $\tilde{h}$ be an extension of h from $\mathbb{R}^d_+$ to $\mathbb{R}^d$ such that $\tilde{h}$ satisfies the conditions of local Lipschitz and linear growth in $\mathbb{R}^d$; we thus have the existence and uniqueness of global solutions for

$$dX_t = [AX_t + \tilde{h}(X_t)]dt + \sum_{k=1}^d \sigma_k X_t dW_t^k$$

(see [28, 32]), which is equivalent to the Stratonovich interpretation of stochastic differential equations

$$dX_t = \left[\left(A - \frac{1}{2}C\right)X_t + \tilde{h}(X_t)\right]dt + \sum_{k=1}^d \sigma_k X_t \circ dW_t^k,$$

where C is defined in (2.6) and generates a random dynamical system in $\mathbb{R}^d$; see [1, Chapter 2], [6, Chapter 2]. In the same manner of Proposition 6.2.1 in [6], we can see that there exists a unique (indistinguished) random dynamical system (θ, φ) generated by (2.1) such that the set $\mathbb{R}^d_+$ is forward invariant, i.e., $\varphi(t,\omega)\mathbb{R}^d_+ \subset \mathbb{R}^d_+$ for all $t \in \mathbb{R}_+$, $\omega \in \Omega$, and $\varphi(t,\omega)x = x(t,\omega,x)$ is the unique solution of equations (2.1) for each initial value $x(0) = x \in \mathbb{R}^d_+$.

Combining the variation-of-constants formula [28, Chapter 3, Theorem 3.1] and the cocycle property of Φ, it follows that

$$\begin{aligned}\varphi(t,\omega)x &= \Phi(t,\omega)x + \Phi(t,\omega)\int_0^t \Phi^{-1}(s,\omega)h(\varphi(s,\omega)x)ds\\ &= \Phi(t,\omega)x + \int_0^t \Phi(t-s,\theta_s\omega)h(\varphi(s,\omega)x)ds, \quad t \geq 0,\ \omega \in \Omega. \qquad (2.8)\end{aligned}$$

By the definition of θ, a similar analysis as in [21] shows that the pull-back trajectories of (θ, φ) are as follows

$$\begin{aligned}\varphi(t,\theta_{-t}\omega)x &= \Phi(t,\theta_{-t}\omega)x + \int_0^t \Phi(t-s,\theta_{s-t}\omega)h(\varphi(s,\theta_{-t}\omega)x)ds\\ &= \Phi(t,\theta_{-t}\omega)x + \int_{-t}^0 \Phi(-s,\theta_s\omega)h(\varphi(t+s,\theta_{-t}\omega)x)ds,\ t \geq 0,\ \omega \in \Omega. \qquad (2.9)\end{aligned}$$

Regarding the feedback function h as an input term, we define the *input-to-state characteristic* map $\mathcal{K}$ associated with given inputs in $\mathbb{R}^d_+$ as follows:

$$[\mathcal{K}(u)](\omega) = \int_{-\infty}^0 \Phi(-s,\theta_s\omega)u(\theta_s\omega)ds, \quad \omega \in \Omega, \qquad (2.10)$$

where u is an $\mathbb{R}^d_+$-valued and tempered random variable with respect to θ.

Remark 1. Since the top Lyapunov exponent of Φ is negative, it is evident that $\mathcal{K}$ is well defined. We first observe the fact that $\|\Phi(t,\omega)\|_2 \leq d\|\Phi(t,\omega)\| \leq dR(\omega)\mathrm{e}^{-\lambda t}$, $\lambda > 0$. According to the above definition, we have

$$\begin{aligned}&\left|\int_{-\infty}^0 \Phi(-s,\theta_s\omega)u(\theta_s\omega)ds\right|_2\\ &\quad\leq \int_{-\infty}^0 |\Phi(-s,\theta_s\omega)u(\theta_s\omega)|_2\, ds\\ &\quad\leq d\int_{-\infty}^0 R(\theta_s\omega)\mathrm{e}^{-\lambda|s|}|u(\theta_s\omega)|_2 ds\\ &\quad\leq d\sup_{t\in\mathbb{R}}\left\{\mathrm{e}^{-\frac{\lambda}{4}|t|}\,|u(\theta_t\omega)|_2\right\}\sup_{t\in\mathbb{R}}\left\{\mathrm{e}^{-\frac{\lambda}{4}|t|}R(\theta_t\omega)\right\}\cdot\int_{-\infty}^0 e^{-\frac{\lambda}{2}|s|}ds,\\ &\quad< \infty, \qquad \omega \in \Omega,\end{aligned}$$

which together with the Lebesgue's monotone convergence theorem [7] implies that

$$\lim_{t\to\infty}\int_{-t}^0 \Phi(-s,\theta_s\omega)u(\theta_s\omega)ds$$

exists for all $\omega \in \Omega$ and $\mathbb{R}^d_+$-valued u by the order-preserving property of Φ. Furthermore, due to the boundedness of h and (L), we similarly have that $\{\varphi(t,\theta_{-t}\omega)x : t \geq 0\}$ is a bounded set for all $\omega \in \Omega$ and $x \in \mathbb{R}^d_+$, which plays an important role in the subsequent sections.

3. Measurability and behavior of random dynamical system generated by SDEs. In this section, we will divide the proof of our main results into a sequence of lemmas and establish some propositions related to the measurability of the metric dynamical system θ with respect to the product σ-algebra $\mathscr{B}(\mathbb{R}_-) \otimes \mathscr{F}_-$ and the dynamical behavior of pull-back trajectories. In what follows, we may repeat some known results without proof to make our exposition self-contained. We start with definitions of future and past σ-algebras, which can be found in [6, 8].

PROPOSITION 3.1. *Define the future and the past σ-algebras for (θ,Φ) and (θ,φ) as follows:*

$$\mathscr{F}^1_+ = \sigma\{\omega \mapsto \Phi(\tau,\theta_t\omega)x : x \in \mathbb{R}^d_+,\ t,\tau \geq 0\},$$

$$\mathscr{F}^1_- = \sigma\{\omega \mapsto \Phi(\tau,\theta_{-t}\omega)x : x \in \mathbb{R}^d_+,\ 0 \leq \tau \leq t\},$$

$$\mathscr{F}^2_+ = \sigma\{\omega \mapsto \varphi(\tau,\theta_t\omega)x : x \in \mathbb{R}^d_+,\ t,\tau \geq 0\},$$

$$\mathscr{F}^2_- = \sigma\{\omega \mapsto \varphi(\tau,\theta_{-t}\omega)x : x \in \mathbb{R}^d_+,\ 0 \leq \tau \leq t\}.$$

Then, we have

$$\mathscr{F}^1_+ \subset \mathscr{F}_+, \qquad \mathscr{F}^1_- \subset \mathscr{F}_-, \tag{3.1}$$

and

$$\mathscr{F}^2_+ \subset \mathscr{F}_+, \qquad \mathscr{F}^2_- \subset \mathscr{F}_-. \tag{3.2}$$

Here, $\mathscr{F}_+$ and $\mathscr{F}_-$ are defined by

$$\mathscr{F}_+ = \sigma\{\omega \mapsto W_t(\omega) : t \geq 0\} \quad \textit{and} \quad \mathscr{F}_- = \sigma\{\omega \mapsto W_t(\omega) : t \leq 0\}.$$

Proof. We only give the proof of (3.1), (3.2) can be obtained analogously. By the theory of stochastic differential equations, it is clear that $\Phi(t,\omega)x$ is adapted to the filtration $\mathscr{F}^t_0 = \sigma\{\omega \mapsto W_s(\omega) : 0 \leq s \leq t\}$, $t \geq 0$, $x \in \mathbb{R}^d_+$. Consequently, for fixed $x \in \mathbb{R}^d_+$, $t,\tau \geq 0$, it follows that

$$\begin{aligned}\sigma\{\omega \mapsto \Phi(\tau,\theta_t\omega)x\} &\subset \theta_t^{-1}\mathscr{F}^\tau_0 \\ &= \sigma\{\omega \mapsto W_s(\theta_t\omega) : 0 \leq s \leq \tau\} \\ &= \sigma\{\omega \mapsto W_{s+t}(\omega) - W_t(\omega) : 0 \leq s \leq \tau\} \\ &\subset \mathscr{F}_+,\end{aligned}$$

which implies that $\mathscr{F}^1_+ \subset \mathscr{F}_+$. Similarly, for any given $x \in \mathbb{R}^d_+$, $0 \leq \tau \leq t$, we have

$$\begin{aligned}\sigma\{\omega \mapsto \Phi(\tau,\theta_{-t}\omega)x\} &\subset \theta_{-t}^{-1}\mathscr{F}^\tau_0 \\ &= \sigma\{\omega \mapsto W_s(\theta_{-t}\omega) : 0 \leq s \leq \tau\} \\ &= \sigma\{\omega \mapsto W_{s-t}(\omega) - W_{-t}(\omega) : 0 \leq s \leq \tau\} \\ &\subset \mathscr{F}_-,\end{aligned}$$

which gives that $\mathscr{F}^1_- \subset \mathscr{F}_-$, and (3.1) is proved. The same proof works for (3.2). ∎

By definitions of θ and the metric ϱ on the space $\Omega = C_0(\mathbb{R}, \mathbb{R}^d)$, it follows immediately that $\theta : \mathbb{R} \times \Omega \mapsto \Omega$ is continuous; see [1, Chapter 2, pp. 74–75]. For the purpose of readability and making this paper self-contained, we present a proof of the continuity of $\theta(\cdot, \omega) : \mathbb{R} \mapsto \Omega$ for all $\omega \in \Omega$, which is sufficient for our discussion.

PROPOSITION 3.2. *For any $\omega \in \Omega$, $\theta(\cdot, \omega) : (\mathbb{R}, |\cdot|) \mapsto (\Omega, \varrho)$ is continuous.*

Proof. Given fixed $t_0 \in \mathbb{R}$ and $\omega \in \Omega$, let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $t_k \to t_0$ as $k \to \infty$. We only need to show that $\varrho(\theta_{t_k}\omega, \theta_{t_0}\omega) \to 0$, i.e., $\varrho\big(\omega(t_k + \cdot) - \omega(t_k), \omega(t_0 + \cdot) - \omega(t_0)\big) \to 0$. Observe that for all $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} \frac{1}{2^n} \frac{\varrho_n\big(\omega(t_k + \cdot) - \omega(t_k), \omega(t_0 + \cdot) - \omega(t_0)\big)}{1 + \varrho_n\big(\omega(t_k + \cdot) - \omega(t_k), \omega(t_0 + \cdot) - \omega(t_0)\big)} \le \sum_{n=N}^{\infty} \frac{1}{2^n} < \varepsilon.$$

The proof is completed by showing that for all $1 \le n \le N$, we have

$$\varrho_n\big(\omega(t_k + \cdot) - \omega(t_k), \omega(t_0 + \cdot) - \omega(t_0)\big) \to 0, \qquad as \quad k \to \infty.$$

Since $t_k \to t_0$, it is obvious that $\{t_k\}_{k\in\mathbb{N}}$ is bounded, which yields that there exists $M_N > 0$ such that $|t_k + t| \le M_N$ and $|t_0 + t| \le M_N$ uniformly for all $k \in \mathbb{N}$, $t \in [-n, n]$, $1 \le n \le N$. On the other hand, we note that any continuous function on a closed and bounded interval $[a, b]$ is uniformly continuous, which reveals that for all $1 \le n \le N$,

$$\begin{aligned}
&\varrho_n\big(\omega(t_k + \cdot) - \omega(t_k), \omega(t_0 + \cdot) - \omega(t_0)\big) \\
&= \max_{t\in[-n,n]} |\omega(t_k + t) - \omega(t_k) - \omega(t_0 + t) + \omega(t_0)| \\
&\le \max_{t\in[-n,n]} |\omega(t_k + t) - \omega(t_0 + t)| + |\omega(t_k) - \omega(t_0)| \\
&\to 0, \quad \text{as} \quad k \to \infty.
\end{aligned}$$

The proof is complete. ∎

PROPOSITION 3.3. *$\theta : \mathbb{R}_- \times \Omega \mapsto \Omega$ is $(\mathscr{B}(\mathbb{R}_-) \otimes \mathscr{F}_-, \mathscr{F}_-)$-measurable and $\theta : \mathbb{R}_+ \times \Omega \mapsto \Omega$ is $(\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F}_+, \mathscr{F}_+)$-measurable.*

Proof. The proof of this proposition is mainly motivated by the proof of Lemma 3.14 in [2]. For convenience, we only deal with the case of time $\mathbb{R}_-$, and the rest of this proposition can be obtained analogously. First, for any $t \le 0$, we have

$$\begin{aligned}
\theta_t^{-1}\mathscr{F}_- &= \theta_t^{-1}\sigma\{\omega \mapsto W_s(\omega) : s \le 0\} \\
&= \sigma\{\omega \mapsto W_s(\theta_t\omega) : s \le 0\} \\
&= \sigma\{\omega \mapsto W_{s+t}(\omega) - W_t(\omega) : s \le 0\} \\
&\subset \mathscr{F}_-,
\end{aligned}$$

which implies that $\theta(t, \cdot) : (\Omega, \mathscr{F}_-) \mapsto (\Omega, \mathscr{F}_-)$ is measurable for any $t \in \mathbb{R}_-$. Moreover, let $\{t_n\}_{n=1}^{\infty}$ denote a dense sequence in $\mathbb{R}_-$. For any $p \ge 1$, $p \in \mathbb{N}$, define

