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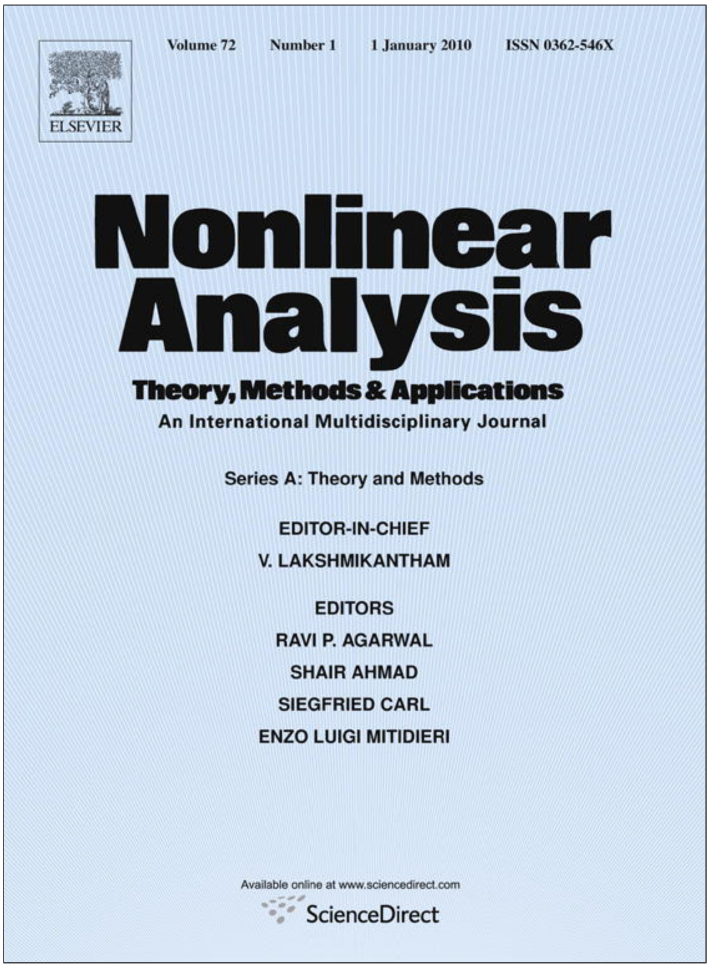

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Existence of homoclinic solutions for a class of second-order Hamiltonian systems[☆]

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ABSTRACT

A new result for existence of homoclinic orbits is obtained for the second-order Hamiltonian systems under a class of new superquadratic conditions. A homoclinic orbit is obtained as a limit of solutions of a certain sequence of boundary-value problems which are obtained by the minimax methods.

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1. Introduction and main results

Consider the second-order nonautonomous Hamiltonian systems

$$\ddot{u}(t) - L(t)u(t) + \nabla W\big(t, u(t)\big) = 0 \tag{1.1}$$

where $t \in R$, $u \in R^n$, $L \in C(R, R^{n\times n})$ is a symmetric matrix valued function and $W : R \times R^n \to R$. As usual we say that a nonzero solution $u(t)$ of (1.1) is homoclinic (to 0) if $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to +\infty$.

Recently the existence and multiplicity of homoclinic orbits for (1.1) have been extensively studied in many papers via critical theory (see [1,3–9,11–20]). For (1.1), the case where $L(t)$ and $W(t, x)$ are either independent of t or periodic in t is studied by several authors (see [7,9,16,17]). Rabinowitz [16] has shown the existence of homoclinic orbits as a limit of $2kT$-periodic solutions of (1.1). By the same method, several results for general Hamiltonian systems were obtained by Felmer et al. [7], Izydorek and Janczewska [9], Tang and Xiao [19]. The related results can be referred to in [15] for the case where $L(t)$ and $W(t, x)$ are either independent of t.

