A New Approach to Stability Analysis for Stochastic Hopfield Neural Networks With Time Delays

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***Abstract*—This article is devoted to the existence and the global stability of stationary solutions for stochastic Hopfield neural networks with time delays and additive white noises. Using the method of random dynamical systems, we present a new approach to guarantee that the infinite-dimensional stochastic flow generated by stochastic delay differential equations admits a globally attracting random equilibrium in the state-space of continuous functions. An example is given to illustrate the effectiveness of our results, and the forward trajectory synchronization will occur.**

***Index Terms*—Random dynamical systems, stability, stationary solutions, stochastic delay neural networks.**

I. Introduction

THE analysis of stability in artificial neural networks plays an important role in the control theory, due to its many applications in physics, ecology, biology, and engineering, such as image recognition and stock market prediction. During the past 35 years, the study on the dynamics of neural networks has advanced greatly. Among the existing works, one of the most popular models was proposed by Hopfield [28], which can be described by the following ordinary differential equations (ODEs):

$$C_i \frac{dx_i(t)}{dt} = -\frac{x_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij} g_j\big(x_j(t)\big) + I_i, \quad i = 1, \ldots, n \tag{1}$$

on $t \geq 0$, where $n \geq 2$ is the number of neurons. Here, the variable $x_i(t)$ represents the voltage on the input of the ith neuron; $C_i > 0$ and $R_i > 0$ are the input capacitance and resistance associated with the ith neuron, respectively; I_i is the constant external input; the matrix $T = (T_{ij})_{n\times n}$ shows the connection strengths between neurons; and g_j are neuron activation functions. Motivated by [28], Hopfield neural networks have been widely studied and applied to many areas, such as signal processing, pattern recognition, combinatorial optimization, and associative memory; see, for examples, [13], [15], [34], [38], [51].

In the pioneer work [28], neurons are assumed to communicate and feedback instantaneously. With the in-depth research, time delays are introduced in various models, due to the finite switching speed of amplifiers, the axonal signal transmission time, and the distance between neurons, which may lead to some undesired dynamical behaviours of neural networks, such as instability and oscillation. Therefore, the global stability problem for delayed neural networks has attracted a lot of attention; see [2], [3], [8], [9], [11], [12], [17], [20]–[23], [25], [27], [37], [40], [41], [44], [45], [52], [53]. They mainly considered the following system of delay differential equations (DDEs):

$$C_i \frac{dx_i(t)}{dt} = -\frac{x_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij} g_j\big(x_j(t-\tau_j)\big) + I_i, \ i = 1, \ldots, n. \tag{2}$$

Besides the effect of time delays, Hopfield [28] pointed out that "...the time evolution of the state of such systems should be represented by a differential equation (perhaps with added noise)" (p. 3088). Actually, in real nervous systems, the synaptic transmission between neurons is a noisy process due to random fluctuations from the release of neurotransmitters and other causes, see [26]. Contrary to intuition, in some nonlinear systems driven by weak inputs, noise generated externally or intrinsically may have beneficial effects to spontaneous activity [19] and stability of neural networks [4], [5], [39]. Therefore, one of the main concerns in the study of neural networks is to consider the presence of the noise and the global stability of stochastic neural networks, which has received considerable attention in the past few years, such as [4], [5], [7], [10], [29]–[31], [39], [48]–[50].

In [6], [29], the authors pointed that the external drive can be decomposed into a constant input and a white noise term,

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such as (14) and (52) in [6]. Moreover, evidence for such randomly fluctuating inputs has been found in experimental studies of oscillations in the olfactory bulb [24]. Motivated by their works, this article is devoted to investigate the global stability of stochastic Hopfield neural networks with time delays

$$C_i dx_i(t) = \left[-\frac{x_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij} g_j\big(x_j(t-\tau_j)\big) + I_i \right] dt + \sum_{j=1}^{m} \sigma_{ij} dW_j(t), \qquad i = 1, \ldots, n \tag{3}$$

where the external input I_i in (2) is perturbed by the Gaussian noise $I_i \to I_i + \sum_{j=1}^{n} \sigma_{ij} \dot{W}_j(t)$. It is well known that in the existing literature, the analysis of the global stability for stochastic functional (or delay) differential equations is mainly based on the construction of Lyapunov functions; see [35], [42]. Especially, for various stability of trivial stationary solutions for stochastic delay neural networks, there have been a lot of related results; see [4], [5], [10], [30], [31]. The main purpose of this article is to study the existence and the global stability of nontrivial stationary solutions for (3), based on the theory of random dynamical systems (RDS). Therefore, there are two main difficulties. One is that the method of constructing suitable Lyapunov functions cannot be applied here. The other is that the fundamental theory of the infinite dimensional random dynamical systems generated by (3) is incomplete. To the best of our knowledge, up to now, there are no results on the existence and the global stability of nontrivial stationary solutions for (3). The main contributions of this article are summarized as follows.

1) Motivated by our recent works [32], [33], the main purpose of this article is to further develop the methods presented in [32], [33], and apply them to consider the global stability of stochastic DDEs (SDDEs). However, this is not an easy job. The main reason is that the classical theory of random dynamical systems established by Arnold [1] is mainly for processing stochastic ODEs, which is finite dimensional. In this article, we shall first develop the theory of infinite dimensional random dynamical systems generated by (3), such as the continuity of the pull-back trajectories, the measurability, and the compactness of some random sets; see Appendix A. Moreover, since the state-space C_τ is an infinite dimensional Banach space, we need to deal with the compactness and the positive cone C_τ^+ is not a strongly minihedral cone.
2) For SDDEs with additive white noises, we will give a new program to prove the existence and the global stability of nontrivial stationary solutions (random equilibria) in the space of continuous functions. In contrast, we do not need to construct some proper Lyapunov functions.
3) Some explicit conditions are given to guarantee the global stability of (3), which are easy to verify.

Notations: Throughout this article, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^n_+$ denotes its non-negative orthant, and $\mathbb{R}^{n\times m}$ denotes the set of all $n \times m$-dimensional real matrices. For any vector $x \in \mathbb{R}^n$ and matrix $A = (a_{ij})_{n\times m} \in \mathbb{R}^{n\times m}$, define the Euclidean norm $|x| := (\sum_{i=1}^{n} |x_i|^2)^{1/2}$ and $\|A\| := (\sum_{j=1}^{m} \sum_{i=1}^{n} |a_{ij}|^2)^{1/2}$. For all $x, y \in \mathbb{R}^n$, $x \geq y$ means that $x - y \in \mathbb{R}^n_+$. X denotes a complete separable metric space (i.e., Polish space) equipped with the Borel σ-algebra $\mathscr{B}(X)$. Let $\tau = \max_{1\leq i\leq n} \tau_i$, $C_\tau := C([-\tau, 0]; \mathbb{R}^n)$ denote the Banach space of continuous functions $\xi : [-\tau, 0] \to \mathbb{R}^n$ equipped with the norm $\|\xi\|_{C_\tau} = \sup_{-\tau\leq s\leq 0} |\xi(s)|$. Let $W(t) = (W_1(t), \ldots, W_m(t))^{\mathrm{T}}$ be an m-dimensional two-sided Brownian motion on the canonical probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Here, $\mathscr{F}$ is the Borel σ-algebra of $\Omega = C_0(\mathbb{R}, \mathbb{R}^m) = \{\omega = (\omega_1, \omega_2, \ldots, \omega_m) \in C(\mathbb{R}, \mathbb{R}^m),\ \omega(0) = 0\}$, which is equipped with the following metric:

$$\varrho(\omega, \omega^*) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\varrho_k(\omega, \omega^*)}{1 + \varrho_k(\omega, \omega^*)}$$

where

$$\varrho_k(\omega, \omega^*) = \max_{t\in[-k,k]} |\omega(t) - \omega^*(t)|$$

and $\mathbb{P}$ is the corresponding Wiener measure.

II. Problem Formulation and Preliminaries

In this section, we consider the following SDDEs with additive white noise:

$$C_i dx_i(t) = \left[-\frac{x_i(t)}{R_i} + \sum_{j=1}^{n} T_{ij} g_j\big(x_j(t-\tau_j)\big) + I_i \right] dt + \sum_{j=1}^{m} \sigma_{ij} dW_j(t), \qquad i = 1, \ldots, n \tag{4}$$

with the initial condition $x(s) = \xi(s)$ for all $s \in [-\tau, 0]$, where $\tau = \max_{1\leq i\leq n} \tau_i$ and $\xi \in C([-\tau, 0]; \mathbb{R}^n)$. System (4) can describe the evolution of a neural network under stochastic perturbations, where the time delay of information transmission between neurons is supposed to be independent of the state and the time. Define

$$a_i = \frac{1}{C_i R_i}, \qquad b_{ij} = \frac{T_{ij}}{C_i}, \qquad \bar{\sigma}_{ij} = \frac{\sigma_{ij}}{C_i}, \qquad d_i = \frac{I_i}{C_i}$$

and then system (4) can be rewritten as

$$dx(t) = [-Ax(t) + h(x_t)]dt + \sigma dW(t) \tag{5}$$

where $x_t = \{x(t+s) : -\tau \leq s \leq 0\}$,

$$A = \mathrm{diag}(a_1, \ldots, a_n),\ x(t) = (x_1(t), \ldots, x_n(t))^T$$

$$B = (b_{ij})_{n\times n},\ g(x) = (g_1(x_1), \ldots, g_n(x_n))^T$$

$$f(x) = Bg(x) + d,\ h(x_t) = f[(x_1(t-\tau_1), \ldots, x_n(t-\tau_n))^T]$$

$$\sigma = (\bar{\sigma}_{ij})_{n\times m},\ d = (d_1, \ldots, d_n)^T$$

$$W(t) = (W_1(t), \ldots, W_m(t))^T.$$

Before stating our main results, we present some basic definitions; see [1], [14].