$\theta_p(t,\omega) = \theta(t_n,\omega)$, where n is the smallest integer such that t belongs to the open interval $B(t_n, \frac{1}{p}) := \{s \in \mathbb{R}_- : |s - t_n| < \frac{1}{p}\}$. Note that θ_p is equal to the map $(t,\omega) \mapsto \theta(t_n,\omega)$ on $[B(t_n,\frac{1}{p}) - \bigcup_{m<n} B(t_m,\frac{1}{p})] \times \Omega$, and so for any $F \in \mathscr{F}_-$, it follows that

$$\begin{aligned}\theta_p^{-1}F &= \bigcup_{n=1}^{\infty}\left\{\theta_p^{-1}F \bigcap \left\{\left[B\left(t_n,\frac{1}{p}\right) - \bigcup_{m<n} B\left(t_m,\frac{1}{p}\right)\right] \times \Omega\right\}\right\}\\ &= \bigcup_{n=1}^{\infty}\left\{\left[B\left(t_n,\frac{1}{p}\right) - \bigcup_{m<n} B\left(t_m,\frac{1}{p}\right)\right] \times \theta_{t_n}^{-1}F\right\}\\ &\in \mathscr{B}(\mathbb{R}_-) \otimes \mathscr{F}_-.\end{aligned}$$

This yields that θ_p is $(\mathscr{B}(\mathbb{R}_-) \otimes \mathscr{F}_-, \mathscr{F}_-)$-measurable. It is well known that $\mathscr{F}_- = \mathscr{B}_{\varrho^-}(\Omega)$, which is the Borel σ-algebra generated by open sets with respect to the pseudometric ϱ^-; see [1, 22]. Here, the pseudometric ϱ^- is defined as follows:

$$\varrho^-(\omega,\omega^*) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\varrho_n^-(\omega,\omega^*)}{1+\varrho_n^-(\omega,\omega^*)}, \quad \varrho_n^-(\omega,\omega^*) = \max_{t\in[-n,0]} |\omega(t) - \omega^*(t)|.$$

Therefore, θ_p is $(\mathscr{B}(\mathbb{R}_-) \otimes \mathscr{F}_-, \mathscr{B}_{\varrho^-}(\Omega))$-measurable for all $p \in \mathbb{N}$. By Proposition 3.2, it is clear that under the metric ϱ, $\theta_p(t,\omega) \to \theta(t,\omega)$ as $p \to \infty$ for all $t \in \mathbb{R}_-$ and $\omega \in \Omega$, which together with the fact that $\varrho^-(\omega,\omega^*) \le \varrho(\omega,\omega^*)$ imply that under the pseudometric ϱ^-, $\theta_p(t,\omega) \to \theta(t,\omega)$ as $p \to \infty$ for all $t \in \mathbb{R}_-$ and $\omega \in \Omega$. Then by Theorem 21.3 in [35], it follows that $\theta : \mathbb{R}_- \times \Omega \mapsto \Omega$ is $(\mathscr{B}(\mathbb{R}_-) \otimes \mathscr{F}_-, \mathscr{B}_{\varrho^-}(\Omega))$-measurable, i.e., $(\mathscr{B}(\mathbb{R}_-) \otimes \mathscr{F}_-, \mathscr{F}_-)$-measurable. The proof is complete. ∎

Proposition 3.4. *For any $\mathscr{F}_-$-measurable tempered random variable u in $\mathbb{R}_+^d$, $\mathcal{K}(u)$ is a random variable with respect to the σ-algebra $\mathscr{F}_-$.*

Proof. By Proposition 3.3, it is immediate that $\theta : \mathbb{R}_- \times \Omega \mapsto \Omega$ is $(\mathscr{B}(\mathbb{R}_-) \otimes \mathscr{F}_-, \mathscr{F}_-)$-measurable, which shows that $u(\theta_t\omega)$ is $(\mathscr{B}(\mathbb{R}_-) \otimes \mathscr{F}_-, \mathscr{B}(\mathbb{R}_+^d))$-measurable. Moreover, it is known that

$$(t,x) \mapsto \Phi(t,\theta_{-t}\omega)x \text{ is continuous}, \quad \omega \in \Omega,$$

from $\mathbb{R}_+ \times \mathbb{R}_+^d$ into $\mathbb{R}_+^d$; see Remark 1.5.1 in [6]. Thus, $t \mapsto \Phi(t,\theta_{-t}\omega)$ is also continuous from $\mathbb{R}_+$ into $\mathbb{R}_+^{d\times d}$, $\omega \in \Omega$. This, together with the fact that $\omega \mapsto \Phi(t,\theta_{-t}\omega)$, $t \in \mathbb{R}_+$, is $\mathscr{F}_-$-measurable by Proposition 3.1, yields that $\Phi(t,\theta_{-t}\omega)$ is $(\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F}_-, \mathscr{B}(\mathbb{R}_+^{d\times d}))$-measurable by Lemma 3.14 in [2]. Combining the definition of $\mathcal{K}$ and Fubini's theorem, it is easy to get the measurability of $\mathcal{K}(u)$. We complete the proof. ∎

Proposition 3.5. *For any $\tau > 0$, define*

$$\xi_\tau^h(\omega) = \inf \overline{\{h(\varphi(t,\theta_{-t}\omega)x) : t \ge \tau\}}$$

and

$$\eta_\tau^h(\omega) = \sup \overline{\{h(\varphi(t,\theta_{-t}\omega)x) : t \ge \tau\}}, \quad x \in \mathbb{R}_+^d,\ \omega \in \Omega.$$

Here inf *and* sup *represent the greatest lower bound and the least upper bound, respec-*

tively. Then $\xi^h_\tau(\omega)$ *and* $\eta^h_\tau(\omega)$ *are* $\mathscr{F}_-$*-measurable random variables. More precisely, they are random variables with respect to* $\mathscr{F}^2_-$.

Proof. First, we claim that $\xi^h_\tau(\omega)$ and $\eta^h_\tau(\omega)$ are random variables with respect to $\mathscr{F}$. The proof is similar in spirit to Proposition 3.2 in [21], and we omit it here. Since $h(\varphi(t,\theta_{-t}\omega)x)$ is an $\mathscr{F}^2_-$-measurable random variable, $t \geq 0$, $x \in \mathbb{R}^d_+$, by the virtue of the proof of Proposition 3.2 in [21], we have $\xi^h_\tau(\omega)$ and $\eta^h_\tau(\omega)$ are $\mathscr{F}^2_-$-measurable, and due to Proposition 3.1 we can obtain the required conclusion. The proof is complete. ∎

LEMMA 3.6 (see [14, Lemma A.2]). *Assume that* $(x_\alpha)_{\alpha\in\Lambda}$ *is a net in a normed space* X *with a solid, normal cone* $X_+ \subseteq X$*, which defines a partial order on* X*. Furthermore, suppose that the net converges to an element* $x_\infty \in X$*, and*

$$x^-_\alpha := \inf\{x_{\alpha'} : \alpha' \geq \alpha\} \quad \textit{and} \quad x^+_\alpha := \sup\{x_{\alpha'} : \alpha' \geq \alpha\}$$

exist for every $\alpha \in \Lambda$*. Then the nets* $(x^-_\alpha)_{\alpha\in\Lambda}$ *and* $(x^+_\alpha)_{\alpha\in\Lambda}$ *also converge to* x_∞.

LEMMA 3.7. *Let assumptions* (A)*,* (L)*, and* (H$_1$) *hold. Then*

$$\mathcal{K}(\theta - \underline{\lim}\, h(\varphi)) \leq \theta - \underline{\lim}\, \varphi \leq \theta - \overline{\lim}\, \varphi \leq \mathcal{K}(\theta - \overline{\lim}\, h(\varphi)) \quad \text{for all } \omega \in \Omega, \tag{3.3}$$

where

$$[\theta - \underline{\lim}\, h(\varphi)](\omega) := \lim_{\tau\to\infty} \xi^h_\tau(\omega) = \lim_{\tau\to\infty} \inf\{h(\varphi(t,\theta_{-t}\omega)x) : t \geq \tau\}, \quad x \in \mathbb{R}^d_+,\ \omega \in \Omega,$$

and

$$[\theta - \overline{\lim}\, h(\varphi)](\omega) := \lim_{\tau\to\infty} \eta^h_\tau(\omega) = \lim_{\tau\to\infty} \sup\{h(\varphi(t,\theta_{-t}\omega)x) : t \geq \tau\}, \quad x \in \mathbb{R}^d_+,\ \omega \in \Omega.$$

In this way, $\theta - \underline{\lim}\, \varphi$ *and* $\theta - \overline{\lim}\, \varphi$ *can be defined similarly.*

Proof. To prove (3.3), we start with the first inequality in (3.3). We first observe that

$$\inf\{\varphi(t,\theta_{-t}\omega)x : t \geq \tau\} = \inf \overline{\{\varphi(t,\theta_{-t}\omega)x : t \geq \tau\}}, \quad x \in \mathbb{R}^d_+,\ \omega \in \Omega,$$

and

$$\inf\{h(\varphi(t,\theta_{-t}\omega)x) : t \geq \tau\} = \inf \overline{\{h(\varphi(t,\theta_{-t}\omega)x) : t \geq \tau\}}, \quad x \in \mathbb{R}^d_+,\ \omega \in \Omega,$$

by Lemma A.1 in [14]. By the boundedness of pull-back trajectories for (2.1) and h, in the same manner as the proof of Proposition 3.5, we can see that well-defined $\theta - \underline{\lim}\, h(\varphi)$ and $\theta - \underline{\lim}\, \varphi$ are two $(\mathscr{F}_-, \mathscr{B}(\mathbb{R}^d_+))$-measurable random variables. Due to the boundedness of h, $\theta - \underline{\lim}\, h(\varphi)$ is a tempered random variable. Hence, $\mathcal{K}(\theta - \underline{\lim}\, h(\varphi))$ is an $(\mathscr{F}_-, \mathscr{B}(\mathbb{R}^d_+))$-measurable random variable by Proposition 3.4.

It follows from the definition of $\theta - \underline{\lim}\, h(\varphi)$ that

$$[\theta - \underline{\lim}\, h(\varphi)](\omega) = \lim_{\tau\to\infty} \xi^h_\tau(\omega), \quad \omega \in \Omega,$$

which together with the Lebesgue's dominated convergence theorem [7] implies that

$$\mathcal{K}(\theta - \underline{\lim}\, h(\varphi)) = \lim_{\tau\to\infty} \mathcal{K}(\xi_\tau),$$

where $\xi_\tau(\omega) := \inf\{h(\varphi(t,\theta_{-t}\omega)x) : t \geq \tau\}$, $x \in \mathbb{R}^d_+$, $\omega \in \Omega$. Now we choose an increasing sequence of time $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \uparrow \infty$ and it is sufficient to show that

$$\mathcal{K}(\xi_{\tau_n}) \leq \theta - \underline{\lim}\, \varphi, \quad \omega \in \Omega,\ n \in \mathbb{N}.$$

It is evident that for all $x \in \mathbb{R}^d_+$, $\omega \in \Omega$, $\lim_{t\to\infty} \Phi(t,\theta_{-t}\omega)x = 0$. Therefore, for any $\tau_n \geq 0$, by the definition of $\mathcal{K}$, it follows that

$$\begin{aligned}
&[\mathcal{K}(\xi_{\tau_n})](\omega)\\
&= \int_{-\infty}^{0} \Phi(-s,\theta_s\omega) \inf\{h(\varphi(t,\theta_{-t}\bullet)x) : t \geq \tau_n\}(\theta_s\omega)ds\\
&= \int_{-\infty}^{0} \Phi(-s,\theta_s\omega) \inf\{h(\varphi(t,\theta_{-t+s}\omega)x) : t \geq \tau_n\}ds\\
&= \lim_{\substack{\tilde{t}\to\infty\\ \tilde{t}\geq\tau_n}} \left\{\Phi(\tilde{t},\theta_{-\tilde{t}}\omega)x + \int_{\tau_n-\tilde{t}}^{0} \Phi(-s,\theta_s\omega) \inf\{h(\varphi(t,\theta_{-t+s}\omega)x) : t \geq \tau_n\}ds\right\}\\
&= \lim_{\substack{\tau\to\infty\\ \tau\geq\tau_n}} \inf \left\{\Phi(\tilde{t},\theta_{-\tilde{t}}\omega)x + \int_{\tau_n-\tilde{t}}^{0} \Phi(-s,\theta_s\omega) \inf\{h(\varphi(t,\theta_{-t+s}\omega)x) : t \geq \tau_n\}ds : \tilde{t} \geq \tau\right\}\\
&\leq \lim_{\substack{\tau\to\infty\\ \tau\geq\tau_n}} \inf \left\{\Phi(\tilde{t},\theta_{-\tilde{t}}\omega)x + \int_{\tau_n-\tilde{t}}^{0} \Phi(-s,\theta_s\omega) h(\varphi(\tilde{t}+s,\theta_{-\tilde{t}}\omega)x)ds : \tilde{t} \geq \tau\right\}\\
&\leq \lim_{\tau\to\infty} \inf \left\{\Phi(\tilde{t},\theta_{-\tilde{t}}\omega)x + \int_{-\tilde{t}}^{0} \Phi(-s,\theta_s\omega) h(\varphi(\tilde{t}+s,\theta_{-\tilde{t}}\omega)x)ds : \tilde{t} \geq \tau\right\}\\
&= [\theta - \underline{\lim}\, \varphi](\omega) \quad \text{for all } \omega \in \Omega.
\end{aligned}$$