If $L(t)$ and $W(t, x)$ are neither autonomous nor periodic in t, the problem of existence of homoclinic orbits for (1.1) is quite different from the one just described, because of the lack of compactness of the Sobolev embedding. In [17], Rabinowitz and Tanaka studied system (1.1) without a periodicity assumption, both for L and W. More precisely, they assumed that the smallest eigenvalue of $L(t)$ tends to $+\infty$ as $|t| \to \infty$, using a variant of the Mountain Pass theorem without the Palais–Smale condition, and proved that system (1.1) possesses a homoclinic orbit.

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Theorem A *(See [17]). Assume that L and W satisfy the following conditions:*

(L) *$L(t)$ is positive definite symmetric matrix for all $t \in R$ and there exists an $l \in C(R, (0,\infty))$ such that $l(t) \to +\infty$ as $|t| \to \infty$ and*

$$\big(L(t)x, x\big) \geq l(t)|x|^2$$

for all $t \in R$ and $x \in R^n$;

(W1) *$W \in C^1(R \times R^n, R)$ and there is a constant $\mu > 2$ such that*

$$0 < \mu W(t,x) \leq \big(x, \nabla W(t,x)\big)$$

for all $t \in R$ and $x \in R^n \setminus \{0\}$;

(W2) *$|\nabla W(t,x)| = o(|x|)$ as $|x| \to 0$ uniformly with respect to $t \in R$;*

(W3) *there is a $\overline{W} \in C(R^n, R)$ such that*

$$|W(t,x)| + |\nabla W(t,x)| \leq |\overline{W}(x)|$$

for all $t \in R$ and $x \in R^n$.

Then system (1.1) possesses a nontrivial homoclinic solution.

Motivated by the above papers [8,9,13,14], we will obtain a new criterion for guaranteeing that (1.1) has one nontrivial homoclinic solution without any periodicity or coercivity condition, especially, $W(t,x)$ satisfies a kind of new superquadratic condition which is different from the corresponding condition in the known literature. We prove the existence of one homoclinic solution as the limit of solutions of a certain sequence of boundary-value problems which are obtained by the minimax methods. The main results are the following theorems.

Theorem 1.1. *Assume that L and W satisfy assumption* (L) *and the following conditions:*

(H1) *$W(t,0) \equiv 0$, $W \in C^1(R \times R^n, R)$ and $|\nabla W(t,x)| = o(|x|)$ as $|x| \to 0$ uniformly in $t \in R$;*

(H2) *there are two constants $\mu > 2$ and $\nu \in [0, \frac{\mu}{2} - 1)$ and $\beta \in L^1(R, R^+)$ such that*

$$\big(\nabla W(t,x), x\big) - \mu W(t,x) \geq -\nu\big(L(t)x, x\big) - \beta(t)$$

for all $t \in R$ and $x \in R^n \setminus \{0\}$;

(H3) *there exists $T_0 > 0$ such that*

$$\liminf_{|x|\to\infty} \frac{W(t,x)}{|x|^2} > \frac{\pi^2}{2T_0^2} + \frac{l_1}{2}$$

uniformly in $t \in [-T_0, T_0]$, where l_1 is the biggest eigenvalue of $L(t)$ on $[-T_0, T_0]$.

Then system (1.1) possesses a nontrivial homoclinic solution.

Remark 1.1. For system (1.1), Theorem 1.1 gives a new criterion for the existence of homoclinic solutions by relaxing condition (W1) and changing condition (W3).

2. Proof of theorems

By the similar idea of [8], we approximate an homoclinic orbit of (1.1) by the following problem

$$\begin{cases} \ddot{u}(t) - L(t)u(t) + \nabla W\big(t, u(t)\big) = 0, & \text{for } t \in [-T, T] \\ u(-T) = u(T) = 0. \end{cases} \tag{2.1}$$

Let

$$E_T = \left\{ u \in W^{1,2}\big([-T,T], R^n\big) \,\Big|\, \int_{-T}^{T} \left[|\dot{u}(t)|^2 + \big(L(t)u(t), u(t)\big)\right] \mathrm{d}t < +\infty \right\}$$

where

$$W^{1,2}\big([-T,T], R^n\big) = \{u : [-T,T] \longrightarrow R^n \,|\, u \text{ is absolutely continuous}, u(-T) = u(T) = 0, \dot{u} \in L^2([-T,T], R^n)\}$$

and for $u \in E_T$, let

$$\|u\| = \left\{ \int_{-T}^{T} \left[|\dot{u}(t)|^2 + \big(L(t)u(t), u(t)\big)\right] \mathrm{d}t \right\}^{\frac{1}{2}},$$

then E_T is a Hilbert space on the above norm.