Definition 1: A quadruple $(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{R}\})$ is called a metric dynamical system if θ is a measurable flow

$$\theta : \mathbb{R} \times \Omega \mapsto \Omega, \qquad \theta_0 = \mathrm{id}, \qquad \theta_{t_2} \circ \theta_{t_1} = \theta_{t_1+t_2}$$

for all $t_1, t_2 \in \mathbb{R}$, which is $(\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}, \mathscr{F})$-measurable. In addition, we assume that $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 2: An random dynamical systems (RDS) on the state-space X induced by a metric dynamical system $(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{R}\})$ is a mapping

$$\varphi : \mathbb{R}_+ \times \Omega \times X \mapsto X, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x)$$

which is $(\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F} \otimes \mathscr{B}(X), \mathscr{B}(X))$-measurable such that for any $\omega \in \Omega$,

1) $\varphi(0, \omega, \cdot)$ is the identity on X;
2) $\varphi(t_1 + t_2, \omega, x) = \varphi(t_2, \theta_{t_1}\omega, \varphi(t_1, \omega, x))$ for all $t_1, t_2 \in \mathbb{R}_+$ and $x \in X$;
3) the mapping $\varphi(t, \omega, \cdot) : X \to X$ is continuous for all $t \in \mathbb{R}_+$.

Definition 3: A family $\{D(\omega), \omega \in \Omega\}$ of nonempty subsets of the state-space X is said to be a random closed (respectively, compact) set if for each $\omega \in \Omega$, it is closed (respectively, compact) and $\omega \to d(x, D(\omega))$ is measurable for each $x \in X$. Here, $d(x, B)$ is the distance in X between point x and the set $B \subset X$.

Definition 4: A random variable $u : \Omega \to X$ is said to be an equilibrium of the RDS (θ, φ) if for all $t \geq 0$ and $\omega \in \Omega$, $\varphi(t, \omega)u(\omega) = u(\theta_t \omega)$.

Definition 5: Let X be a Banach space with a closed convex cone X_+, which gives a partial order relation on X via $x \leq y$ if $y - x \in X_+$. An element $x \in X$ is called an upper bound for a subset $A \subset X$ if $y \leq x$ for all $y \in A$. An upper bound x_0 is called the least upper bound (or supremum), denoted by $x_0 = \sup A$, if $x_0 \leq x$ for any other upper bound x. Moreover, lower bound and the greatest lower bound (or infimum) can be defined similarly.

In this article, we assume the following.

1) $f_i : \mathbb{R}^n \to \mathbb{R}$ is globally Lipschitz continuous with Lipschitz constant L_i, monotone (or anti-monotone), $|f_i(x)| \leq M_i$, $\forall x \in \mathbb{R}^n$ for some constant $M_i > 0$ and $\frac{\sqrt{n}L}{\lambda} < 1$, where $L = \sqrt{n} \max_{1 \leq i \leq n} L_i$ is the global Lipschitz constant for $h : C_\tau \to \mathbb{R}^n$ and $\lambda = \min_{1 \leq i \leq n} a_i > 0$.

Here, monotone means that

$$x \leq_{\mathbb{R}^n_+} y \quad \Rightarrow \quad f(x) \leq_{\mathbb{R}^n_+} f(y), \qquad \forall x, y \in \mathbb{R}^n$$

and anti-monotone means that

$$x \leq_{\mathbb{R}^n_+} y \quad \Rightarrow \quad f(x) \geq_{\mathbb{R}^n_+} f(y), \qquad \forall x, y \in \mathbb{R}^n$$

where $x \leq_{\mathbb{R}^n_+} y$ stands for $y - x \in \mathbb{R}^n_+$. Moreover, f can be seen as sigmoidal functions, which have been widely used in the engineering literature; see [16], [18].

Next, set $y(t, \omega, x_0 - z_0(\omega)) = x(t, \omega, x_0) - z(\theta_t \omega)$ for all $\omega \in \Omega$, where $x(t, \omega, x_0)$ is the solution of (5) with the initial value $x_0 \in C_\tau$ and

$$z(t, \omega) \equiv z(\theta_t \omega) = \int_{-\infty}^{t} \exp\{-A(t-s)\} \sigma dW(s)$$

is the Ornstein–Uhlenbeck process, i.e., the stationary solution of SDEs

$$dz(t) = -Az(t)dt + \sigma dW(t).$$

Here, the shift operator θ is defined by $\theta_t \omega(\bullet) = \omega(t + \bullet) - \omega(t)$ for all $t \in \mathbb{R}$ and $\omega \in \Omega$. Using the standard method, define $\varphi(t, \omega, x_0) : \mathbb{R}_+ \times \Omega \times C_\tau \to C_\tau$ by

$$\varphi(t, \omega, x_0) := x_t(\omega, x_0) = y_t\big(\omega, x_0 - z_0(\omega)\big) + z_t(\omega) \quad (6)$$

where $x_t(\omega, x_0) = \{x(t+s, \omega, x_0) : -\tau \leq s \leq 0\}$, $y_t(\omega, y_0) = \{y(t+s, \omega, y_0) : -\tau \leq s \leq 0\}$ and $z_t(\omega) = \{z(\theta_{t+s}\omega) : -\tau \leq s \leq 0\}$; it is easily seen that y satisfies the corresponding random functional differential equations

$$\frac{dy}{dt} = -Ay + h\big(y_t + z_t(\omega)\big) \quad (7)$$

with the initial value

$$y_0(\omega) = x_0 - z_0(\omega) \quad (8)$$

and then φ is an RDS generated by (5) in C_τ.

For $t \geq \tau$ and $-\tau \leq s \leq 0$, using the variation of constants formula [42, Th. 3.1], one can have that

$$\begin{aligned}
&\varphi(t, \omega, x_0)[s] \\
&= e^{-A(t+s)} x_0(0) + e^{-A(t+s)} \int_0^{t+s} e^{Au} h\big(\varphi(u, \omega, x_0)\big) du \\
&\quad + e^{-A(t+s)} \int_0^{t+s} e^{Au} \sigma dW_u \\
&= e^{-A(t+s)} x(0) + \int_0^{t+s} e^{-A(t+s-u)} h\big(\varphi(u, \omega, x_0)\big) du \\
&\quad + \int_0^{t+s} e^{-A(t+s-u)} \sigma dW_u, \quad t + s \geq 0 \qquad (9)
\end{aligned}$$

where $x_0(s) := x(s)$. Combining the definition of θ and (9), it follows immediately that for all $t \geq \tau$,

$$\begin{aligned}
&\varphi(t, \theta_{-t}\omega, x_0)[s] \\
&= e^{-A(t+s)} x(0) + \int_{-t}^{s} e^{-A(s-u)} h\big(\varphi(u + t, \theta_{-t}\omega, x_0)\big) du \\
&\qquad + \int_{-t}^{s} e^{-A(s-u)} \sigma dW_u, \quad t + s \geq 0. \qquad (10)
\end{aligned}$$

At the end of this section, motivated by the recent work [32], we will define an important characteristic operator associated with (10), which is defined by

$$[\mathcal{K}(r)](s, \omega) = \int_{-\infty}^{s} e^{-A(s-u)} r(\theta_u \omega) du + \int_{-\infty}^{s} e^{-A(s-u)} \sigma dW_u \quad (11)$$

for all $-\tau \leq s \leq 0$ and $\omega \in \Omega$. Here, $r : \Omega \to \mathbb{R}^n$ is a tempered random variable with respect to the ergodic metric dynamical system θ, i.e.,

$$\sup_{t \in \mathbb{R}} \left\{ e^{-\gamma|t|} \, |r(\theta_t \omega)| \right\} < \infty \quad \text{for all } \omega \in \Omega \text{ and } \gamma > 0.$$

By (A1), it is easy to see that the operator $\mathcal{K}$ is well defined and $[\mathcal{K}(r)](s,\omega)$ is continuous with respect to $s \in [-\tau, 0]$ for all tempered random variable r and $\omega \in \Omega$.