Here, the above inequality follows by Lemma 3.6, the order-preserving property of Φ and the positivity of h. The rest proof of (3.3) runs as before. The proof is complete. ∎

LEMMA 3.8. *Let assumptions* (A), (L), *and* (H$_1$) *hold. Then h possesses the following properties:*

(i) *If h is monotone, then*

(3.4)
$$h(\theta - \underline{\lim}\, \varphi) \leq \theta - \underline{\lim}\, h(\varphi) \leq \theta - \overline{\lim}\, h(\varphi) \leq h(\theta - \overline{\lim}\, \varphi) \quad \text{for all } \omega \in \Omega.$$

1. *If h is anti-monotone, then*

(3.5)
$$h(\theta - \overline{\lim}\, \varphi) \leq \theta - \underline{\lim}\, h(\varphi) \leq \theta - \overline{\lim}\, h(\varphi) \leq h(\theta - \underline{\lim}\, \varphi) \quad \text{for all } \omega \in \Omega.$$

Proof. The proof of this lemma is very similar to that of Lemma 3.4 in [21], and we omit it here. ∎

LEMMA 3.9. *Let assumptions* (A), (L), *and* (H$_1$) *hold. Then*

$$\mathcal{K}(\xi^h_\tau) \leq \theta - \underline{\lim}\, \varphi \leq \theta - \overline{\lim}\, \varphi \leq \mathcal{K}(\eta^h_\tau) \quad \text{for all } \omega \in \Omega \text{ and } \tau \geq 0. \tag{3.6}$$

Here $\xi^h_\tau(\omega)$ and $\eta^h_\tau(\omega)$ are defined in Proposition 3.5. *Moreover, let $\mathcal{K}^h := h\circ\mathcal{K}$. Then the following hold:*

(i) *If h is monotone, then for any $\tau \geq 0$ and $k \in \mathbb{N}$*

$$(3.7)\quad (\mathcal{K}^h)^k(\xi^h_\tau) \leq \theta - \underline{\lim}\, h(\varphi) \leq \theta - \overline{\lim}\, h(\varphi) \leq (\mathcal{K}^h)^k(\eta^h_\tau) \text{ for all } \omega \in \Omega.$$

1. *If h is anti-monotone, then for any $\tau \geq 0$ and $k \in \mathbb{N}$*

$$(3.8)\quad (\mathcal{K}^h)^{2k}(\xi^h_\tau) \leq \theta - \underline{\lim}\, h(\varphi) \leq \theta - \overline{\lim}\, h(\varphi) \leq (\mathcal{K}^h)^{2k}(\eta^h_\tau) \text{ for all } \omega \in \Omega.$$

Proof. The proof of this lemma is very similar to that of Lemma 3.5 in [21], and we omit it here. ∎

4. Global stability theorems. In this section, we mainly consider two kinds of output functions (feedback functions): one is that derivatives of h are bounded and the other is that h is uniformly bounded away from zero. We shall establish two theorems for guaranteeing the existence and uniqueness of positive random equilibrium and almost surely global stability of pull-back trajectories.

4.1. Type one: Derivatives of h are bounded. In this subsection, we will state a global stability theorem in the case that derivatives of h are bounded and apply it to well-known stochastic feedback systems. In what follows, we shall propose a natural condition that the tempered random variable $R(\omega)$ given in (L) is independent of the σ-algebra $\mathscr{F}_- = \sigma\{\omega \mapsto W_t(\omega) : t \leq 0\}$. Assume that (L) holds. Then it is easy to see that the random variable $R(\omega) := \sup\{e^{\lambda t}\|\Phi(t,\omega)\| : t \geq 0\}$ is measurable with respect to the σ-algebra $\mathscr{F}_+ = \sigma\{\omega \mapsto W_t(\omega) : t \geq 0\}$, which is independent of $\mathscr{F}_- = \sigma\{\omega \mapsto W_t(\omega) : t \leq 0\}$ by the definition of the two-sided Wiener process; see [1, Chapter 2, p. 107]. This implies that such an R is independent of $\mathscr{F}_-$. Nevertheless, we still put this independence in our condition (R) because other choices of $R(\omega)$ may not be $\mathscr{F}_+$-measurable. Moreover, we assume that the tempered random variable $R \in \mathcal{L}^1(\Omega, \mathscr{F}_+, \mathbb{P})$, which will be illustrated in our examples.

LEMMA 4.1. *Let assumptions* (A), (L), *and* (H_1) *hold. Assume additionally that the following conditions on R and h are satisfied:*

(R) *Let $R \in \mathcal{L}^1(\Omega, \mathscr{F}_+, \mathbb{P}; \mathbb{R}_+)$ and be independent of the σ-algebra $\mathscr{F}_- = \sigma\{\omega \mapsto W_t(\omega) : t \leq 0\}$.*

(H_2) *Let $M = \max\{\sup_{x\in\mathbb{R}^d_+} |\frac{\partial h_i(x)}{\partial x_j}|, i,j = 1,\dots,d\}$ such that $\frac{Md^2\|R\|_{\mathcal{L}^1}}{\lambda} < 1$, where $\|R\|_{\mathcal{L}^1} = \mathbb{E}R = \int_\Omega R(\omega)\mathbb{P}(d\omega)$.*

Denote by $\mathcal{L}^1_{\mathscr{F}_-} := \mathcal{L}^1(\Omega, \mathscr{F}_-, \mathbb{P}; [0,\Gamma])$ the space of all $\mathscr{F}_-$-measurable functions $f : \Omega \to [0,\Gamma]$ (which must be integrable), where $\Gamma = (\Gamma_1,\dots,\Gamma_d)$, $\Gamma_i = \sup_{x\in\mathbb{R}^d_+} |h_i(x)|$, $i = 1,\dots,d$. Then the space is complete under the metric $\|u\|_{\mathcal{L}^1} = \mathbb{E}|u|$, $u \in \mathcal{L}^1_{\mathscr{F}_-}$ and the operator $\mathcal{K}^h = h \circ \mathcal{K} : (\mathcal{L}^1_{\mathscr{F}_-}, \|\cdot\|_{\mathcal{L}^1}) \to (\mathcal{L}^1_{\mathscr{F}_-}, \|\cdot\|_{\mathcal{L}^1})$ is a contraction mapping.

Proof. Since $[0,\Gamma]$ is closed in $\mathbb{R}^d_+$, it is clear that $\|\cdot\|_{\mathcal{L}^1}$ defines a metric on $\mathcal{L}^1(\Omega, \mathscr{F}_-, \mathbb{P}; [0,\Gamma])$ and the space endowed with this metric is complete.

We now turn to prove that $\mathcal{K}^h : \mathcal{L}^1_{\mathscr{F}_-} \to \mathcal{L}^1_{\mathscr{F}_-}$ is a contraction mapping. Let us first point out that $\mathcal{K}^h : \mathcal{L}^1_{\mathscr{F}_-} \to \mathcal{L}^1_{\mathscr{F}_-}$ is well defined by Propositions 3.3 and 3.4, i.e., given any $\mathscr{F}_-$-measurable random variable u in $\mathcal{L}^1_{\mathscr{F}_-}$, $\mathcal{K}^h(u)$ is an $\mathscr{F}_-$-measurable random variable in $\mathcal{L}^1_{\mathscr{F}_-}$. Next, we observe that

$$\sup_{x\in\mathbb{R}^d_+} \|Dh(x)\| \leq M,$$

where $Dh(x)$ is the Jacobian of h. Note that $|\Phi x| \leq d\|\Phi\| \cdot |x|$ for all $x \in \mathbb{R}^d$ and

$\Phi \in \mathbb{R}^{d\times d}$, and then for any $f_1, f_2 \in \mathcal{L}^1(\Omega, \mathscr{F}_-, \mathbb{P}; [0,\Gamma])$, we have

$$\begin{aligned}
\|\mathcal{K}^h(f_1) - \mathcal{K}^h(f_2)\|_{\mathcal{L}^1} &= \mathbb{E}\Big|\int_0^1 Dh[\mathcal{K}(f_2) + \mu(\mathcal{K}(f_1) - \mathcal{K}(f_2))]d\mu \cdot [\mathcal{K}(f_1) - \mathcal{K}(f_2)]\Big| \\
&\le d \sup_{x\in\mathbb{R}^d_+} \|Dh(x)\| \cdot \mathbb{E}|\mathcal{K}(f_1) - \mathcal{K}(f_2)| \\
&\le Md\mathbb{E}\left|\int_{-\infty}^0 \Phi(-s,\theta_s\omega) f_1(\theta_s\omega)ds - \int_{-\infty}^0 \Phi(-s,\theta_s\omega) f_2(\theta_s\omega)ds\right| \\
&\le Md^2\mathbb{E}\int_{-\infty}^0 \|\Phi(-s,\theta_s\omega)\| \cdot |f_1(\theta_s\omega) - f_2(\theta_s\omega)|ds \\
&\le Md^2\mathbb{E}\int_{-\infty}^0 \mathrm{e}^{\lambda s} R(\theta_s\omega)|f_1(\theta_s\omega) - f_2(\theta_s\omega)|ds \\
&= Md^2\int_{-\infty}^0 \mathrm{e}^{\lambda s}\int_\Omega R(\omega)|f_1(\omega) - f_2(\omega)|\mathbb{P}(d\omega)ds \\
&= Md^2\mathbb{E}R \cdot \mathbb{E}|f_1 - f_2|\int_{-\infty}^0 \mathrm{e}^{\lambda s}ds \\
&= \frac{Md^2\|R\|_{\mathcal{L}^1}}{\lambda}\|f_1 - f_2\|_{\mathcal{L}^1},
\end{aligned}$$

where the third-to-last equality holds because of $\theta_t\mathbb{P} = \mathbb{P}$, $t \in \mathbb{R}$, while the second-to-last equality has used the independence between R and $\mathscr{F}_-$. The proof is complete. ∎

Suppose that there exists a random equilibrium $v(\omega)$ such that $\lim_{t\to\infty}\varphi(t,\theta_{-t}\omega)x = v(\omega)$. Then v must be $\mathscr{F}_-$-measurable. Thus, the most right choice is to choose input space to be a subspace of $\mathscr{F}_-$-measurable space. Measurability with respect to $(\mathscr{B}(\mathbb{R}_-)\otimes\mathscr{F}_-, \mathscr{F}_-)$ for $\theta : \mathbb{R}_- \times \Omega \mapsto \Omega$ (see Proposition 3.3) makes us obtain $\mathscr{F}_-$-measurability for the input-to-state characteristic operator $\mathcal{K}(u)$ if u is tempered and $\mathscr{F}_-$-measurable (see Proposition 3.4). By choosing $\mathcal{L}^1_{\mathscr{F}_-}$ to be the input space, the operator $\mathcal{K}^h : \mathcal{L}^1_{\mathscr{F}_-} \to \mathcal{L}^1_{\mathscr{F}_-}$ is well defined. With the help of the condition (H_2) and the independence between R and the past σ-algebra $\mathscr{F}_-$, we get the contraction for the operator $\mathcal{K}^h$. Let u be the fixed point of the operator $\mathcal{K}^h$. Then the image of the input-to-state characteristic operator $\mathcal{K}(u)$ for this fixed point will be a globally attracting positive random equilibrium for the pull-back flow of (2.1), which will be confirmed in the following global stability theorem.

THEOREM 4.2 (global stability theorem I). *Let assumptions* (A)*,* (L)*,* (H_1)*,* (H_2)*, and* (R) *hold. Then there exists a unique fixed point $u \in \mathcal{L}^1(\Omega, \mathscr{F}_-, \mathbb{P}; [0,\Gamma])$ for the operator $\mathcal{K}^h$ such that for any $x \in \mathbb{R}^d_+$*

$$\lim_{t\to\infty}\varphi(t,\theta_{-t}\omega)x = [\mathcal{K}(u)](\omega) \quad \mathbb{P}\text{-}a.s. \tag{4.1}$$

Furthermore, we have $\varphi(t,\omega)[\mathcal{K}(u)](\omega) = [\mathcal{K}(u)](\theta_t\omega)$, $\mathbb{P}$-a.s., $t > 0$, i.e., $[\mathcal{K}(u)](\cdot)$ is a globally attracting positive random equilibrium in $\mathbb{R}^d_+$.