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We consider a functional $I : E_T \to R$, defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{-T}^{T} W\big(t, u(t)\big)\mathrm{d}t. \tag{2.2}$$

Then we can easily check that $I \in C^1(E_T, R)$ and

$$\langle I'(u), v\rangle = \int_{-T}^{T} \big[(\dot{u}(t), \dot{v}(t)) + (L(t)u(t), v(t)) - \big(\nabla W\big(t, u(t)\big), v(t)\big)\big]\,\mathrm{d}t \tag{2.3}$$

for all $u, v \in E_T$. Furthermore, it is well known that the critical points of I in E_T are classical solutions of (2.1) (see [2,10]).

We will obtain a critical point of I by using a standard of the Mountain Pass theorem. It provides the minimax characterization for the critical value which is important for what follows. Therefore, we state this theorem precisely.

Lemma 2.1 *(See [16]). Let E be a real Banach space and $I \in C^1(E, R)$ satisfy the Palais–Smale condition. If I satisfies the following conditions:*

(i) $I(0) = 0$;
(ii) *there exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$;*
(iii) *there exists $e \in E \setminus \bar{B}_\rho(0)$ such that $I(e) \leq 0$,*

then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I\big(g(s)\big),$$

where $B_\rho(0)$ is an open ball in E of radius ρ about at 0, and

$$\Gamma = \big\{g \in C\big([0, 1], E\big) : g(0) = 0, g(1) = e\big\}.$$

Lemma 2.2. *Let $a > 0$ and $u \in W^{1,2}\big([-T, T], R^n\big)$. Then for every $t \in [-T, T]$, the following inequality holds:*

$$|u(t)| \leq (2a)^{-\frac{1}{2}}\left(\int_{t-a}^{t+a} |u(s)|^2\mathrm{d}s\right)^{\frac{1}{2}} + \sqrt{\frac{a}{2}}\left(\int_{t-a}^{t+a} |\dot{u}(s)|^2\mathrm{d}s\right)^{\frac{1}{2}}. \tag{2.4}$$

Proof. Fix $t \in [-T, T]$. For every $\tau \in [-T, T]$,

$$|u(t)| \leq |u(\tau)| + \left|\int_{\tau}^{t} \dot{u}(s)\mathrm{d}s\right|. \tag{2.5}$$

Since u can be extended by zero in $R \setminus [-T, T]$, integrating (2.5) over $[t - a, t + a]$ and using the Hölder inequality, we obtain

$$\begin{aligned}
2a\,|u(t)| &\leq \int_{t-a}^{t+a} |u(\tau)|\,\mathrm{d}\tau + \int_{t-a}^{t+a}\left|\int_{\tau}^{t} \dot{u}(s)\mathrm{d}s\right|\mathrm{d}\tau \\
&\leq \int_{t-a}^{t+a} |u(\tau)|\,\mathrm{d}\tau + \int_{t-a}^{t}\int_{t-a}^{t} |\dot{u}(s)|\,\mathrm{d}s\mathrm{d}\tau + \int_{t}^{t+a}\int_{t}^{t+a} |\dot{u}(s)|\,\mathrm{d}s\mathrm{d}\tau \\
&\leq (2a)^{\frac{1}{2}}\left(\int_{t-a}^{t+a} |u(s)|^2\mathrm{d}s\right)^{\frac{1}{2}} + a\int_{t-a}^{t+a} |\dot{u}(s)|\,\mathrm{d}s \\
&\leq (2a)^{\frac{1}{2}}\left(\int_{t-a}^{t+a} |u(s)|^2\mathrm{d}s\right)^{\frac{1}{2}} + a(2a)^{\frac{1}{2}}\left(\int_{t-a}^{t+a} |\dot{u}(s)|^2\,\mathrm{d}s\right)^{\frac{1}{2}},
\end{aligned}$$