III. Main Results

In this section, we will use the approach of random dynamical systems to establish some useful inequalities in the sense of partial order, which play the key role in the presentation of the dynamical behavior of stochastic flow φ generated by (5). For our purpose, we introduce the partial order $\leq_{C_\tau^+}$ in the way

$$\xi \leq_{C_\tau^+} \eta \qquad \Longleftrightarrow \qquad \eta - \xi \in C_\tau^+$$

where $C_\tau^+ = \{\xi | \xi \in C_\tau, \xi_i(s) \geq 0, i=1,\ldots,n, \forall s \in [-\tau, 0]\}$ is a solid normal minihedral cone in the Banach space C_τ.

Lemma 3.1 [43, Lemma A.2]: Let $(x_t)_{t\in\mathbb{R}_+}$ is a net in a normed space X associated with a solid, normal cone $X_+ \subseteq X$. Suppose that the net converges to a point $x \in X$, and that

$$x_t^- := \inf\{x_{t'} : t' \geq t\} \quad \text{and} \quad x_t^+ := \sup\{x_{t'} : t' \geq t\}$$

exist for every $t \in \mathbb{R}_+$. Then, the nets $(x_t^-)_{t\in\mathbb{R}_+}$ and $(x_t^+)_{t\in\mathbb{R}_+}$ also converge to x.

Lemma 3.2: For any $t \geq \tau$, let

$$\begin{aligned}\alpha_t^h(\omega) &= \inf \overline{\{h(\varphi(u, \theta_{-u}\omega, x_0)) : u \geq t\}} \\ &= \inf\{h(\varphi(u, \theta_{-u}\omega, x_0)) : u \geq t\}\end{aligned}$$

and

$$\begin{aligned}\beta_t^h(\omega) &= \sup \overline{\{h(\varphi(u, \theta_{-u}\omega, x_0)) : u \geq t\}} \\ &= \sup\{h(\varphi(u, \theta_{-u}\omega, x_0)) : u \geq t\}\end{aligned}$$

where $x_0 \in C_\tau$ and $\omega \in \Omega$. Then, $\alpha_t^h(\omega)$ and $\beta_t^h(\omega)$ are $\mathscr{F}_-$-measurable random variables for all $t \geq \tau$, where $\mathscr{F}_- = \sigma\{\omega \mapsto W_t(\omega) : t \leq 0\}$ is the past σ-algebra.

Proof: For any $t \geq \tau$, by (A1) and Proposition 1 in the Appendix, it is clear that $\overline{\{h(\varphi(u, \theta_{-u}\omega, x_0)) : u \geq t\}}$ is a compact set in $\mathbb{R}^n$, it follows that $\alpha_t^h(\omega)$ and $\beta_t^h(\omega)$ are well defined for all $\omega \in \Omega$ and $t \geq \tau$. Here, we use the fact that $\inf \overline{A} = \inf A$ and $\sup \overline{A} = \sup A$, where A is a bounded set in $\mathbb{R}^n$; see Lemma A.1 in [43]. Moreover, by Proposition 2 in the Appendix, it is easily seen that

$$u \mapsto h(\varphi(u, \theta_{-u}\omega, x_0)) \text{ is continuous}$$

from $\mathbb{R}_+$ into $\mathbb{R}^n$ for all $\omega \in \Omega$ and $x_0 \in C_\tau$. The rest of the proof can be followed by the same arguments in [33, Proposition 3.5]; we omit it here. The proof is complete. ■

Lemma 3.3: Assume that (A1) holds, we have

$$\begin{aligned}[\mathcal{K}(\underline{\lim}_\theta h(\varphi))](\bullet,\omega) &\leq [\underline{\lim}_\theta \varphi](\bullet,\omega) \\ &\leq [\overline{\lim}^\theta \varphi](\bullet,\omega) \leq [\mathcal{K}(\overline{\lim}^\theta h(\varphi))](\bullet,\omega)\end{aligned} \tag{12}$$

for all $\omega \in \Omega$. Here, $\leq$ means $\leq_{C_\tau^+}$

$$[\underline{\lim}_\theta \varphi](\bullet,\omega) := \lim_{t\to\infty} \inf\{\varphi(u, \theta_{-u}\omega, x_0)[\bullet] : u \geq t\}$$

$$[\overline{\lim}^\theta \varphi](\bullet,\omega) := \lim_{t\to\infty} \sup\{\varphi(u, \theta_{-u}\omega, x_0)[\bullet] : u \geq t\}$$

$$[\underline{\lim}_\theta h(\varphi)](\omega) := \lim_{t\to\infty} \inf\{h(\varphi(u, \theta_{-u}\omega, x_0)) : u \geq t\}$$

and

$$[\overline{\lim}^\theta h(\varphi)](\omega) := \lim_{t\to\infty} \sup\{h(\varphi(u, \theta_{-u}\omega, x_0)) : u \geq t\}$$

for all $\omega \in \Omega$ and $x_0 \in C_\tau$.

Proof: In order to show that (12) holds, we need to prove three inequalities. For simplicity, we only prove the first inequality in (12), and others can be obtained similarly. By Lemma 3.2, we can immediately have that $\underline{\lim}_\theta h(\varphi)$ and $\overline{\lim}^\theta h(\varphi)$ exist, which are two $\mathscr{F}_-$-measurable random variables. Moreover, using Proposition 1 and 2 in the Appendix, it is clear that $\overline{\{\varphi(t, \theta_{-t}\omega, x_0) : t \geq \tau\}}$ is a compact set in C_τ and for any $\omega \in \Omega$ and $x_0 \in C_\tau$

$$u \mapsto \varphi(u, \theta_{-u}\omega, x_0) \text{ is a continuous mapping}$$

from $\mathbb{R}_+$ into C_τ, which together with Proposition 1.5.3 and Theorem 3.2.1 in [14] implies that $\underline{\lim}_\theta \varphi : \Omega \to C_\tau$ is a well-posed $\mathscr{F}_-$-measurable function, and the same conclusion holds for $\overline{\lim}^\theta \varphi$. Besides, combining (11), Proposition 3.3 in [33], and Fubini's theorem, we can show that $[\mathcal{K}(\underline{\lim}_\theta h(\varphi))] : \Omega \to C_\tau$ is also an $\mathscr{F}_-$-measurable function. Here, the well-posedness of $[\mathcal{K}(\underline{\lim}_\theta h(\varphi))]$ is based on the assumption that $h : C_\tau \to \mathbb{R}^n$ is bounded, and so $\underline{\lim}_\theta h(\varphi)$ is uniformly bounded with respect to all $\omega \in \Omega$. According to Proposition 3, it follows immediately that

$$[\mathcal{K}(\underline{\lim}_\theta h(\varphi))](\bullet,\omega) = \lim_{t\to\infty} [\mathcal{K}(\alpha_t^h)](\bullet,\omega) \text{ in } C_\tau.$$

This implies that we only need to prove that for any $t \geq \tau$, the following inequality

$$[\mathcal{K}(\alpha_t^h)](\bullet,\omega) \leq [\underline{\lim}_\theta \varphi](\bullet,\omega) \text{ in } C_\tau \tag{13}$$

is true for all $\omega \in \Omega$. At the same time, by (A1), it is evident that $h(\xi) \in [-M, M]$ for all $\xi \in C_\tau$, where $M = (M_1, \ldots, M_n)^T \in \text{int}\mathbb{R}_+^n$. Now, let us begin to prove (13). By Proposition 4, it is obvious that for any $t \geq \tau$, we have

$$\begin{aligned}&[\mathcal{K}(\alpha_t^h)](\bullet,\omega) \\ &= \int_{-\infty}^{\bullet} e^{-A(\bullet-u)} \inf\{h(\varphi(v, \theta_{-v}\diamond, x_0)) : v \geq t\}(\theta_u\omega)du \\ &\quad + \int_{-\infty}^{\bullet} e^{-A(\bullet-u)}\sigma dW_u \\ &= \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\geq t+\tau}} \Big\{ e^{-A(\tilde{t}+\bullet)}x(0) + \int_{t-\tilde{t}}^{\bullet} e^{-A(\bullet-u)} \inf \\ &\quad \{h(\varphi(v, \theta_{-v+u}\omega, x_0)) + M : v \geq t\}du \\ &\quad - \int_{-\tilde{t}}^{\bullet} e^{-A(\bullet-u)}Mdu + \int_{-\tilde{t}}^{\bullet} e^{-A(\bullet-u)}\sigma dW_u \Big\} \\ &\overset{\text{Lem 2.1}}{=\!=\!=} \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\geq t+\tau}} \inf \Big\{ e^{-A(\tilde{t}+\bullet)}x(0) + \int_{t-\tilde{t}}^{\bullet} e^{-A(\bullet-u)} \inf \\ &\quad \{h(\varphi(v, \theta_{-v+u}\omega, x_0)) + M : v \geq t\}du\end{aligned}$$

$$- \int_{-\tilde{t}}^{\bullet} e^{-A(\bullet-u)} M du + \int_{-\tilde{t}}^{\bullet} e^{-A(\bullet-u)} \sigma dW_u : \tilde{t} \geq \tilde{\tilde{t}} \Big\}$$

$$\leq \lim_{\substack{\tilde{\tilde{t}} \to \infty \\ \tilde{\tilde{t}} \geq t+\tau}} \inf \Big\{ e^{-A(\tilde{t}+\bullet)} x(0) + \int_{t-\tilde{t}}^{\bullet} e^{-A(\bullet-u)}$$