Proof. Fix $\tau \ge 0$. Whenever h is monotone or antimonotone, for convenience, by Lemma 3.9, it follows immediately that

$$(\mathcal{K}^h)^{2k}(\xi^h_\tau) \le \theta - \underline{\lim}\, h(\varphi) \le \theta - \overline{\lim}\, h(\varphi) \le (\mathcal{K}^h)^{2k}(\eta^h_\tau) \quad \text{for all } \omega\in\Omega \text{ and } k\in\mathbb{N}. \tag{4.2}$$

Clearly, ξ^h_τ and η^h_τ are bounded $\mathscr{F}_-$-measurable variables in $\mathcal{L}^1(\Omega, \mathscr{F}_-, \mathbb{P}; [0,\Gamma])$ by Proposition 3.5. Combining the Banach fixed point theorem [41] and Lemma 4.1, there exists a unique random variable $u \in \mathcal{L}^1(\Omega, \mathscr{F}_-, \mathbb{P}; [0, \Gamma])$ such that

$$[\mathcal{K}^h(u)](\omega) = u(\omega) \quad \mathbb{P}\text{-a.s.}$$

and

$$\lim_{k\to\infty} \mathbb{E}\left|(\mathcal{K}^h)^{2k}(\xi_\tau^h) - u\right| = \lim_{k\to\infty} \mathbb{E}\left|(\mathcal{K}^h)^{2k}(\eta_\tau^h) - u\right| = 0, \tag{4.3}$$

i.e., $(\mathcal{K}^h)^{2k}(\xi_\tau^h) \xrightarrow{\mathcal{L}^1} u$ and $(\mathcal{K}^h)^{2k}(\eta_\tau^h) \xrightarrow{\mathcal{L}^1} u$, which implies that $(\mathcal{K}^h)^{2k}(\xi_\tau^h) \xrightarrow{\mathbb{P}} u$ and $(\mathcal{K}^h)^{2k}(\eta_\tau^h) \xrightarrow{\mathbb{P}} u$. Therefore, for any $x \in \mathbb{R}_+^d$, there exists a subsequence $\{k_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} [(\mathcal{K}^h)^{2k_j}(\xi_\tau^h)](\omega) = u(\omega) = \lim_{j\to\infty} [(\mathcal{K}^h)^{2k_j}(\eta_\tau^h)](\omega) \quad \mathbb{P}\text{-a.s.} \tag{4.4}$$

The proof of (4.1) follows by the similar arguments as in Theorem 4.2 in [21]. Finally, we show that $[\mathcal{K}(u)](\omega) > 0$ for all $\omega \in \Omega$. Combining the fact that $u(\omega) = [\mathcal{K}^h(u)](\omega) > 0$ for all $\omega \in \Omega$ and Proposition 6.2.2 in [6], it follows that $[\mathcal{K}(u)](\omega) > 0$ for all $\omega \in \Omega$. The proof is complete. ∎

Remark 2. If $h \in C^1(\mathbb{R}_+^d, \text{int}\mathbb{R}_+^d)$, by Corollary 6.3.1 in [6], it is clear that $\varphi(t,\omega)(\mathbb{R}_+^d \setminus \{0\}) \subset \text{int}\mathbb{R}_+^d$. This implies that $\varphi(t, \theta_{-t}\omega)[\mathcal{K}(u)](\theta_{-t}\omega) = [\mathcal{K}(u)](\omega) \gg 0$, $\omega \in \Omega$, $t > 0$. Here, $x \gg y$ means that $x - y \in \text{int}\mathbb{R}_+^n$ for all $x, y \in \mathbb{R}^n$.

Remark 3. Assume that $h \in C^1(\mathbb{R}_+^d, \mathbb{R}_+^d)$ and $h(0) = 0$. Then the origin is an equilibrium for (2.1). Theorem 4.2 still holds in this case, that is, if all conditions in Theorem 4.2 are satisfied except $h(x) \neq 0$, then the origin is globally attracting. The proof is the same.

Usually, almost all feedback systems in gene regulation mainly focus on the existence and global stability of nontrivial positive stationary motion, such as the positive equilibrium or closed orbit, rather than the trivial solution zero. For these kinds of positive and negative feedback systems perturbed by multiplicative white noise, researchers are mainly interested in the existence and global stability of nontrivial positive stationary solution. In what follows, we will show the efficiency of our result. Our Theorem 4.2 works for the stochastic Goodwin negative feedback system, Othmer–Tyson positive feedback system, and Griffith positive feedback system as well as other stochastic competitive systems with multiplicative noise. Our main task is to check the conditions (L) and (R) in order to use Theorem 4.2; in other words, we need to choose a suitable $\lambda > 0$ and $\mathscr{F}_+$-measurable random variable $R \in \mathcal{L}^1(\Omega, \mathscr{F}_+, \mathbb{P}; \mathbb{R}_+)$ such that (H_2) holds. During this process, the key point is to estimate the expectation of R.

Now we consider stochastic single loop feedback system

$$\begin{cases} dx_1 = [-\alpha_1 x_1 + f(x_n)]dt + \sigma_1 x_1 dW_t^1, \\ dx_i = [x_{i-1} - \alpha_i x_i]dt + \sigma_i x_i dW_t^i, \quad 2 \le i \le n, \end{cases} \tag{4.5}$$

where $\alpha_i > 0$ for $i = 1, \dots, n$ and $f \in C_b^1(\mathbb{R}_+, \mathbb{R}_+)$, i.e., f and its derivative are both bounded in $\mathbb{R}_+$. Moreover, we assume that f is increasing or decreasing in $\mathbb{R}_+$. The corresponding linear homogeneous stochastic Itô type differential equations is

$$\begin{cases} dx_1 = -\alpha_1 x_1 dt + \sigma_1 x_1 dW_t^1, \\ dx_i = [x_{i-1} - \alpha_i x_i]dt + \sigma_i x_i dW_t^i, \quad 2 \le i \le n. \end{cases} \tag{4.6}$$

By the variation-of-constants formula, we can easily calculate the fundamental matrix $\Phi(t,\omega)$ of (4.6) as follows.

$$\Phi(t,\omega) = \begin{bmatrix} \Phi_{11}(t,\omega) & 0 & \cdots & 0 \\ \Phi_{21}(t,\omega) & \Phi_{22}(t,\omega) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n1}(t,\omega) & \Phi_{n2}(t,\omega) & \cdots & \Phi_{nn}(t,\omega) \end{bmatrix} \tag{4.7}$$

for all $t \geq 0$ and $\omega \in \Omega$, where

$$\Phi_{ii}(t,\omega) = e^{-(\alpha_i + \frac{1}{2}\sigma_i^2)t + \sigma_i W_t^i(\omega)}, \tag{4.8}$$

$$\Phi_{ij}(t,\omega) = \Phi_{ii}(t,\omega) \int_0^t \Phi_{ii}^{-1}(s,\omega)\Phi_{i-1,j}(s,\omega)ds, \quad 1 \leq j \leq i-1, \tag{4.9}$$

for all $i = 1, \dots, n$. Let $\lambda = \frac{1}{n+1}\min\{\alpha_1, \dots, \alpha_n\}$. Then it is easy to check that

$$\begin{aligned} \Phi_{ii}(t,\omega) &\leq e^{-[(n+1)\lambda + \frac{1}{2}\sigma_i^2]t + \sigma_i W_t^i(\omega)} \\ &= e^{-(i\lambda + \frac{1}{2}\sigma_i^2)t + \sigma_i W_t^i(\omega)} e^{-(n+1-i)\lambda t} \\ &\leq R_i(\omega) e^{-(n+1-i)\lambda t} \end{aligned} \tag{4.10}$$

for all $t \geq 0$ and $\omega \in \Omega$, where

$$R_i(\omega) = \sup_{t \geq 0} \exp\left(-(i\lambda + \frac{1}{2}\sigma_i^2)t + \sigma_i W_t^i(\omega)\right), \qquad i = 1, \dots, n. \tag{4.11}$$

Next, we show that $R_i(\omega)$ is a tempered random variable for all $i = 1, \dots, n$. For any $\omega \in \Omega$ and $\gamma > 0$, by (4.11), it follows that

$$\begin{aligned} &\sup_{\tau \in \mathbb{R}} \left\{ e^{-\gamma|\tau|} |R_i(\theta_\tau \omega)| \right\} \\ &= \sup_{\tau \in \mathbb{R}} \left\{ \sup_{t \geq 0} \exp\left(-\gamma|\tau| - \left(i\lambda + \frac{1}{2}\sigma_i^2 \right) t + \sigma_i W_{t+\tau}^i(\omega) - \sigma_i W_\tau^i(\omega) \right) \right\} \\ &\leq \sup_{\tau \in \mathbb{R}} \left\{ \sup_{t \geq 0} \exp\left(-\frac{\gamma \wedge \beta_i}{2}|t+\tau| + \sigma_i W_{t+\tau}^i(\omega) \right) \exp\left(-\frac{\gamma \wedge \beta_i}{2}|\tau| - \sigma_i W_\tau^i(\omega) \right) \right\} \\ &\leq \sup_{t \in \mathbb{R}} \exp\left(-\frac{\gamma \wedge \beta_i}{2}|t| + \sigma_i W_t^i(\omega) \right) \sup_{\tau \in \mathbb{R}} \exp\left(-\frac{\gamma \wedge \beta_i}{2}|\tau| - \sigma_i W_\tau^i(\omega) \right) \\ &< \infty, \end{aligned}$$

where $\beta_i = i\lambda + \frac{1}{2}\sigma_i^2$ for all $i = 1, \dots, n$ and the last inequality holds because of the law of the iterated logarithm of Brownian motion. In what follows, we claim that

$$\Phi_{ij}(t,\omega) \leq R_j(\omega)\widetilde{R}_{j+1}(\omega) \cdots \widetilde{R}_i(\omega) e^{-(n+1-i)\lambda t} \tag{4.12}$$

for all $1 \leq j \leq i-1$, where

$$\widetilde{R}_i(\omega) = \int_0^\infty R_i(\theta_s \omega) e^{-\lambda s} ds \tag{4.13}$$

is also a tempered random variable for all $i = 1, \dots, n$. Indeed, given any $\omega \in \Omega$ and $\gamma > 0$, from (4.13), we can see that

$$\begin{aligned}\sup_{\tau\in\mathbb{R}}\left\{e^{-\gamma|\tau|}\left|\widetilde{R}_i(\theta_\tau\omega)\right|\right\} &= \sup_{\tau\in\mathbb{R}}\left\{e^{-\gamma|\tau|}\int_0^\infty e^{-\lambda s}R_i(\theta_{s+\tau}\omega)ds\right\}\\ &\le \sup_{\tau\in\mathbb{R}}\left\{\int_0^\infty e^{-\frac{\lambda s}{2}}e^{-\frac{\lambda\wedge\gamma}{2}|s+\tau|}R_i(\theta_{s+\tau}\omega)ds\right\}\\ &\le \sup_{\tau\in\mathbb{R}}\left\{e^{-\frac{\lambda\wedge\gamma}{2}|\tau|}\left|R_i(\theta_\tau\omega)\right|\right\}\int_0^\infty e^{-\frac{\lambda s}{2}}ds\\ &< \infty,\end{aligned}$$

where the last inequality holds because of the temperedness of $R_i(\omega)$ for all $i = 1, \dots, n$. In order to check (4.12), we only present the proof of $\Phi_{21}(t,\omega)$ and $\Phi_{31}(t,\omega)$; the rest can be analogously completed by induction. Combining (4.8), (4.9), (4.10), and (4.11), it is clear that

$$\begin{aligned}\Phi_{21}(t,\omega) &= \int_0^t \Phi_{22}(t-s,\theta_s\omega)\Phi_{11}(s,\omega)ds\\ &\le \int_0^t e^{-[(n+1)\lambda+\frac{1}{2}\sigma_2^2](t-s)+\sigma_2 W^2_{t-s}(\theta_s\omega)}R_1(\omega)e^{-n\lambda s}ds\\ &\le \int_0^t R_2(\theta_s\omega)e^{-(n-1)\lambda(t-s)}R_1(\omega)e^{-n\lambda s}ds\\ &\le e^{-(n-1)\lambda t}R_1(\omega)\int_0^\infty R_2(\theta_s\omega)e^{-\lambda s}ds\\ &= R_1(\omega)\widetilde{R}_2(\omega)e^{-(n-1)\lambda t}\end{aligned}$$

and

$$\begin{aligned}\Phi_{31}(t,\omega) &= \int_0^t \Phi_{33}(t-s,\theta_s\omega)\Phi_{21}(s,\omega)ds\\ &\le \int_0^t e^{-[(n+1)\lambda+\frac{1}{2}\sigma_3^2](t-s)+\sigma_3 W^3_{t-s}(\theta_s\omega)}R_1(\omega)\widetilde{R}_2(\omega)e^{-(n-1)\lambda s}ds\\ &\le \int_0^t R_3(\theta_s\omega)e^{-(n-2)\lambda(t-s)}R_1(\omega)\widetilde{R}_2(\omega)e^{-(n-1)\lambda s}ds\\ &\le e^{-(n-2)\lambda t}R_1(\omega)\widetilde{R}_2(\omega)\int_0^\infty R_3(\theta_s\omega)e^{-\lambda s}ds\\ &= R_1(\omega)\widetilde{R}_2(\omega)\widetilde{R}_3(\omega)e^{-(n-2)\lambda t}\end{aligned}$$

for all $t \ge 0$ and $\omega \in \Omega$. Furthermore, we note that for all $\omega \in \Omega$, $R_i(\omega) \ge 1$, which implies that $\widetilde{R}_i(\omega) \ge \frac{1}{\lambda}$ and

$$\begin{aligned}\Phi_{ij}(t,\omega) &\le R_j(\omega)\widetilde{R}_{j+1}(\omega)\cdots\widetilde{R}_i(\omega)e^{-(n+1-i)\lambda t}\\ &\le \lambda^{n-i}R_j(\omega)\widetilde{R}_{j+1}(\omega)\cdots\widetilde{R}_i(\omega)\widetilde{R}_{i+1}(\omega)\cdots\widetilde{R}_n(\omega)e^{-\lambda t}\\ &\le \max\{1,\lambda^{n-1}\}R_j(\omega)\widetilde{R}_{j+1}(\omega)\cdots\widetilde{R}_i(\omega)\widetilde{R}_{i+1}(\omega)\cdots\widetilde{R}_n(\omega)e^{-\lambda t}\end{aligned}$$

for all $1 \le j \le i-1$, $i = 1, \dots, n$. Let

$$R(\omega) := \max\{1,\lambda^{n-1}\}\bigvee_{j=1}^n R_j(\omega)\widetilde{R}_{j+1}(\omega)\cdots\widetilde{R}_n(\omega), \tag{4.14}$$

which yields that $R(\omega)$ is tempered and

$$\|\Phi(t,\omega)\| := \max\{|\Phi_{ij}(t,\omega)| : i,j=1,\dots,n\} \le R(\omega)e^{-\lambda t}, \quad t\ge 0,\ \omega\in\Omega.$$