$$[h(\varphi(\tilde{t}+u, \theta_{-\tilde{t}}\omega, x_0)) + M] du - \int_{-\tilde{t}}^{\bullet} e^{-A(\bullet-u)} M du$$

$$+ \int_{-\tilde{t}}^{\bullet} e^{-A(\bullet-u)} \sigma dW_u : \tilde{t} \geq \tilde{\tilde{t}} \Big\}$$

$$\leq \lim_{\substack{\tilde{\tilde{t}} \to \infty \\ \tilde{\tilde{t}} \geq t+\tau}} \inf \Big\{ e^{-A(\tilde{t}+\bullet)} x(0) + \int_{-\tilde{t}}^{\bullet} e^{-A(\bullet-u)}$$

$$[h(\varphi(\tilde{t}+u, \theta_{-\tilde{t}}\omega, x_0)) + M] du - \int_{-\tilde{t}}^{\bullet} e^{-A(\bullet-u)} M du$$

$$+ \int_{-\tilde{t}}^{\bullet} e^{-A(\bullet-u)} \sigma dW_u : \tilde{t} \geq \tilde{\tilde{t}} \Big\}$$

$$= \lim_{\tilde{\tilde{t}} \to \infty} \inf \Big\{ \varphi(\tilde{t}, \theta_{-\tilde{t}}\omega, x_0)[\bullet] : \tilde{t} \geq \tilde{\tilde{t}} \Big\} = [\underline{\lim}_\theta \varphi](\bullet, \omega)$$

for all $\omega \in \Omega$, where the second-to-last inequality holds due to that $e^{-At}x \geq 0$ for all $t \geq 0$ and $x \in \mathbb{R}^n_+$. The proof is complete. ■

Remark 1: In this lemma, we remove the positivity of h, which is weaker than that in [32].

Lemma 3.4: Assume that (A1) holds. It follows that for all $\omega \in \Omega$,

1) If f is monotone, then

$$h(\underline{\lim}_\theta \varphi) \leq \underline{\lim}_\theta h(\varphi) \leq \overline{\lim}^\theta h(\varphi) \leq h(\overline{\lim}^\theta \varphi). \quad (14)$$

2) If f is anti-monotone, then

$$h(\overline{\lim}^\theta \varphi) \leq \underline{\lim}_\theta h(\varphi) \leq \overline{\lim}^\theta h(\varphi) \leq h(\underline{\lim}_\theta \varphi). \quad (15)$$

Proof: By Lemma 3.3, it is easy to check that $\underline{\lim}_\theta \varphi$ and $\overline{\lim}^\theta \varphi$ are well defined for all $\omega \in \Omega$. The rest proof is similar to Lemma 3.4 in [32]; we omit it here. ■

Lemma 3.5: Assume that (A1) holds. Define the operator $\mathcal{K}^h$ to be $h \circ \mathcal{K}$, which means that $[\mathcal{K}^h(r)](\omega) = h\{[\mathcal{K}(r)](\bullet, \omega)\}$ for any tempered random variable r, thus

1) If f is monotone, then for all $t \geq \tau$, $\omega \in \Omega$ and $k \in \mathbb{N}$,

$$[(\mathcal{K}^h)^k(\alpha_t^h)](\omega) \leq [\underline{\lim}_\theta h(\varphi)](\omega) \leq [\overline{\lim}^\theta h(\varphi)](\omega) \leq [(\mathcal{K}^h)^k(\beta_t^h)](\omega). \quad (16)$$

2) If f is anti-monotone, then for all $t \geq \tau$, $\omega \in \Omega$ and $k \in \mathbb{N}$,

$$[(\mathcal{K}^h)^{2k}(\alpha_t^h)](\omega) \leq [\underline{\lim}_\theta h(\varphi)](\omega) \leq [\overline{\lim}^\theta h(\varphi)](\omega) \leq [(\mathcal{K}^h)^{2k}(\beta_t^h)](\omega). \quad (17)$$

Proof: The proof is similar to Lemma 3.5 in [32]; we omit it here. ■

Lemma 3.6: Assume that (A1) holds. Furthermore, let $\mathcal{M}^b_{\mathscr{F}_-}(\Omega; [-M, M])$ be the space of all $\mathscr{F}_-$-measurable functions $r : \Omega \to [-M, M]$, where $M = (M_1, \ldots, M_n)^T$ is a strongly positive vector in $\mathbb{R}^n_+$ such that $h(\xi) \in [-M, M]$ for all $\xi \in C_\tau$. Next, we define a metric on $\mathcal{M}^b_{\mathscr{F}_-}(\Omega; [-M, M])$ to be

$$d(r_1, r_2) := |r_1 - r_2|_\infty = \sup_{\omega \in \Omega} |r_1(\omega) - r_2(\omega)|$$

where $r_1, r_2 \in \mathcal{M}^b_{\mathscr{F}_-}(\Omega; [-M, M])$. Therefore, we have that $(\mathcal{M}^b_{\mathscr{F}_-}, d)$ is a complete metric space and the operator $\mathcal{K}^h := h \circ \mathcal{K} : \mathcal{M}^b_{\mathscr{F}_-} \to \mathcal{M}^b_{\mathscr{F}_-}$ is a contraction mapping.

Proof: Using the same arguments in [32, Lemma 4.1], it is easy to see that $(\mathcal{M}^b_{\mathscr{F}_-}, d)$ is a complete metric space. In order to prove the conclusion, we first need to show that the operator $\mathcal{K}^h := h \circ \mathcal{K} : \mathcal{M}^b_{\mathscr{F}_-} \to \mathcal{M}^b_{\mathscr{F}_-}$ is well defined. For any $r \in \mathcal{M}^b_{\mathscr{F}_-}$, it is clear that $[\mathcal{K}(r)](\bullet, \omega) = \{[\mathcal{K}(r)](s, \omega), -\tau \leq s \leq 0\} \in C_\tau$ for all $\omega \in \Omega$. Moreover, set

$$[\mathcal{K}(r)](t, \omega) = \int_{-\infty}^{t} e^{-A(t-u)} r(\theta_u \omega) du + \int_{-\infty}^{t} e^{-A(t-u)} \sigma dW_u \quad (18)$$

for all $\omega \in \Omega$, which is also continuous with respect to $t \in \mathbb{R}_-$. Note that $\theta : \mathbb{R}_- \times \Omega \mapsto \Omega$ is $(\mathscr{B}(\mathbb{R}_-) \otimes \mathscr{F}_-, \mathscr{F}_-)$-measurable; see Proposition 3.3 in [33]. Combining this and Fubini's theorem, we can easily have that $[\mathcal{K}(r)](t, \bullet)$ is $\mathscr{F}_-$-measurable for all $t \leq 0$ and $r \in \mathcal{M}^b_{\mathscr{F}_-}$, which together with Lemma II.2.1 in [46] implies that the mapping

$$\omega \longmapsto [\mathcal{K}(r)]_t(\bullet, \omega) = \{[\mathcal{K}(r)](t+s, \omega), -\tau \leq s \leq 0\}$$

is $\mathscr{F}_-$-measurable from Ω into C_τ for all $t \leq 0$, where $[\mathcal{K}(r)](t+s, \omega)$ is defined as (18). Let $t = 0$, it follows that

$$\omega \longmapsto [\mathcal{K}(r)]_0(\bullet, \omega) = \{[\mathcal{K}(r)](s, \omega), -\tau \leq s \leq 0\}$$

is $(\mathscr{F}_-, \mathscr{B}(C_\tau))$-measurable. Furthermore, since $h : C_\tau \to \mathbb{R}^n$ is continuous, then it is immediate that $\mathcal{K}^h : \mathcal{M}^b_{\mathscr{F}_-} \to \mathcal{M}^b_{\mathscr{F}_-}$ is well defined.

Finally, we will show that the operator $\mathcal{K}^h : \mathcal{M}^b_{\mathscr{F}_-} \to \mathcal{M}^b_{\mathscr{F}_-}$ is contracted. For any $r_1, r_2 \in \mathcal{M}^b_{\mathscr{F}_-}$, we have that

$$\left|\mathcal{K}^h(r_1) - \mathcal{K}^h(r_2)\right|_\infty$$

$$= \sup_{\omega \in \Omega} |h\{[\mathcal{K}(r_1)](\bullet, \omega)\} - h\{[\mathcal{K}(r_2)](\bullet, \omega)\}|$$

$$\leq L \sup_{\omega \in \Omega} \|[\mathcal{K}(r_1)](\bullet, \omega) - [\mathcal{K}(r_2)](\bullet, \omega)\|_{C_\tau}$$

$$= L \sup_{\omega \in \Omega} \sup_{-\tau \leq s \leq 0} \left| \int_{-\infty}^{s} e^{-A(s-u)} [r_1(\theta_u \omega) - r_2(\theta_u \omega)] du \right|$$

$$\leq L \sup_{-\tau \leq s \leq 0} \int_{-\infty}^{s} \|e^{-A(s-u)}\| \cdot |r_1 - r_2|_\infty du$$

$$\leq \sqrt{n} L \sup_{-\tau \leq s \leq 0} \int_{-\infty}^{s} e^{\lambda(u-s)} du \cdot |r_1 - r_2|_\infty$$

$$= \sqrt{n} L \int_{-\infty}^{0} e^{\lambda u} du \cdot |r_1 - r_2|_\infty$$

$$= \frac{\sqrt{n} L}{\lambda} |r_1 - r_2|_\infty, \quad \text{where } \frac{\sqrt{n} L}{\lambda} < 1.$$

The proof is complete. ■

Theorem 3.1: Assume that (A1) holds. It follows that there exists a unique fixed point $r \in \mathcal{M}^b_{\mathscr{F}_-}$ for the operator $\mathcal{K}^h : \mathcal{M}^b_{\mathscr{F}_-} \to \mathcal{M}^b_{\mathscr{F}_-}$, which satisfies that

$$\lim_{t\to\infty} \varphi(t, \theta_{-t}\omega, x_0)[\bullet] = [\mathcal{K}(r)](\bullet, \omega) \tag{19}$$

in C_τ for all $x_0 \in C_\tau$ and $\omega \in \Omega$. In addition, $\varphi(t, \omega, [\mathcal{K}(r)](\bullet, \omega)) = [\mathcal{K}(r)](\bullet, \theta_t\omega)$, $t \geq 0$. That is, $[\mathcal{K}(r)](\bullet, \omega)$ is a random equilibrium in C_τ for the stochastic flow φ, which generates the stationary solution $[\mathcal{K}(r)](\bullet, \theta_t\omega)$ for (5).

Proof: Using Lemma 3.5, whether f is monotone or anti-monotone, we can always get that

$$[(\mathcal{K}^h)^{2k}(\alpha_t^h)](\omega) \leq [\underline{\lim}_\theta h(\varphi)](\omega)$$
$$\leq [\overline{\lim}^\theta h(\varphi)](\omega) \leq [(\mathcal{K}^h)^{2k}(\beta_t^h)](\omega) \tag{20}$$

for all $t \geq \tau$, $\omega \in \Omega$ and $k \in \mathbb{N}$. Moreover, by Lemma 3.2, it is clear that α_t^h and β_t^h belong to the complete metric space $\mathcal{M}^b_{\mathscr{F}_-}$ for all $t \geq 0$. Note that $\mathcal{K}^h$ is a contraction mapping on $\mathcal{M}^b_{\mathscr{F}_-}$, see Lemma 3.6, which together with Banach's fixed point theorem yields that there exists an $\mathscr{F}_-$-measurable random variable $r : \Omega \to [-M, M]$ such that $[\mathcal{K}^h(r)](\omega) = r(\omega)$ and

$$\lim_{k\to\infty} [(\mathcal{K}^h)^{2k}(\alpha_t^h)](\omega) = r(\omega) = \lim_{k\to\infty} [(\mathcal{K}^h)^{2k}(\beta_t^h)](\omega)$$

for all $t \geq \tau$ and $\omega \in \Omega$. Using this and (20), we have that

$$[\underline{\lim}_\theta h(\varphi)](\omega) = [\overline{\lim}^\theta h(\varphi)](\omega) = r(\omega). \tag{21}$$

Combining (21) and Lemma 3.3, it follows easily that

$$[\underline{\lim}_\theta \varphi](\bullet, \omega) = [\overline{\lim}^\theta \varphi](\bullet, \omega) = [\mathcal{K}(r)](\bullet, \omega). \tag{22}$$

By Proposition 1 and definitions of inf and sup in C_τ, it is clear that

$$\inf\{\varphi(u, \theta_{-u}\omega, x_0)[\bullet] : u \geq t\} \leq \varphi(t, \theta_{-t}\omega, x_0)[\bullet]$$

and

$$\varphi(t, \theta_{-t}\omega, x_0)[\bullet] \leq \sup\{\varphi(u, \theta_{-u}\omega, x_0)[\bullet] : u \geq t\}$$

for all $t \geq \tau$ and $\omega \in \Omega$, which together with (22) implies that (19) holds. Furthermore, by the definition of the cocycle φ, we conclude that $[\mathcal{K}(r)](\bullet, \omega)$ is an $\mathscr{F}_-$-measurable random equilibrium in C_τ. The proof is complete. ■

Remark 2: By Theorem 3.1, for the global stability of (5), we do not need to construct Lyapunov functions and condition (A1) is easy to verify. Furthermore, the boundedness of the nonlinear term h may be restricted. Actually, we believe that the global Lipschitz assumption is enough, which yields that the proofs in Appendix A will be more complex and the well-posedness of the operator $\mathcal{K}$ should be checked.

Corollary 3.1: Assume that (A1) holds. It follows that

$$\lim_{t\to\infty} \|\varphi(t, \omega, x_0) - \varphi(t, \omega, y_0)\|_{C_\tau} = 0 \tag{23}$$

in probability, for any different initial values $x_0, y_0 \in C_\tau$. Thus, we have that

$$\lim_{t\to\infty} |x(t, \omega, x_0) - x(t, \omega, y_0)| = 0 \tag{24}$$

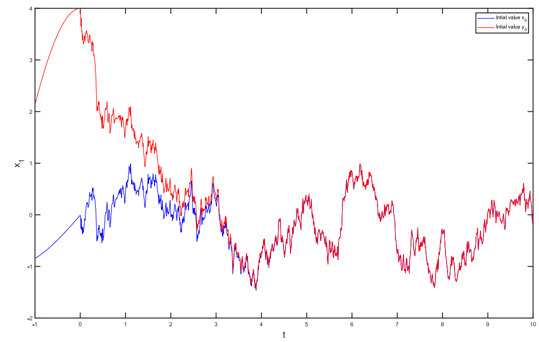

Fig. 1. Numerical simulation for $x^1(t)$ of (26) with differential initial values $x_0 = (\sin t, 2\sin t, 3\sin t)^T$ and $y_0 = (4\cos t, 5\cos t, 6\cos t)^T$ for all $t \in [-1, 0]$.

in probability, $x(t, \omega, x_0) = \varphi(t, \omega, x_0)[0]$ is the solution of (5) with the initial value $x_0 \in C_\tau$.

Proof: By (19), it is obvious that

$$\lim_{t\to\infty} \|\varphi(t, \theta_{-t}\omega, x_0) - \varphi(t, \theta_{-t}\omega, y_0)\|_{C_\tau} = 0 \tag{25}$$

for all $\omega \in \Omega$. Since $\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$, (25) gives that (23) and (24) hold. The proof is complete. ■

IV. Examples

In this section, we will present an example to illustrate the effect of Theorem 3.1 and Corollary 3.1.

Example 4.1: Let $W(t) = (W_1(t), W_2(t), W_3(t))^{\mathrm{T}}$ be a three-dimensional Brownian motion, $I_1 = I_2 = I_3 = 0$ and $\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 3$. We consider the following SDDEs with additive white noise:

$$dx^i(t) = \left[-(Ax)^i(t) + \frac{1}{3}\arctan(x^{i-1}(t-1))\right] dt + \sigma_i dW_i(t) \tag{26}$$

$i = 1, 2, 3$, where

$$A = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \tag{27}$$

$x = (x^1, x^2, x^3)^{\mathrm{T}} \in \mathbb{R}^3$ and x^0 represents x^3. Direct computation shows that $\lambda = \frac{3}{2}$ and $L = \frac{\sqrt{3}}{3}$; it is clear that

$$\frac{\sqrt{3}L}{\lambda} = \frac{2}{3} < 1.$$

Hence, there exists a globally attracting stationary solution for the stochastic flow generated by (26) in C_τ and trajectory synchronization will occur based on Theorem 3.1 and Corollary 3.1; see Figs. 1 – 3.

Remark 3: Example 4.1 and Figs. 1– 3 show that there is no trivial stationary solutions for (26) and the forward trajectories can only be synchronized, which implies that the method of constructing Lyapunov functions is invalid here. However, by Theorem 3.1, we prove that all the pull-back trajectories will converge to a globally attracting random equilibrium in the infinite dimensional Banach space C_τ.

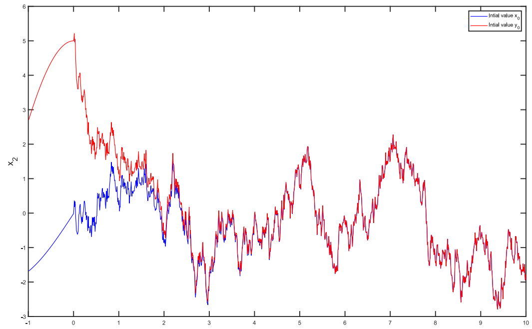

Fig. 2. Numerical simulation for $x^2(t)$ of (26) with differential initial values $x_0 = (\sin t, 2\sin t, 3\sin t)^T$ and $y_0 = (4\cos t, 5\cos t, 6\cos t)^T$ for all $t \in [-1, 0]$.