Next, we will show that $R \in \mathcal{L}^1(\Omega, \mathscr{F}_+, \mathbb{P}; \mathbb{R}_+)$. In fact, for all $i=1,\dots,n$, it is obvious that R_i is $\mathscr{A}_i$-measurable, where $\mathscr{A}_i = \sigma\{\omega \mapsto W_t^i(\omega) : t\ge 0\}$. Consequently, R_i is $\mathscr{F}_+$-measurable for all $i=1,\dots,n$. Furthermore, for fixed $t\ge 0$, it is obvious that $e^{-(i\lambda+\frac{1}{2}\sigma_i^2)t+\sigma_i(W_{t+\cdot}^i(\omega)-W_\cdot^i(\omega))}$ is continuous for all $\omega\in\Omega$ and $e^{-(i\lambda+\frac{1}{2}\sigma_i^2)t+\sigma_i(W_{t+s}^i(\cdot)-W_s^i(\cdot))}$ is $\mathscr{A}_i$-measurable for all $s\ge 0$ and $i=1,\dots,n$, which together with Lemma 3.14 in [2] implies that for any $t\ge 0$

$$(s,\omega)\mapsto e^{-(i\lambda+\frac{1}{2}\sigma_i^2)t+\sigma_i\left(W_{t+s}^i(\omega)-W_s^i(\omega)\right)}, \quad s\ge 0,\ \omega\in\Omega,$$

is $(\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{A}_i, \mathscr{B}(\mathbb{R}_+))$-measurable. Then, by (4.11), it follows that

$$(s,\omega)\mapsto R_i(\theta_s\omega) = \sup_{t\ge 0} e^{-(i\lambda+\frac{1}{2}\sigma_i^2)t+\sigma_i\left(W_{t+s}^i(\omega)-W_s^i(\omega)\right)}, \quad s\ge 0,\ \omega\in\Omega$$

is also $(\mathscr{B}(\mathbb{R}_+)\otimes\mathscr{A}_i, \mathscr{B}(\mathbb{R}_+))$-measurable. Combining this and Fubini's theorem, it is clear that $\widetilde{R}_i(\omega)$ is $\mathscr{A}_i$-measurable, and so is $\mathscr{F}_+$-measurable for all $i=1,\dots,n$. The above analysis implies that R is $\mathscr{F}_+$-measurable. Now, we will prove that R is $\mathcal{L}^1$-integrable. Combining the fact that an n-dimensional Brownian motion has n independent components, (4.13) and (4.14), it follows that

$$\begin{aligned}\mathbb{E}R &\le \max\{1,\lambda^{n-1}\}\sum_{j=1}^n \mathbb{E}\{R_j\widetilde{R}_{j+1}\cdots\widetilde{R}_n\}\\ &= \max\{1,\lambda^{n-1}\}\sum_{i=1}^n \frac{1}{\lambda^{n-i}}\prod_{j=i}^n \mathbb{E}R_j\\ &= \max\{1,\lambda^{n-1}\}\sum_{i=1}^n \frac{1}{\lambda^{n-i}}\prod_{j=i}^n\left(1+\frac{\sigma_j^2}{2j\lambda}\right),\end{aligned} \tag{4.15}$$

where the last equality holds because of the property of geometric Brownian motion, i.e., $\mathbb{E}\sup_{t\ge 0}\exp\left(-(\mu+\frac{1}{2}\sigma^2)t+\sigma W_t(\omega)\right) = 1+\frac{\sigma^2}{2\mu}$, where $\mu>0$ and $\sigma\in\mathbb{R}$; see [17, p. 585] and [34, p. 1639].

Let $h(x) = (f(x_n),0,\dots,0)^T$, $x\in\mathbb{R}_+^n$, $\Gamma_1 = \sup_{x_n\in\mathbb{R}_+}|f(x_n)|$, $\Gamma_i = 0$ for all $2\le i\le n$ and $M = \sup_{x_n\in\mathbb{R}_+}\frac{\mathrm{d}f(x_n)}{\mathrm{d}x_n}$. Then employing Theorem 4.2 and Remark 3, we conclude the following.

Corollary 4.3. *Let $\alpha_i>0$ for $i=1,\dots,n$ and $f\in C_b^1(\mathbb{R}_+,\mathbb{R}_+)$. Assume that f is increasing or decreasing in $\mathbb{R}_+$. If*

$$\frac{Mn^2\|R\|_{\mathcal{L}^1}}{\lambda} \le \frac{Mn^2}{\lambda}\max\{1,\lambda^{n-1}\}\sum_{i=1}^n\frac{1}{\lambda^{n-i}}\prod_{j=i}^n\left(1+\frac{\sigma_j^2}{2j\lambda}\right) < 1 \tag{4.16}$$

holds, then the stochastic single loop feedback system (4.5) *admits a unique globally attracting random equilibrium in $\mathbb{R}_+^n$.*

Example 4.1 (stochastic Goodwin system). Consider n-dimensional *stochastic Goodwin negative feedback system*

$$\begin{cases} dx_1 = [-\alpha_1 x_1 + \frac{V}{K+x_n^m}]dt + \sigma_1 x_1 dW_t^1,\\ dx_i = [x_{i-1}-\alpha_i x_i]dt + \sigma_i x_i dW_t^i, \quad 2\le i\le n,\end{cases} \tag{4.17}$$

where $m > 1$, $K > 1$, $V > 0$, and $\alpha_i > 0$ for $i = 1, \dots, n$. It is clear that (4.17) is a nonmonotone stochastic system, which can be regarded as the stochastic Goodwin model; see [15, 19]. Moreover, an easy computation shows that

$$M = \sup_{x_n \in \mathbb{R}_+} \left\{ \frac{mVx_n^{m-1}}{(K + x_n^m)^2} \right\} \le \sup_{x_n \in \mathbb{R}_+} \left\{ \frac{mV(1 + x_n^m)}{(K + x_n^m)^2} \right\} \le \frac{mV}{K}.$$

Applying Corollary 4.3, we get that if

$$\frac{mn^2V}{\lambda K} \max\{1, \lambda^{n-1}\} \sum_{i=1}^{n} \frac{1}{\lambda^{n-i}} \prod_{j=i}^{n} \left(1 + \frac{\sigma_j^2}{2j\lambda}\right) < 1 \tag{4.18}$$

is satisfied, then stochastic Goodwin negative feedback system (4.17) possesses a globally attracting nontrivial random equilibrium in $\mathbb{R}_+^n$. Here, (4.18) holds for V sufficiently small or K sufficiently large. Moreover, we can have that the unique random equilibrium $\mathcal{K}(u)$ is strongly positive, i.e., $[\mathcal{K}(u)](\omega) \gg 0$ for all $\omega \in \Omega$. Noting that $f(x_n) = \frac{V}{K+x_n^m} > 0$ for all $x_n \in \mathbb{R}_+$, it follows that $u(\omega) = [\mathcal{K}^h(u)](\omega) = (u_1(\omega), 0, \dots, 0)$ and $u_1(\omega) > 0$ for all $\omega \in \Omega$. Combining (4.7), (4.8), and (4.9), it is clear that given any $t > 0$, $\Phi_{i1}(t, \omega) > 0$ for all $i = 1, \dots, n$ and $\omega \in \Omega$. This together with the definition of $\mathcal{K}$ implies that $[\mathcal{K}(u)](\omega) \gg 0$ for all $\omega \in \Omega$.

Example 4.2 (stochastic Othmer–Tyson system). Consider the following n-dimensional stochastic Othmer–Tyson positive feedback system:

$$\begin{cases} dx_1 = [-\alpha_1 x_1 + \frac{k_0(1+x_n^m)}{K+x_n^m}]dt + \sigma_1 x_1 dW_t^1, \\ dx_i = [x_{i-1} - \alpha_i x_i]dt + \sigma_i x_i dW_t^i, \quad 2 \le i \le n, \end{cases} \tag{4.19}$$

where $k_0 > 0$, $K > 1$, $m > 1$, and $\alpha_i > 0$ for $i = 1, \dots, n$. This model can be found in [33, 38], which is a stochastic cooperative system. By the direct calculation, it is obvious that

$$M = \sup_{x_n \in \mathbb{R}_+} \left\{ \frac{mk_0(K-1)x_n^{m-1}}{(K + x_n^m)^2} \right\} \le \sup_{x_n \in \mathbb{R}_+} \left\{ \frac{mk_0(K-1)(1 + x_n^m)}{(K + x_n^m)^2} \right\} \le \frac{mk_0(K-1)}{K}.$$

As long as the condition (H_2)

$$\frac{mk_0n^2(K-1)}{\lambda K} \max\{1, \lambda^{n-1}\} \sum_{i=1}^{n} \frac{1}{\lambda^{n-i}} \prod_{j=i}^{n} \left(1 + \frac{\sigma_j^2}{2j\lambda}\right) < 1 \tag{4.20}$$

holds, stochastic Othmer–Tyson positive feedback system (4.19) possesses a unique globally attracting nontrivial random equilibrium in $\mathbb{R}_+^n$ for pull-back flow by Corollary 4.3. It is easy to see that the condition (4.20) is true for k_0 small enough. Furthermore, the strong positivity of the unique random equilibrium $\mathcal{K}(u)$ can be obtained by the same argument in Example 4.1.

Example 4.3 (stochastic Griffith system). Next, we study the following n-dimensional *stochastic Griffith positive feedback system*:

$$\begin{cases} dx_1 = [-\alpha_1 x_1 + \frac{Kx_n^m}{1+Kx_n^m}]dt + \sigma_1 x_1 dW_t^1, \\ dx_i = [x_{i-1} - \alpha_i x_i]dt + \sigma_i x_i dW_t^i, \quad 2 \le i \le n, \end{cases} \tag{4.21}$$

where $m > 1$, $K > 0$ and $\alpha_i > 0$ for $i = 1, \dots, n$; see [18]. An easy computation shows that

$$M = \sup_{x_n \in \mathbb{R}_+} \left\{ \frac{mKx_n^{m-1}}{(1 + Kx_n^m)^2} \right\} = \frac{mKx_n^{m-1}}{(1 + Kx_n^m)^2} \bigg|_{x_n^m = \frac{m-1}{K(m+1)}} = \frac{\sqrt[m]{K}(m+1)^2}{4m} \left(\frac{m-1}{m+1} \right)^{\frac{m-1}{m}}.$$

If K is small enough, the condition (H$_2$)

$$\frac{\sqrt[m]{K}n^2(m+1)^2}{4m\lambda} \left(\frac{m-1}{m+1} \right)^{\frac{m-1}{m}} \max\{1, \lambda^{n-1}\} \sum_{i=1}^{n} \frac{1}{\lambda^{n-i}} \prod_{j=i}^{n} \left(1 + \frac{\sigma_j^2}{2j\lambda} \right) < 1 \tag{4.22}$$

holds. Using Corollary 4.3 and Remark 3, it follows that the zero solution is the unique globally attracting random equilibrium in $\mathbb{R}_+^n$ for stochastic Griffith positive feedback system (4.21).