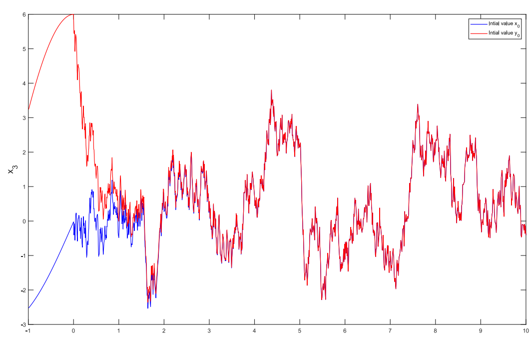

Fig. 3. Numerical simulation for $x^3(t)$ of (26) with differential initial values $x_0 = (\sin t, 2\sin t, 3\sin t)^T$ and $y_0 = (4\cos t, 5\cos t, 6\cos t)^T$ for all $t \in [-1, 0]$.

Appendix A
Proofs of the Compactness and Continuity of the Pull-Back Trajectories

Proposition 1: Suppose that (A1) holds. For any initial value $x_0 \in C_\tau$ and $\omega \in \Omega$, the pull-back trajectory $\{\varphi(t, \theta_{-t}\omega, x_0) : t \geq \tau\}$ is a relative compact set in C_τ.

Proof: The proof will be divided into two parts. First, we will show that $\{\varphi(t, \theta_{-t}\omega, x_0) : t \geq \tau\}$ is uniformly bounded in C_τ. By (10), it is obvious that for all $\omega \in \Omega$ and $x_0 \in C_\tau$, we have

$$
\begin{aligned}
&\sup_{t\geq\tau} \|\varphi(t, \theta_{-t}\omega, x_0)\|_{C_\tau} \\
&= \sup_{t\geq\tau} \sup_{-\tau\leq s\leq 0} |\varphi(t, \theta_{-t}\omega, x_0)[s]| \\
&\leq \sup_{0\leq u<\infty} |e^{-Au}x(0)| + \sup_{-\tau\leq s\leq 0} \|e^{-As}\| \cdot |M| \cdot \int_{-\infty}^{0} \|e^{Au}\| du \\
&\quad + \sup_{-\tau\leq s\leq 0} \|e^{-As}\| \cdot \sup_{t\geq\tau} \sup_{-\tau\leq s\leq 0} \left| \int_{-t}^{s} e^{Au}\sigma dW_u \right| \\
&\leq \sqrt{n}|x(0)| + \sqrt{n}|M| \cdot \sup_{-\tau\leq s\leq 0} \|e^{-As}\| \cdot \int_{-\infty}^{0} e^{\lambda u} du \\
&\quad + 2 \sup_{-\tau\leq s\leq 0} \|e^{-As}\| \cdot \sup_{\tilde{t}\geq 0} \left| \int_{-\infty}^{-\tilde{t}} e^{Au}\sigma dW_u \right| \\
&< \infty
\end{aligned}
$$

where $M = (M_1, \ldots, M_n)^T$ and the last inequality holds due to that $\lim_{\tilde{t}\to\infty} \int_{-\infty}^{-\tilde{t}} e^{Au}\sigma dW_u = 0$ and $\int_{-\infty}^{-\tilde{t}} e^{Au}\sigma dW_u$ is continuous with respect to $\tilde{t} \geq 0$, see [36, Problem 3.20 in Ch. 1]. This shows that $\{\varphi(t, \theta_{-t}\omega, x_0) : t \geq \tau\}$ is a bounded set in C_τ for all $\omega \in \Omega$ and $x_0 \in C_\tau$.

Next, we will prove that the family of functions $\{\varphi(t, \theta_{-t}\omega, x_0) : t \geq \tau\}$ is equicontinuous for all $\omega \in \Omega$ and $x_0 \in C_\tau$. Given any $-\tau \leq s_1 < s_2 \leq 0$, it is easy to see that

$$
\begin{aligned}
&|\varphi(t, \theta_{-t}\omega, x_0)[s_2] - \varphi(t, \theta_{-t}\omega, x_0)[s_1]| \\
&\leq \|e^{-A(t+s_2)} - e^{-A(t+s_1)}\| \cdot |x(0)| \\
&\quad + \left| \int_{-t}^{s_2} e^{-A(s_2-u)} \cdot \triangle du - \int_{-t}^{s_1} e^{-A(s_1-u)} \cdot \triangle du \right| \\
&\quad + \left| \int_{-t}^{s_2} e^{-A(s_2-u)}\sigma dW_u - \int_{-t}^{s_1} e^{-A(s_1-u)}\sigma dW_u \right|
\end{aligned}
$$

where $\triangle = h\big(\varphi(u+t, \theta_{-t}\omega, x_0)\big)$. For the convenience, the proof will be divided into three steps. First, it is clear that

$$
\begin{aligned}
&\|e^{-A(t+s_2)} - e^{-A(t+s_1)}\| \cdot |x(0)| \\
&\leq \|e^{-A(t+s_1)}\| \cdot |x(0)| \cdot \|e^{-A(s_2-s_1)} - \mathbb{I}_{n\times n}\| \\
&\leq \sqrt{n} \cdot |x(0)| \cdot \sqrt{n}|1 - e^{-\bar{\lambda}(s_2-s_1)}|
\end{aligned} \tag{28}
$$

where $\bar{\lambda} = \max_{1\leq i\leq n} a_i > 0$ and $\mathbb{I}_{n\times n} = \mathrm{diag}(1, \ldots, 1)$. Second, let $\alpha = \sup_{-\tau\leq s\leq 0} \|e^{-As}\|$, we observe that

$$
\begin{aligned}
&\left| \int_{-t}^{s_2} e^{-A(s_2-u)} \cdot \triangle du - \int_{-t}^{s_1} e^{-A(s_1-u)} \cdot \triangle du \right| \\
&\leq \left| e^{-As_2} \int_{-t}^{s_2} e^{Au} \cdot \triangle du - e^{-As_1} \int_{-t}^{s_2} e^{Au} \cdot \triangle du \right| \\
&\quad + \left| e^{-As_1} \int_{-t}^{s_2} e^{Au} \cdot \triangle du - e^{-As_1} \int_{-t}^{s_1} e^{Au} \cdot \triangle du \right| \\
&\leq \alpha|M| \cdot \int_{-\infty}^{0} \|e^{Au}\| du \cdot \|e^{-A(s_2-s_1)} - \mathbb{I}_{n\times n}\| \\
&\quad + \alpha|M| \cdot \int_{s_1}^{s_2} \|e^{Au}\| du \\
&\leq \alpha|M| \cdot \int_{-\infty}^{0} \|e^{Au}\| du \cdot \sqrt{n}|1 - e^{-\bar{\lambda}(s_2-s_1)}| \\
&\quad + \sqrt{n}\alpha|M| \cdot |s_2 - s_1|.
\end{aligned} \tag{29}
$$

Finally, direct calculation shows that

$$
\begin{aligned}
&\left| \int_{-t}^{s_2} e^{-A(s_2-u)}\sigma dW_u - \int_{-t}^{s_1} e^{-A(s_1-u)}\sigma dW_u \right| \\
&\leq \left| e^{-As_2} \int_{-t}^{s_2} e^{Au}\sigma dW_u - e^{-As_1} \int_{-t}^{s_2} e^{Au}\sigma dW_u \right| \\
&\quad + \left| e^{-As_1} \int_{-t}^{s_2} e^{Au}\sigma dW_u - e^{-As_1} \int_{-t}^{s_1} e^{Au}\sigma dW_u \right| \\
&\leq 2\alpha \sup_{\tilde{t}\geq 0} \left| \int_{-\infty}^{-\tilde{t}} e^{Au}\sigma dW_u \right| \cdot \|e^{-A(s_2-s_1)} - \mathbb{I}_{n\times n}\|
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha\left|\int_{s_1}^{s_2} e^{Au}\sigma dW_u\right| \\
&\leq 2\alpha \sup_{\tilde{t}\geq 0}\left|\int_{-\infty}^{-\tilde{t}} e^{Au}\sigma dW_u\right|\cdot\sqrt{n}|1-e^{-\bar{\lambda}(s_2-s_1)}| \\
&+\alpha\left|\int_{s_1}^{s_2} e^{Au}\sigma dW_u\right|. \qquad (30)
\end{aligned}
$$

Note that $\int_{-\infty}^{s} e^{Au}\sigma dW_u$ is continuous with respect to $s\leq 0$ and then it is uniformly continuous on $[-\tau, 0]$. This together with (28)–(30) implies that $\{\varphi(t,\theta_{-t}\omega,x_0) : t\geq\tau\}$ is equicontinuous for all $\omega\in\Omega$ and $x_0\in C_\tau$. Using the Arzela–Ascoli theorem, we complete the proof. ■

Remark 4: Note that C_τ^+ is a solid normal minihedral cone, which together with the compactness of the pull-back trajectories can guarantee that inf and sup in C_τ exist; see Theorem 3.1.2 and Theorem 3.2.1 in [14].

Proposition 2: Assume that (A1) holds. It follows that the mapping

$$t\mapsto\varphi(t,\theta_{-t}\omega,x_0)$$

is continuous from $\mathbb{R}_+$ into C_τ for all $\omega\in\Omega$ and $x_0\in C_\tau$.

Proof: By (6), it is clear that

$$
\begin{aligned}
\varphi(t,\theta_{-t}\omega,x_0) &= y_t\big(\theta_{-t}\omega,x_0-z_0(\theta_{-t}\omega)\big)+z_t(\theta_{-t}\omega)\\
&= y_t\big(\theta_{-t}\omega,x_0-z_{-t}(\omega)\big)+z_0(\omega).
\end{aligned}
$$

This shows that it is enough to prove the continuity of $y_t\big(\theta_{-t}\omega,x_0-z_{-t}(\omega)\big)=y_t\big(\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)$ for any fixed $\omega\in\Omega$ and $x_0\in C_\tau$. Let $y^k\big(t,\omega,x_0-z_0(\omega)\big)=y^k\big(t,\omega,y_0(\omega)\big)$ be the Picard approximations of the solution for (7) and (8), i.e.,

$$y_0^0\big(\omega,x_0-z_0(\omega)\big)=y_0(\omega)=x_0-z_0(\omega)$$

and

$$y^0\big(t,\omega,x_0-z_0(\omega)\big)=[y_0(\omega)](0)=[x_0-z_0(\omega)](0)$$

for all $t\geq 0$. Moreover, for each $k\geq 1$, $y_0^k\big(\omega,x_0-z_0(\omega)\big)=y_0(\omega)=x_0-z_0(\omega)$ and

$$
\begin{aligned}
y^k\big(t,\omega,y_0(\omega)\big) &= [y_0(\omega)](0)-\int_0^t Ay^{k-1}\big(u,\omega,y_0(\omega)\big)du\\
&+\int_0^t h\left(y_u^{k-1}\big(\omega,y_0(\omega)\big)+z_u(\omega)\right)du.
\end{aligned}
$$

It follows that for any $t\geq 0$ and $\omega\in\Omega$, we have

$$y_0^0\big(\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)=y_0(\theta_{-t}\omega)=x_0-z_{-t}(\omega)$$

and

$$y^0\big(u,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)=[y_0(\theta_{-t}\omega)](0)=x_0(0)-z(-t,\omega)$$

for all $u\geq 0$. Analogously,

$$y_0^k\big(\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)=y_0(\theta_{-t}\omega)=x_0-z_{-t}(\omega)$$

and

$$
\begin{aligned}
&y^k\big(u,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\\
&=[y_0(\theta_{-t}\omega)](0)-\int_0^u Ay^{k-1}\big(\widetilde{u},\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)d\widetilde{u}\\
&+\int_0^u h\left(y_{\widetilde{u}}^{k-1}\big(\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)+z_{\widetilde{u}}(\theta_{-t}\omega)\right)d\widetilde{u} \qquad (31)
\end{aligned}
$$

where $u\geq 0$. Now, we claim that

$$
\begin{aligned}
&\sup_{0\leq u\leq\bar{u}}\left|y^{k+1}\big(u,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)-y^k\big(u,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\right|\\
&\leq\frac{N(\widetilde{L}\bar{u})^k}{k!} \qquad (32)
\end{aligned}
$$

for all $k\in\mathbb{N}$ and $\bar{u},t\in[0,T]$, $T>0$, where $\widetilde{L}=\|A\|+L$ and $N=N(T,\omega)$ will be defined below. For $k=0$, direct computation shows that

$$
\begin{aligned}
&\sup_{0\leq u\leq T}\left|y^1\big(u,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)-y^0\big(u,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\right|\\
&\leq\sup_{0\leq u\leq T}\int_0^u\left|Ay^0\big(\widetilde{u},\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\right|d\widetilde{u}+T\sup_{\xi\in C_\tau}|h(\xi)|\\
&\leq T\sup_{0\leq t\leq T}|A[x_0(0)-z(-t,\omega)]|+T|M|\\
&:=N(T,\omega) \qquad (33)
\end{aligned}
$$

which implies that (32) holds for $k=0$. Next, we assume that (32) holds for some $k\in\mathbb{N}$. Then, we have

$$
\begin{aligned}
&\sup_{0\leq u\leq\bar{u}}\left|y^{k+2}\big(u,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\right.\\
&\quad\left.-y^{k+1}\big(u,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\right|\\
&\leq\sup_{0\leq u\leq\bar{u}}\int_0^u\widetilde{L}\|y_{\widetilde{u}}^{k+1}\big(\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\\
&\quad-y_{\widetilde{u}}^{k}\big(\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\|_{C_\tau}d\widetilde{u}\\
&\leq\widetilde{L}\int_0^{\bar{u}}\frac{N(\widetilde{L}\widetilde{u})^k}{k!}d\widetilde{u}=\frac{N(\widetilde{L}\bar{u})^{k+1}}{(k+1)!}.
\end{aligned}
$$

This yields that (32) holds for $k+1$. Consequently, by induction, we conclude that (32) holds for all $k\in\mathbb{N}$. Next, we will prove that for any $\omega\in\Omega$, the mapping

$$(u,t)\longmapsto y_u^k\big(\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big) \qquad (34)$$

is continuous from $\mathbb{R}_+\times\mathbb{R}_+$ into C_τ for all $k\in\mathbb{N}$. The proof of this property can be divided into two parts. Let $(u,t)\in[0,T]\times[0,T]$, $T>0$. First, from (32), it is easy to check that

$$
\begin{aligned}
&\sup_{-\tau\leq u\leq T}\left|y^k\big(u,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\right|\\
&\leq\sup_{-\tau\leq u\leq T}\left|y^0\big(u,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\right|+\sum_{i=1}^k\sup_{-\tau\leq u\leq T}\\
&\quad\left|y^i\big(u,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)-y^{i-1}\big(u,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\right|\\
&=\sup_{-\tau\leq u\leq T}\left|y^0\big(u,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\right|+\sum_{i=1}^k\sup_{0\leq u\leq T}\\
&\quad\left|y^i\big(u,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)-y^{i-1}\big(u,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\right|
\end{aligned}
$$

$$\begin{aligned}
&\le \sum_{i=1}^{k} \frac{N(\widetilde{L}T)^{i-1}}{(i-1)!} + \sup_{-\tau\le u\le 0} |x_0(u) - z(-t+u,\omega)| \\
&\quad + \sup_{0\le t\le T} |x_0(0) - z(-t,\omega)| \\
&\le N\exp(\widetilde{L}T) + \sup_{-\tau\le u\le 0} |x_0(u)| + \sup_{-T-\tau\le u\le 0} |z(u,\omega)| \\
&\quad + \sup_{0\le t\le T} |x_0(0) - z(-t,\omega)| := K(T,\omega)
\end{aligned}$$

for all $\omega \in \Omega, k \in \mathbb{N}$ and $t \in [0,T]$. Therefore, for any $0 \le u_1 \le u_2 \le T$, we have that

$$\begin{aligned}
&\|y^k_{u_2}\big(\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big) - y^k_{u_1}\big(\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big)\|_{C_\tau} \\
&= \sup_{-\tau\le s\le 0} \big| y^k\big(u_2+s, \theta_{-t}\omega, y_0(\theta_{-t}\omega)\big) \\
&\quad - y^k\big(u_1+s, \theta_{-t}\omega, y_0(\theta_{-t}\omega)\big)\big| \\
&\le \sup_{-\tau\le s\le 0} \int_{u_1+s}^{u_2+s} \big(\big|Ay^{k-1}\big(\widetilde{u}, \theta_{-t}\omega, y_0(\theta_{-t}\omega)\big)\big| \\
&\quad + \big|h\left(y^{k-1}_{\widetilde{u}}\big(\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big) + z_{\widetilde{u}}(\theta_{-t}\omega)\right)\big|\big)\, d\widetilde{u} \\
&\le \left[\|A\|K(T,\omega) + \sup_{\xi\in C_\tau} |h(\xi)|\right] \cdot |u_2 - u_1|
\end{aligned}$$

which induces that for all $\omega \in \Omega$ and $k \in \mathbb{N}$,

$$u \longmapsto y^k_u\big(\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big), \quad u \in [0,T] \tag{35}$$

is continuous uniformly with respect to $t \in [0,T]$. The remaining task is to show that

$$t \longmapsto y^k_u\big(\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big), \quad t \in [0,T] \tag{36}$$

is continuous for any $u \in [0,T]$, $\omega \in \Omega$ and $k \in \mathbb{N}$. For $k = 0$, choose $0 \le t_1 \le t_2 \le T$, it follows that

$$\begin{aligned}
&\|y^0_u\big(\theta_{-t_2}\omega, y_0(\theta_{-t_2}\omega)\big) - y^0_u\big(\theta_{-t_1}\omega, y_0(\theta_{-t_1}\omega)\big)\|_{C_\tau} \\
&= \sup_{-\tau\le s\le 0} \big| y^0\big(u+s, \theta_{-t_2}\omega, y_0(\theta_{-t_2}\omega)\big) \\
&\quad - y^0\big(u+s, \theta_{-t_1}\omega, y_0(\theta_{-t_1}\omega)\big)\big| \\
&\le \sup_{-\tau\le s\le 0} |z(s-t_2,\omega) - z(s-t_1,\omega)|
\end{aligned}$$