Example 4.4. We consider an n-dimensional *stochastic competitive system*

$$dx_i = [-\alpha_i x_i + h_i(x)]dt + \sigma_i x_i dW_t^i, \tag{4.23}$$

where $\alpha_i > 0$ for all $i = 1, \dots, n$ and

$$h_i(x) := \frac{1}{K_i + x_1^m + \cdots + x_n^m}, \qquad x \in \mathbb{R}_+^n,\ i = 1, \dots, n, \tag{4.24}$$

where $m > 1$ and $K_i > 1$ for all $i = 1, \dots, n$. Then, h is a C^1-decreasing function from $\mathbb{R}_+^n$ to $\mathbb{R}_+^n \backslash \{0\}$. It follows immediately that (4.23) is a stochastic competitive system. By the direct computation, we obtain

$$\Phi(t, \omega) = \mathrm{diag}\,[\Phi_{11}(t, \omega), \dots, \Phi_{nn}(t, \omega)] \tag{4.25}$$

for all $t \geq 0$ and $\omega \in \Omega$, where

$$\Phi_{ii}(t, \omega) = e^{-(\alpha_i + \frac{1}{2}\sigma_i^2)t + \sigma_i W_t^i(\omega)}. \tag{4.26}$$

Consequently, it is evident that

$$\|\Phi(t, \omega)\| := \max\{|\Phi_{ij}(t, \omega)| : i, j = 1, \dots, n\} \leq R(\omega)e^{-\lambda t}, \quad t \geq 0,\ \omega \in \Omega, \tag{4.27}$$

where $\lambda = \frac{1}{2} \min\{\alpha_1, \dots, \alpha_n\}$ and

$$R(\omega) = \bigvee_{i=1}^{n} \sup_{t \geq 0} \exp\left(-\left(\lambda + \frac{1}{2}\sigma_i^2 \right) t + \sigma_i W_t^i(\omega) \right). \tag{4.28}$$

It follows that $R(\omega)$ is $\mathscr{F}_+$-measurable and independent of $\mathscr{F}_-$. Similar to the analysis for system (4.5), we claim that $R(\omega)$ is tempered and belongs to $\mathcal{L}^1(\Omega, \mathscr{F}_+, \mathbb{P}; \mathbb{R}_+)$. In fact, a simple calculation shows that

$$\mathbb{E}R \leq \sum_{i=1}^{n} \mathbb{E} \sup_{t \geq 0} \exp\left(-\left(\lambda + \frac{1}{2}\sigma_i^2 \right) t + \sigma_i W_t^i(\omega) \right) = \sum_{i=1}^{n} \left(1 + \frac{\sigma_i^2}{2\lambda} \right).$$

Furthermore, we see that

$$M = \max\left\{ \sup_{x \in \mathbb{R}_+^n} \frac{mx_j^{m-1}}{(K_i + x_1^m + \cdots + x_n^m)^2}, i, j = 1, \dots, n \right\} \leq \frac{m}{K_i} \leq \frac{m}{K},$$

where $K = \min\{K_i, i = 1, \dots, n\}$. Therefore, the condition (H_2) is

$$\frac{Mn^2\|R\|_{\mathcal{L}^1}}{\lambda} \leq \frac{mn^2}{\lambda K}\sum_{i=1}^{n}\left(1+\frac{\sigma_i^2}{2\lambda}\right) < 1,$$

which must hold when λ or K is large enough. According to Theorem 4.2 and Remark 2, stochastic competitive system (4.23) admits a unique globally asymptotically stable positive random equilibrium in $\mathbb{R}_+^n$, to which all the pull-back trajectories of (4.23) converge.

Remark 4. Observe that the condition (H_2) involves in the expectation of the random variable $R(\omega)$. So we need to provide the exact representation of $R(\omega)$ in the definition (L) of the top Lyapunov exponent. We point out that the choice of λ and R in the condition (L) is not unique. Usually, the bigger λ makes $\mathbb{E}R$ bigger. As for the linear homogeneous stochastic differential equations (4.6), we choose $\lambda = \frac{1}{n+1}\min\{\alpha_1, \dots, \alpha_n\}$ and R as (4.14) whose estimation is given in (4.15). We note that during this process we lose a lot. For a concrete example, even if the condition (4.16) fails, we can trace our idea by choosing other λ and R such that Theorem 4.2 still holds, which is shown in the following three-dimensional *stochastic Othmer–Tyson positive feedback system*

$$\begin{cases} dx_1 = [-8x_1 + \frac{1}{6}\cdot\frac{1+x_3^3}{\frac{4}{3}+x_3^3}]dt + \frac{1}{2}x_1 dW_t^1, \\ dx_2 = [x_1 - 9x_2]dt + \frac{1}{4}x_2 dW_t^2, \\ dx_3 = [x_2 - 10x_3]dt + \frac{1}{3}x_3 dW_t^3. \end{cases} \tag{4.29}$$

In fact, an easy computation shows that

$$M = \sup_{x_n\in\mathbb{R}_+}\left\{\frac{mk_0(K-1)x_n^{m-1}}{(K+x_n^m)^2}\right\} = \frac{x_n^2}{6(\frac{4}{3}+x_n^3)^2}\bigg|_{x_n^3=\frac{2}{3}} = \frac{1}{24}\left(\frac{2}{3}\right)^{2/3}.$$

In Corollary 4.3, $\lambda = 2$ in this case. Thus

$$\max\{1,\lambda^2\}\sum_{i=1}^{3}\frac{1}{\lambda^{3-i}}\prod_{j=i}^{3}\left(1+\frac{\sigma_j^2}{2j\lambda}\right)\Big|_{\lambda=2} \geq 2^2+2+1 = 7,$$

which implies that $\frac{1}{24}\left(\frac{2}{3}\right)^{2/3}\times\frac{9}{2}\times 7 > 1$. That is, the condition (4.16) does not work. In what follows, we will prove that the condition (4.16) can hold by changing the choice of λ and R suitably. It is clear that $\Phi_{11}(t,\omega) = e^{(-8-\frac{1}{8})t+\frac{1}{2}W_t^1(\omega)}$, $\Phi_{22}(t,\omega) = e^{(-9-\frac{1}{32})t+\frac{1}{4}W_t^2(\omega)}$, $\Phi_{33}(t,\omega) = e^{(-10-\frac{1}{18})t+\frac{1}{3}W_t^3(\omega)}$, and

$$\Phi_{21}(t,\omega) = \int_0^t e^{(-9-\frac{1}{32})(t-s)+\frac{1}{4}\left(W_t^2(\omega)-W_s^2(\omega)\right)}\Phi_{11}(s,\omega)ds,$$

$$\Phi_{3i}(t,\omega) = \int_0^t e^{(-10-\frac{1}{18})(t-s)+\frac{1}{3}\left(W_t^3(\omega)-W_s^3(\omega)\right)}\Phi_{2i}(s,\omega)ds, \qquad i=1,2.$$

Hence, it is easy to check that

$$\Phi_{ii}(t,\omega) \leq R_i(\omega)e^{-(4-i)t}, \quad i=1,2,3, \tag{4.30}$$

for all $t \geq 0$ and $\omega \in \Omega$, where

$$R_1(\omega) = \sup_{t\geq 0} \exp\left(\left(-5-\frac{1}{8}\right)t + \frac{1}{2}W_t^1(\omega)\right),$$
$$R_2(\omega) = \sup_{t\geq 0} \exp\left(\left(-7-\frac{1}{32}\right)t + \frac{1}{4}W_t^2(\omega)\right)$$

and

$$R_3(\omega) = \sup_{t\geq 0} \exp\left(\left(-9-\frac{1}{18}\right)t + \frac{1}{3}W_t^3(\omega)\right).$$

It follows that

$$\begin{aligned}\Phi_{21}(t,\omega) &= \int_0^t e^{(-9-\frac{1}{32})(t-s)+\frac{1}{4}W^2_{t-s}(\theta_s\omega)}\Phi_{11}(s,\omega)ds \\ &\leq e^{-2t}R_1(\omega)\int_0^t e^{-s}e^{(-7-\frac{1}{32})(t-s)+\frac{1}{4}W^2_{t-s}(\theta_s\omega)}ds \\ &\leq e^{-2t}R_1(\omega)\int_0^\infty e^{-s}R_2(\theta_s\omega)ds \\ &= e^{-2t}R_1(\omega)\widetilde{R}_2(\omega),\end{aligned}$$

$$\Phi_{31}(t,\omega) \leq e^{-t}R_1(\omega)\widetilde{R}_2(\omega)\int_0^\infty e^{-s}R_3(\theta_s\omega)ds = e^{-t}R_1(\omega)\widetilde{R}_2(\omega)\widetilde{R}_3(\omega),$$

and

$$\Phi_{32}(t,\omega) \leq e^{-t}R_2(\omega)\int_0^\infty e^{-s}R_3(\theta_s\omega)ds = e^{-t}R_2(\omega)\widetilde{R}_3(\omega).$$

In order to verify the condition (H_2), we choose $\lambda = 1$ and

$$\begin{aligned}R(\omega) &= R_1(\omega)\bigvee R_2(\omega)\bigvee R_3(\omega)\bigvee R_1(\omega)\widetilde{R}_2(\omega)\bigvee R_1(\omega)\widetilde{R}_2(\omega)\widetilde{R}_3(\omega)\bigvee R_2(\omega)\widetilde{R}_3(\omega) \\ &= R_3(\omega)\bigvee R_2(\omega)\widetilde{R}_3(\omega)\bigvee R_1(\omega)\widetilde{R}_2(\omega)\widetilde{R}_3(\omega),\end{aligned}$$

where the last equality holds based on the fact that for all $\omega \in \Omega$, $\widetilde{R}_i(\omega) \geq 1$, $i = 2, 3$. Consequently, it is obvious that

$$\mathbb{E}R \leq \mathbb{E}R_3 + \mathbb{E}R_1\cdot\mathbb{E}R_2\cdot\mathbb{E}R_3 + \mathbb{E}R_2\cdot\mathbb{E}R_3 = \frac{163}{162} + \frac{41}{40}\cdot\frac{225}{224}\cdot\frac{163}{162} + \frac{225}{224}\cdot\frac{163}{162} < 3.0528.$$

Therefore, we have that

$$\frac{Mn^2\|R\|_{\mathcal{L}^1}}{\lambda} \leq \frac{1}{24}\left(\frac{2}{3}\right)^{2/3}\times 9\times 3.0528 < 1.$$

That is, the condition (4.16) holds. This reveals that the choice of λ and R plays a key role in the proof of our result.

Remark 5. According to Chueshov [6, p. 221], the stochastic Othmer–Tyson positive feedback system (4.19) with $m = 1$ is sublinear and admits a globally asymptotically attracting positive random equilibrium in $\mathbb{R}^n_+$. As far as we know, all other results in Examples 4.1–4.4 are new.

4.2. Type two: h is uniformly bounded away from zero. In this subsection, we will prove another global stability theorem in the case that h is uniformly bounded away from zero and present some examples. First, we give some notation and preliminaries. Letting V be a real Banach space, a closed subset $V_+ \subset V$ is said to be a *cone* if V_+ is convex and $\alpha V_+ \subset V_+$ for all $\alpha \in \mathbb{R}_+$, and $V_+ \cap (-V_+) = \{0\}$. We denote a partial order on V by $x \le y$ if $y - x \in V_+$, which is compatible with the structure of linear vector space V. A cone V_+ is said to be *solid* if it has nonempty interior points $\mathrm{int}V_+$. A cone V_+ is said to be *normal* if there exists a constant $c > 0$ such that $\|x\| \le c\|y\|$ whenever $0 \le x \le y$. Next, we will introduce definitions of *part* and *part (Birkhoff) metric*.

DEFINITION 4.4 (part (Birkhoff) metric).

(i) *An equivalence relation is defined by $x \sim y$ if there exists $c > 0$ such that $c^{-1}x \le y \le cx$, and then the equivalence classes on the cone V_+ are called the parts of V_+.*

(ii) *Let C be any nonzero part of V_+. Then*

$$p(x, y) := \inf\{\log c : c^{-1}x \le y \le cx\}, \quad x, y \in C, \tag{4.31}$$

is called the part metric (or Birkhoff metric) of C.

It is clear that $\mathrm{int}V_+$ is a part and any part is a cone in V. Let $\mathcal{L}^\infty(\Omega; \mathbb{R}^d) := \mathcal{L}^\infty(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R}^d)$ denote the Banach space of $\mathbb{R}^d$-valued, $\mathscr{F}$-measurable, essentially bounded functions defined on Ω almost surely with the essential supremum norm $\|f\|_\infty := \inf\{B : |f| \le B \ \mathbb{P}\text{-a.s.}\}$. The nonnegative functions in $\mathcal{L}^\infty(\Omega; \mathbb{R}^d)$ form a normal, solid cone $\mathcal{L}^\infty_+(\Omega; \mathbb{R}^d)$; see [23, sections 1.5 and 5.2], where $\mathrm{int}\mathcal{L}^\infty_+(\Omega; \mathbb{R}^d) = \{f : \text{there extis } \epsilon = (\epsilon_1, \dots, \epsilon_d) \in \mathrm{int}\mathbb{R}^d_+ \text{ such that } f \ge \epsilon \ \mathbb{P}\text{-a.s.}\}$, which consists of the family of functions essentially bounded away from zero. To prove our main results, we start with a lemma.

LEMMA 4.5. *Let assumptions* (A), (L), *and* (H_1) *hold. Assume additionally that*

(H_3) *$h : \mathbb{R}^d_+ \to [\delta, \Gamma] \subset \mathrm{int}\mathbb{R}^d_+$, where $\delta = (\delta_1, \dots, \delta_d) \gg 0$. That is, h is uniformly bounded away from zero. Furthermore, we assume that there exists a constant $T > 1$ such that h_T is sublinear, i.e.,*

$$\lambda h_T(x) \le h_T(\lambda x) \quad \text{for all } x \in \mathbb{R}^d_+ \text{ and } \lambda \in [0, 1],$$

where $h_T(x) = h(x) - \frac{1}{T}\delta$, $x \in \mathbb{R}^d_+$, or there exists a constant $S > 1$ such that h_S^{-1} is sublinear, i.e.,

$$\lambda h_S^{-1}(x) \le h_S^{-1}(\lambda x) \quad \text{for all } x \in \mathbb{R}^d_+ \text{ and } \lambda \in [0, 1],$$

where $h_S^{-1}(x) = (\frac{1}{h_1(x)} - \frac{1}{S\Gamma_1}, \dots, \frac{1}{h_d(x)} - \frac{1}{S\Gamma_d})$, $x \in \mathbb{R}^d_+$.
The operator $\mathcal{K}^h = h \circ \mathcal{K} : \mathrm{int}\mathcal{L}^\infty_+(\Omega; \mathbb{R}^d) \to \mathrm{int}\mathcal{L}^\infty_+(\Omega; \mathbb{R}^d)$ is defined by

$$[\mathcal{K}^h(u)](\omega) = \left[h_i\left(\int_{-\infty}^0 \Phi(-s, \theta_s\omega)u(\theta_s\omega)ds\right)\right]_{i=1}^d, \quad u \in \mathrm{int}\mathcal{L}^\infty_+(\Omega; \mathbb{R}^d),$$

where u is the representative such that u is bounded for all $\omega \in \Omega$. Then $\mathrm{int}\mathcal{L}^\infty_+(\Omega; \mathbb{R}^d)$ is complete with respect to the Birkhoff metric p. Moreover, the operator $\mathcal{K}^h = h \circ \mathcal{K} : (\mathrm{int}\mathcal{L}^\infty_+(\Omega; \mathbb{R}^d), p) \to (\mathrm{int}\mathcal{L}^\infty_+(\Omega; \mathbb{R}^d), p)$ is a contraction mapping.

Proof. For any $0 \le x \le y$, $x, y \in \mathcal{L}_+^\infty(\Omega;\mathbb{R}^d)$, it is easily seen that $\|x\|_\infty \le \|y\|_\infty$, which yields that $\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d)$ is a normal cone. Then, it follows that $(\mathrm{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d), p)$ is a complete metric space; see Proposition 3.1.1 in [6] or [23].

Now, we will show that $\mathcal{K}^h = h \circ \mathcal{K} : (\mathrm{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d), p) \to (\mathrm{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d), p)$ is a contraction mapping. By (H$_3$), it is clear that

$$(\Gamma_1, \dots, \Gamma_d) = \Gamma \ge [\mathcal{K}^h(u)](\omega) \ge \delta = (\delta_1, \dots, \delta_d) \quad \text{for all } \omega \in \Omega \text{ and } u \in \mathrm{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d).$$

That is, $\mathcal{K}^h$ is well defined from $\mathrm{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d)$ into itself. In what follows, we will denote by H the mapping

$$H(u) = \left[h_i\left(\int_{-\infty}^0 \Phi(-s, \theta_s\omega)u(\theta_s\omega)ds\right)\right]_{i=1}^d, \quad u \in \mathrm{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d),$$

and let $H^{-1}(u) = \left[\frac{1}{H_i(u)}\right]_{i=1}^d$ for all $u \in \mathrm{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d)$. Given any $u, v \in \mathrm{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d)$, if h_T is sublinear, then there exists a constant $0 \le L_1 = L_1(\delta, \Gamma, T) < 1$ such that

$$\begin{aligned} p\left(H(u), H(v)\right) &= p\left(\frac{\delta}{T} + H(u) - \frac{\delta}{T}, \frac{\delta}{T} + H(v) - \frac{\delta}{T}\right) \\ &\le L_1 p\left(H(u) - \frac{\delta}{T}, H(v) - \frac{\delta}{T}\right); \end{aligned} \tag{4.32}$$

see [14, Lemma 5.2] or [31, Theorem 2.6, p. 59]. Combining the definition of H and the sublinearity of h_T, it is clear that $[H - \frac{\delta}{T}](u) = H(u) - \frac{\delta}{T} : \mathrm{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d) \mapsto \mathrm{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d)$ is also sublinear. That is, $H - \frac{\delta}{T}$ is nonexpansive with respect to the Birkhoff metric p. By (4.31) and (4.32), it follows that

$$p(\mathcal{K}^h(u), \mathcal{K}^h(v)) = p(H(u), H(v)) \le L_1 p\left(H(u) - \frac{\delta}{T}, H(v) - \frac{\delta}{T}\right) \le L_1 p\left(u, v\right)$$

for all $u, v \in \mathrm{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d)$. On the other hand, while h_S^{-1} is sublinear, then there exists a constant $0 \le L_2 = L_2(\delta, \Gamma, S) < 1$ such that

$$\begin{aligned} p\left(H(u), H(v)\right) &= p\left(H^{-1}(u), H^{-1}(v)\right) \\ &= p\left(\frac{1}{S}\Gamma^{-1} + H^{-1}(u) - \frac{1}{S}\Gamma^{-1}, \frac{1}{S}\Gamma^{-1} + H^{-1}(v) - \frac{1}{S}\Gamma^{-1}\right) \\ &\le L_2 p\left(H^{-1}(u) - \frac{1}{S}\Gamma^{-1}, H^{-1}(v) - \frac{1}{S}\Gamma^{-1}\right), \end{aligned} \tag{4.33}$$

where $\Gamma^{-1} = (\frac{1}{\Gamma_1}, \dots, \frac{1}{\Gamma_d})$; see [14, Lemma 5.2] or [31, Theorem 2.6, p. 59]. Since h_S^{-1} is sublinear, this guarantees the sublinearity of $[H^{-1} - \frac{1}{S}\Gamma^{-1}](u) = H^{-1}(u) - \frac{1}{S}\Gamma^{-1} : \mathrm{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d) \mapsto \mathrm{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d)$, i.e., $H^{-1} - \frac{1}{S}\Gamma^{-1}$ is nonexpansive with respect to the Birkhoff metric p. From (4.31) and (4.33), it is easily seen that

$$\begin{aligned} p(\mathcal{K}^h(u), \mathcal{K}^h(v)) = p(H(u), H(v)) &\le L_2 p\left(H^{-1}(u) - \frac{1}{S}\Gamma^{-1}, H^{-1}(v) - \frac{1}{S}\Gamma^{-1}\right) \\ &\le L_2 p\left(u, v\right) \end{aligned}$$

for all $u, v \in \mathrm{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d)$. The proof is complete. ∎

THEOREM 4.6 (global stability theorem II). *Let assumptions* (A), (L), (H_1), *and* (H_3) *hold. Then the operator* $\mathcal{K}^h$ *admits a unique fixed point* $u \in \text{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d)$ *such that for all* $x \in \mathbb{R}_+^d$

$$\lim_{t\to\infty} \varphi(t,\theta_{-t}\omega)x = [\mathcal{K}(u)](\omega) \quad \mathbb{P}\text{-}a.s. \tag{4.34}$$

Moreover, we have $\varphi(t,\omega)[\mathcal{K}(u)](\omega) = [\mathcal{K}(u)](\theta_t\omega)$, $\mathbb{P}$-*a.s.*, $t > 0$, *i.e.*, $[\mathcal{K}(u)](\cdot)$ *is a globally stable strongly positive random equilibrium in* $\mathbb{R}_+^d$.

Proof. Fix $\tau \geq 0$. Without loss of generality, by Lemma 3.9, it is easy to see that

$$[\mathcal{K}^h]^{2k}(\xi_\tau^h) \leq \theta - \underline{\lim}\, h(\varphi) \leq \theta - \overline{\lim}\, h(\varphi) \leq [\mathcal{K}^h]^{2k}(\eta_\tau^h) \quad \text{for all } \omega\in\Omega \text{ and } k \in \mathbb{N}. \tag{4.35}$$

Observe that $h : \mathbb{R}_+^d \to [\delta, \Gamma]$ is uniformly bounded away from zero. This implies that ξ_τ^h and η_τ^h both belong to $\text{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d)$. Using the Banach fixed point theorem [41], by Lemma 4.5, there exists a unique globally attracting fixed point $u \in \text{int}\mathcal{L}_+^\infty(\Omega;\mathbb{R}^d)$ such that $[\mathcal{K}^h(u)](\omega) = u(\omega)$ $\mathbb{P}$-a.s. and

$$\lim_{k\to\infty} p\left([\mathcal{K}^h]^{2k}(\xi_\tau^h), u\right) = \lim_{k\to\infty} p\left([\mathcal{K}^h]^{2k}(\eta_\tau^h), u\right) = 0. \tag{4.36}$$

It is obvious that the norm $\|\cdot\|_\infty$ is monotone, i.e., $0 \leq x \leq y$ implies that $\|x\|_\infty \leq \|y\|_\infty$. Consequently, by Remark 3.1.1 in [6], we have

$$\begin{aligned}\|[\mathcal{K}^h]^{2k}(\xi_\tau^h) - u\|_\infty &\leq \left(2e^{p\left([\mathcal{K}^h]^{2k}(\xi_\tau^h),u\right)} - e^{-p\left([\mathcal{K}^h]^{2k}(\xi_\tau^h),u\right)} - 1\right) \\ &\qquad \cdot \min\{\|[\mathcal{K}^h]^{2k}(\xi_\tau^h)\|_\infty, \|u\|_\infty\} \\ &\leq \left(2e^{p\left([\mathcal{K}^h]^{2k}(\xi_\tau^h),u\right)} - e^{-p\left([\mathcal{K}^h]^{2k}(\xi_\tau^h),u\right)} - 1\right) \cdot \|u\|_\infty \\ &\to 0, \quad \text{as} \quad k\to\infty,\end{aligned}$$

which implies that

$$\lim_{k\to\infty} [(\mathcal{K}^h)^{2k}(\xi_\tau^h)](\omega) = u(\omega) \quad \mathbb{P}\text{-a.s.} \tag{4.37}$$

Applying the same argument, then

$$\lim_{k\to\infty} [(\mathcal{K}^h)^{2k}(\eta_\tau^h)](\omega) = u(\omega) \quad \mathbb{P}\text{-a.s.} \tag{4.38}$$

The remainder of the proof can be handled like that of Theorem 4.2 in [21]. Furthermore, by the fact that h is uniformly bounded away from zero and Remark 2, it is clear that $\mathcal{K}(u)$ is a strongly positive random equilibrium. The proof is complete. ∎

Let us now illustrate Theorem 4.6 by discussing a few examples. In what follows, we will explain that our main results can be applied to stochastic cooperative, competitive and predator-prey systems with multiplicative noise, and other nonmonotone stochastic systems. For the sake of convenience, we only present some third-dimensional stochastic systems here.

Example 4.5. First, we consider the three-dimensional *stochastic cooperative system*

$$dX_t = [AX_t + h(X_t)]dt + \sum_{i=1}^{3} G_i X_t dW_t^i, \tag{4.39}$$

where

$$A = \begin{bmatrix} -1 & 1 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{3} \end{bmatrix}, \ G_1 = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \ G_2 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \ G_3 = 2I_{3\times 3},$$

and

$$h_i(x) := 2 + g_i(x_i), \qquad x \in \mathbb{R}^3_+, \ i = 1,2,3, \tag{4.40}$$

where $g_i(x_i) = \frac{x_i}{1+x_i}$ is a C^1-increasing sublinear function from $\mathbb{R}_+$ to $\mathbb{R}_+$, $i = 1,2,3$. It is clear that (4.39) is a stochastic cooperative system. Let $\delta = (2,2,2)$ and $\Gamma = (3,3,3)$. It is easy to check that $h : \mathbb{R}^3_+ \to [\delta, \Gamma]$ is an order-preserving and bounded function. Moreover, choose $T = 2$, and it is clear that $h_T(x) = h(x) - \frac{1}{2}\delta = \frac{1}{2}\delta + \tilde{g}(x)$ is sublinear, where $\tilde{g}_i(x) = g_i(x_i)$, $i = 1,2,3$. In order to use Theorem 4.6, we need to prove that the top Lyapunov exponent is negative. By Theorem 2.4.4 in [6], it follows that for any $x \in \mathbb{R}^d \setminus \{0\}$, there exists the Lyapunov exponent

$$\lambda(x) := \lim_{t\to+\infty} \frac{1}{t} \log |\Phi(t,\omega)x| \quad \text{for all } \omega \in \Omega^*, \tag{4.41}$$

where Ω^* is a θ-invariant set of full measure. In fact, we can choose the indistinguishable from of $\Phi(t,\omega)$ and extend the existence of Lyapunov exponents to the whole Ω; see Remark 1.2.1 in [6]. Moreover, it is known that $\lambda := \max_{x\in\mathbb{R}^d\setminus\{0\}} \lambda(x)$ is the top Lyapunov exponent; see Theorem 2.4.4 and Definition 1.9.1 in [6]. Therefore, in order to prove (L), it suffices to show that there exists a constant $L_\lambda > 0$ such that

$$\limsup_{t\to+\infty} \frac{1}{t} \log |\Phi(t,\omega)x| \le -L_\lambda \quad \mathbb{P}\text{-a.s.} \tag{4.42}$$

for all $x \in \mathbb{R}^d \setminus \{0\}$. Let us discuss the corresponding linear homogeneous stochastic Itô equations of (4.39), i.e.,

$$dX_t = AX_t dt + \sum_{i=1}^{3} G_i X_t dW_t^i.$$

Hence,

$$|Ax|_2 \le \|A\|_2 |x|_2 = \sqrt{\frac{125}{36}} |x|_2 \le 2|x|_2,$$

$$\sum_{i=1}^{3} |G_i x|_2^2 = \frac{61}{4} x_1^2 + 12x_2^2 + 12x_3^2 \le \frac{61}{4} |x|_2^2$$

and

$$\sum_{i=1}^{3} |x^T G_i x|^2 \ge \frac{9}{4}|x|_2^4 + 4|x|_2^4 + 4|x|_2^4 = \frac{41}{4}|x|_2^4, \quad x \in \mathbb{R}^d.$$

Then by Theorem 5.1 in [28, Chapter 4], it follows easily that

$$\limsup_{t\to+\infty} \frac{1}{t} \log |\Phi(t,\omega)x| \le -\left(\frac{41}{4} - 2 - \frac{61}{8}\right) = -\frac{5}{8} \quad \mathbb{P}\text{-a.s.},$$

which implies that (4.42) holds. By Theorem 4.6, stochastic cooperative system (4.39) admits a unique globally attracting strongly positive random equilibrium in $\mathbb{R}^3_+$ for all pull-back trajectories.