for all $u \in [0,T]$. This implies that for $k = 1$, we have

$$\begin{aligned}
&\|y^1_u\big(\theta_{-t_2}\omega, y_0(\theta_{-t_2}\omega)\big) - y^1_u\big(\theta_{-t_1}\omega, y_0(\theta_{-t_1}\omega)\big)\|_{C_\tau} \\
&= \sup_{-\tau\le s\le 0} \big| y^1\big(u+s, \theta_{-t_2}\omega, y_0(\theta_{-t_2}\omega)\big) \\
&\quad - y^1\big(u+s, \theta_{-t_1}\omega, y_0(\theta_{-t_1}\omega)\big)\big| \\
&\le \sup_{-\tau\le s\le 0} |z(s-t_2,\omega) - z(s-t_1,\omega)| + \sup_{-\tau\le s\le 0} \int_0^{u+s} \\
&\quad [\widetilde{L}\|y^0_{\widetilde{u}}\big(\theta_{-t_2}\omega, y_0(\theta_{-t_2}\omega)\big) - y^0_{\widetilde{u}}\big(\theta_{-t_1}\omega, y_0(\theta_{-t_1}\omega)\big)\|_{C_\tau} \\
&\quad + L\|z_{\widetilde{u}}(\theta_{-t_2}\omega) - z_{\widetilde{u}}(\theta_{-t_1}\omega)\|_{C_\tau}]d\widetilde{u} \\
&\le (\widetilde{L}T + 1) \sup_{-\tau\le s\le 0} |z(s-t_2,\omega) - z(s-t_1,\omega)| \\
&\quad + LT \sup_{-\tau\le s\le T} |z(s-t_2,\omega) - z(s-t_1,\omega)| \\
&\le (\widetilde{L}T + LT + 1) \sup_{-\tau\le s\le T} |z(s-t_2,\omega) - z(s-t_1,\omega)| \\
&:= D_1 \sup_{-\tau\le s\le T} |z(s-t_2,\omega) - z(s-t_1,\omega)|
\end{aligned}$$

for all $u \in [0,T]$ and $0 \le t_1 \le t_2 \le T$. Now, we assume that for some $k \ge 1$, there exists a constant $D_k > 0$ such that

$$\begin{aligned}
&\|y^k_u\big(\theta_{-t_2}\omega, y_0(\theta_{-t_2}\omega)\big) - y^k_u\big(\theta_{-t_1}\omega, y_0(\theta_{-t_1}\omega)\big)\|_{C_\tau} \\
&\le D_k \sup_{-\tau\le s\le T} |z(s-t_2,\omega) - z(s-t_1,\omega)|
\end{aligned}$$

for all $u \in [0,T]$ and $0 \le t_1 \le t_2 \le T$. Similar as the above analysis, it is evident that

$$\begin{aligned}
&\|y^{k+1}_u\big(\theta_{-t_2}\omega, y_0(\theta_{-t_2}\omega)\big) - y^{k+1}_u\big(\theta_{-t_1}\omega, y_0(\theta_{-t_1}\omega)\big)\|_{C_\tau} \\
&\le (\widetilde{L}TD_k + LT + 1) \sup_{-\tau\le s\le T} |z(s-t_2,\omega) - z(s-t_1,\omega)| \\
&:= D_{k+1} \sup_{-\tau\le s\le T} |z(s-t_2,\omega) - z(s-t_1,\omega)|
\end{aligned}$$

for all $u \in [0,T]$ and $0 \le t_1 \le t_2 \le T$. Therefore, by induction, it is easily seen that (36) holds for any $u \in [0,T]$, $\omega \in \Omega$ and $k \in \mathbb{N}$. Here, we use the fact that for any $\omega \in \Omega$, $z(t,\omega)$ is uniformly continuous on $[-T-\tau, T]$. Combining (35) and (36), it is obvious that

$$y^k_u\big(\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big) : [0,T] \times [0,T] \longmapsto C_\tau$$

is continuous with respect to (u,t) for all $\omega \in \Omega$, $k \in \mathbb{N}$ and $T > 0$. That is, (34) holds. In view of inequality (32), we can obtain that

$$y^k_u\big(\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big) \longrightarrow y_u\big(\theta_{-t}\omega, y_0(\theta_{-t}\omega)$$

in C_τ, as $k \to \infty$, uniformly with respect to $(u,t) \in [0,T] \times [0,T]$, where $T > 0$. Using Theorem 21.6 in [47, Ch. 2], it follows immediately that

$$(u,t) \longmapsto y_u\big(\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big)$$

is continuous from $[0,T] \times [0,T]$ into C_τ for all $\omega \in \Omega$ and $T > 0$. This proves that

$$y_t\big(\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big) : [0,T] \longmapsto C_\tau$$

is continuous with respect to $t \in [0,T]$ for all $\omega \in \Omega$ and $T > 0$. Since $T > 0$ is arbitrary, the proof is complete. ■

Proposition 3: Let $\{r_t : \Omega \to \mathbb{R}^n, t \in \mathbb{R}_+\}$ be a family of uniformly bounded random variables and for all $\omega \in \Omega$,

$$r_t(\omega) \to r(\omega) \quad \text{as} \quad t \to \infty$$

where $|r_t(\omega)| \le |M|$ for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$, it follows that

$$\lim_{t\to\infty} [\mathcal{K}(r_t)](\bullet,\omega) = [\mathcal{K}(r)](\bullet,\omega) \text{ in } C_\tau. \tag{37}$$

Proof: By the definition of $\mathcal{K}$, we have that

$$\begin{aligned}&\sup_{-\tau\le s\le 0}|[\mathcal{K}(r_t)](s,\omega)-[\mathcal{K}(r)](s,\omega)|\\&\le \sup_{-\tau\le s\le 0}\int_{-\infty}^{s}\|e^{-A(s-u)}\|\cdot|r_t(\theta_u\omega)-r(\theta_u\omega)|du\\&\le \sqrt{n}\sup_{-\tau\le s\le 0}\|e^{-As}\|\int_{-\infty}^{0}e^{\lambda u}|r_t(\theta_u\omega)-r(\theta_u\omega)|du\end{aligned}$$

which together with Lebesgue's-dominated convergence theorem yields that (37) holds; the proof is complete. ■

Proposition 4: For any initial value $x(0)\in\mathbb{R}^n$ and bounded random variable $r:\Omega\to\mathbb{R}^n$ such that $r(\omega)\in[-M,M]$ for all $\omega\in\Omega$, we have that

$$\begin{aligned}[\mathcal{K}(r)](\bullet,\omega)=\lim_{t\to\infty}\Big\{&e^{-A(t+\bullet)}x(0)+\int_{-t}^{\bullet}e^{-A(\bullet-u)}r(\theta_u\omega)du\\&+\int_{-t}^{\bullet}e^{-A(\bullet-u)}\sigma dW_u\Big\}\end{aligned}\tag{38}$$

in C_τ.

Proof: First, it is obvious that

$$\sup_{-\tau\le s\le 0}\|e^{-A(t+s)}\|\le\sqrt{n}\sup_{-\tau\le s\le 0}\|e^{-As}\|\cdot e^{-\lambda t}\to 0\tag{39}$$

as $t\to\infty$. Second, we show that

$$\sup_{-\tau\le s\le 0}\left|\int_{-t}^{s}e^{-A(s-u)}r(\theta_u\omega)du-\int_{-\infty}^{s}e^{-A(s-u)}r(\theta_u\omega)du\right|\tag{40}$$

tends to 0 as $t\to\infty$. It is clear that

$$\begin{aligned}&\sup_{-\tau\le s\le 0}\left|\int_{-\infty}^{-t}e^{-A(s-u)}r(\theta_u\omega)du\right|\\&\le\sup_{-\tau\le s\le 0}\|e^{-As}\|\int_{-\infty}^{-t}\|e^{Au}\|\cdot|M|du\\&\le\sqrt{n}\sup_{-\tau\le s\le 0}\|e^{-As}\|\cdot|M|\int_{-\infty}^{-t}e^{\lambda u}du\\&\longrightarrow 0\quad\text{as}\quad t\to\infty.\end{aligned}$$

Third, it is easily seen that

$$\begin{aligned}&\sup_{-\tau\le s\le 0}\left|\int_{-t}^{s}e^{-A(s-u)}\sigma dW_u-\int_{-\infty}^{s}e^{-A(s-u)}\sigma dW_u\right|\\&\le\sup_{-\tau\le s\le 0}\|e^{-As}\|\cdot\left|\int_{-\infty}^{-t}e^{Au}\sigma dW_u\right|\\&\longrightarrow 0\quad\text{as}\quad t\to\infty\end{aligned}$$

which together with (39) and (40) implies that (38) holds; the proof is complete. ■

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