The same conclusion can be obtained if we replace $g_i(x_i)$ by $g_i(x_1 + x_2 + x_3)$ or let $h_i(x) = \frac{1}{2+g_i(x_i)}$, where $g_i(x_i) = \frac{1}{1+x_i}$, $i = 1, 2, 3$.

Example 4.6. Next, we shall study the three-dimensional *stochastic competitive system*

$$dX_t = [AX_t + h(X_t)]dt + \sum_{i=1}^{3} G_i X_t dW_t^i, \tag{4.43}$$

where

$$A = \text{diag}\left[-1, \frac{1}{2}, 1\right],\ \text{G}_1 = \text{diag}\left[1, \frac{3}{2}, 1\right],\ \text{G}_2 = -2\text{I}_{3\times 3},\ \text{G}_3 = \text{diag}\left[-\frac{1}{2}, \frac{1}{4}, \frac{1}{3}\right],$$

and

$$h_i(x) := \frac{1}{1 + g_i(x_{i-1})}, \qquad x \in \mathbb{R}_+^3,\ i = 1, 2, 3, \tag{4.44}$$

where $g_i(x_{i-1}) = \frac{x_{i-1}}{1+x_{i-1}}$ is a C^1-increasing sublinear function from $\mathbb{R}_+$ to $\mathbb{R}_+(x_0 = x_3)$, $i = 1, 2, 3$. It is a simple matter to see that $h : \mathbb{R}_+^3 \to \text{int}\mathbb{R}_+^3$ is an anti-order-preserving and bounded function, which yields that (4.43) is a stochastic competitive system. Furthermore, it is easily seen that $h : \mathbb{R}_+^3 \to [\delta, \Gamma]$, where $\delta = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $\Gamma = (1, 1, 1)$. Let $S = 2$, and it follows that $h_S^{-1}(x) = \Gamma + \tilde{g}(x) - \frac{1}{2}\Gamma = \frac{1}{2}\Gamma + \tilde{g}(x)$ is sublinear, where $\tilde{g}_i(x) = g_i(x_{i-1})$, $i = 1, 2, 3$. For the purpose of using Theorem 4.6, it remains to verify (L). As the analysis in Example 4.5, we are now in a position to show that

$$\limsup_{t\to+\infty} \frac{1}{t} \log |\Phi(t, \omega)x| \le -L_\lambda \quad \mathbb{P}\text{-a.s.}, \tag{4.45}$$

where $L_\lambda > 0$ is independent of $\omega \in \Omega$ and $x \in \mathbb{R}^d \setminus \{0\}$. A simple computation gives that

$$|Ax|_2 \le \|A\|_2 |x|_2 = \sqrt{\frac{9}{4}} |x|_2 = \frac{3}{2}|x|_2,$$

$$\sum_{i=1}^{3} |G_i x|_2^2 = \frac{21}{4} x_1^2 + \frac{101}{16} x_2^2 + \frac{46}{9} x_3^2 \le \frac{101}{16} |x|_2^2$$

and

$$\sum_{i=1}^{3} |x^T G_i x|^2 \ge |x|_2^4 + 4|x|_2^4 = 5|x|_2^4, \quad x \in \mathbb{R}^d.$$

This implies that

$$\limsup_{t\to+\infty} \frac{1}{t} \log |\Phi(t, \omega)x| \le -\left(5 - \frac{101}{32} - \frac{3}{2}\right) = -\frac{11}{32} \quad \mathbb{P}\text{-a.s.}$$

by Theorem 5.1 in [28, Chapter 4]. Applying Theorem 4.6, we conclude that there exists a unique globally stable strongly positive random equilibrium in $\mathbb{R}_+^3$ for stochastic competitive system (4.43).

The same conclusion can be obtained if we replace $g_i(x_{i-1})$ by $g_i(x_1+x_2+x_3)$ or let $h_i(x) = 2 + \frac{1}{1+x_{i-1}^m}$, where $m > 1$, $i = 1, 2, 3$.

Example 4.7. Finally, we investigate the three-dimensional *stochastic predator-prey system*

$$dX_t = [AX_t + h(X_t)]dt + \sum_{i=1}^{3} G_i X_t dW_t^i, \tag{4.46}$$

where

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 1 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{4} \end{bmatrix},\ G_1 = 3I_{3\times 3},\ G_2 = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{5}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix},\ G_3 = \begin{bmatrix} -\frac{5}{2} & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

and

$$h_i(x) := \frac{1}{3 + g_i(x_{i+1})}, \qquad x \in \mathbb{R}_+^3,\ i = 1, 2, 3, \tag{4.47}$$

where $g_i(x_{i+1}) = \frac{1+x_{i+1}}{2+x_{i+1}}$ is a C^1-increasing sublinear function from $\mathbb{R}_+$ to $\mathbb{R}_+ (x_4 = x_1)$, $i = 1, 2, 3$. Write $f(x) = Ax + h(x)$, $x \in \mathbb{R}_+^3$. Then $\frac{\partial f_i}{\partial x_{i-1}}(x) = 1 > 0$ and $\frac{\partial f_i}{\partial x_{i+1}}(x) = -\frac{1}{(3+g_i(x_{i+1}))^2} \cdot \frac{1}{(2+x_{i+1})^2} < 0$ for $i = 1, 2, 3$ $(x_0 = x_3, x_4 = x_1)$. This implies that (4.46) is a stochastic predator-prey system. Consider $\delta = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ and $\Gamma = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. It is obvious that $h : \mathbb{R}_+^3 \to [\delta, \Gamma]$ is an anti-order-preserving and bounded function. Set $S = 2$, it is evident that $h_S^{-1}(x) = \Gamma^{-1} + \tilde{g}(x) - \frac{1}{2}\Gamma^{-1} = \frac{1}{2}\Gamma^{-1} + \tilde{g}(x)$ is sublinear, where $\Gamma^{-1} = (3, 3, 3)$ and $\tilde{g}_i(x) = g_i(x_{i+1})$, $i = 1, 2, 3$. Furthermore, we can see that

$$|Ax|_2 \le \|A\|_2 |x|_2 = \sqrt{\frac{493}{144}} |x|_2 \le 2|x|_2,$$

$$\sum_{i=1}^{3} |G_i x|_2^2 = \frac{35}{2} x_1^2 + \frac{313}{16} x_2^2 + 14 x_3^2 \le \frac{313}{16} |x|_2^2$$

and

$$\sum_{i=1}^{3} |x^T G_i x|^2 \ge 9|x|_2^4 + |x|_2^4 + 4|x|_2^4 = 14|x|_2^4, \quad x \in \mathbb{R}^d,$$

which together with Theorem 5.1 in [28, Chapter 4] implies that

$$\limsup_{t\to+\infty} \frac{1}{t} \log |\Phi(t,\omega)x| \le -\left(14 - \frac{313}{32} - 2\right) = -\frac{71}{32} \quad \mathbb{P}\text{-a.s.}$$

In the same way as above, we have that (L) holds. In view of Theorem 4.6, the stochastic predator-prey system (4.46) admits a unique strongly positive random equilibrium in $\mathbb{R}_+^3$ which attracts all pull-back trajectories.

Furthermore, if we define $h_i(x) = 3 + \frac{1}{1+x_{i+1}^m}$, where $m > 1$, $i = 1, 2, 3$, the same result still holds.

5. Discussion. In this paper, we have considered the stochastic stability of a nonlinear stochastic control system with inputs and outputs driven by multiplicative white noise and established two global stability theorems. That is, there exists a unique globally attracting nonnegative random equilibrium $\mathcal{K}(u)$ in $\mathbb{R}_+^d$ for the random dynamical system generated by those stochastic feedback systems, such as a stochastic Goodwin negative feedback system, Othmer–Tyson positive feedback system, Griffith positive feedback system, and so on. Motivated by the idea in [21], the key point in this paper is to construct a suitable complete metric space as the input space such that the operator $\mathcal{K}^h$ is contractive on it. However, the fundamental matrix $\Phi(t,\omega)$ depends on $\omega \in \Omega$. This yields that we cannot give a uniform estimate of $\Phi(t,\omega)$ for all $\omega \in \Omega$ and the problem will become more difficult to study than that in additive white noise [21]. To overcome this difficulty, in the case that derivatives of output functions are bounded, the joint measurability of the metric dynamical system $\theta : \mathbb{R}_- \times \Omega \mapsto \Omega$ with respect to the product σ-algebra $\mathscr{B}(\mathbb{R}_-) \otimes \mathscr{F}_-$ is first established; see Proposition 3.3. This helps us to successfully receive the $\mathscr{F}_-$-measurability for the input-to-state characteristic operator $\mathcal{K}(u)$ while u is tempered and $\mathscr{F}_-$-measurable; see Proposition 3.4. It is just because these measurabilities are obtained that the operator $\mathcal{K}^h : \mathcal{L}^1_{\mathscr{F}_-} \to \mathcal{L}^1_{\mathscr{F}_-}$ is well defined. Combining the condition (H_2) and the independence between R and the past σ-algebra $\mathscr{F}_-$, we finally proved that the operator $\mathcal{K}^h$ is a contraction mapping on the input space $\mathcal{L}^1_{\mathscr{F}_-}$. Here, the choice of the input space seems to be the best, since any globally attracting random equilibrium $v(\omega)$, i.e., $\lim_{t\to\infty} \varphi(t, \theta_{-t}\omega)x = v(\omega)$, must be $\mathscr{F}_-$-measurable.

In the use of Theorem 4.2, the most important task is to verify the condition (H_2) in Lemma 4.1. For this purpose, we should give suitable estimation of the upper bound for $\frac{\|R\|_{\mathcal{L}^1}}{\lambda}$, where the positive constant λ and the tempered random variable R are defined in the condition (L). In fact, it is interesting and difficult to get the optimal upper bound of $\frac{\|R\|_{\mathcal{L}^1}}{\lambda}$ for high-dimensional stochastic control systems. Even if the exact expression of the solution is given, this is not an easy issue. For example, in the model of stochastic single loop feedback system (4.5), our bound such as (4.16) is conservative. : If $n = 1$, then the optimal bound of $\frac{\|R\|_{\mathcal{L}^1}}{\lambda}$ can be calculated. That is, we study the following scalar SDE:

$$dx = -\alpha x dt + \sigma x dW_t,$$

where $\alpha > 0$ and $\sigma \neq 0$. It is well known that $\Phi(t,\omega) = e^{(-\alpha - \frac{\sigma^2}{2})t + \sigma W_t(\omega)}$, $t \geq 0$, and $\omega \in \Omega$. Consequently, in order to verify the condition (H_2), we can let $0 < \lambda < \alpha$ and $R(\omega) = \sup_{t\geq 0} \exp[-(\alpha - \lambda + \frac{\sigma^2}{2})t + \sigma W_t(\omega)]$, $\omega \in \Omega$. This implies that $\frac{\|R\|_{\mathcal{L}^1}}{\lambda} = \frac{1}{\lambda} + \frac{\sigma^2}{2\lambda(\alpha-\lambda)}$ and $\min_{0<\lambda<\alpha} \frac{\|R\|_{\mathcal{L}^1}}{\lambda} = \frac{1}{\lambda_0} + \frac{\sigma^2}{2\lambda_0(\alpha-\lambda_0)}$, where $\lambda_0 = \frac{(2\alpha+\sigma^2)-\sigma\sqrt{2\alpha+\sigma^2}}{2}$. Then, the condition ($\mathrm{H}_2$) ($n = 1$) can be interpreted as $M \cdot [\frac{1}{\lambda_0} + \frac{\sigma^2}{2\lambda_0(\alpha-\lambda_0)}] < 1$. However, in general ($n \geq 2$), we have no good idea to get this best estimation. This will be left for future consideration. To our knowledge, our new theory provides some new insights to investigate the stochastic stability of stochastic nonmonotone control systems.

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