which implies (2.4) holds. The proof is complete. ■

Corollary 2.1. *Let $u \in W^{1,2}\big([-T, T], R^n\big)$. Then for every $t \in [-T, T]$, the following inequality holds:*

$$|u(t)| \leq \left[\int_{t-1}^{t+1} \big(|\dot{u}(s)|^2 + |u(s)|^2\big)\,\mathrm{d}s\right]^{\frac{1}{2}}. \tag{2.6}$$

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Proof. Let $a = 1$ in (2.4). Then we have

$$|u(t)| \leq \frac{\sqrt{2}}{2}\left[\left(\int_{t-1}^{t+1} |u(s)|^2\, ds\right)^{\frac{1}{2}} + \left(\int_{t-1}^{t+1} |\dot{u}(s)|^2\, ds\right)^{\frac{1}{2}}\right],$$

which together with the inequality $(\sqrt{a} + \sqrt{b})/2 \leq \sqrt{(a+b)/2}$ implies that (2.6) holds. The proof is complete. ■

Lemma 2.3. *For $u \in E_T$,*

$$\|u\|_{L^\infty_{[-T,T]}} \leq \frac{1}{\sqrt{2\sqrt{l_*}}}\|u\| = \frac{1}{\sqrt{2\sqrt{l_*}}}\left\{\int_{-T}^{T}\left[|\dot{u}(s)|^2 + \big(L(s)u(s), u(s)\big)\right] ds\right\}^{\frac{1}{2}}, \tag{2.7}$$

where $l_* = \inf_{t \in R} l(t)$.

Proof. Since $u \in E_T$, so $u \in W^{1,2}\big([-T,T], R^n\big)$, then there exists a $t^* \in [-T, T]$ such that

$$\left|u(t^*)\right| = \max_{t \in [-T,T]} |u(t)|. \tag{2.8}$$

We choose two sequence $\{t_k\}$ and $\{t_{-k}\}$ such that

$$-T \leq \cdots < t_{-3} < t_{-2} < t_{-1} < t^* < t_1 < t_2 < t_3 < \cdots \leq T,$$
$$\lim_{k\to\infty} t_k = T, \qquad \lim_{k\to\infty} t_{-k} = -T,$$

and then

$$\lim_{k\to\infty} |u(t_k)| = \lim_{k\to\infty} |u(t_{-k})| = 0.$$

Note that

$$\left|u(t^*)\right|^2 = |u(t_k)|^2 - 2\int_{t^*}^{t_k} \big(u(s), \dot{u}(s)\big) ds, \tag{2.9}$$

and

$$\left|u(t^*)\right|^2 = |u(t_{-k})|^2 + 2\int_{t_{-k}}^{t^*} \big(u(s), \dot{u}(s)\big) ds. \tag{2.10}$$

For $u \in E_T$, we have by (2.9) and (2.10),

$$\begin{aligned}
\left|u(t^*)\right|^2 &= \frac{1}{2}\big(|u(t_k)|^2 + |u(t_{-k})|^2\big) - \int_{t^*}^{t_k} \big(u(s), \dot{u}(s)\big) ds + \int_{t_{-k}}^{t^*} \big(u(s), \dot{u}(s)\big) ds \\
&\leq \frac{1}{2}\big(|u(t_k)|^2 + |u(t_{-k})|^2\big) + \int_{t_{-k}}^{t_k} |u(s)|\,|\dot{u}(s)|\, ds \\
&\leq \frac{1}{2}\big(|u(t_k)|^2 + |u(t_{-k})|^2\big) + \frac{1}{2}\int_{t_{-k}}^{t_k} \frac{1}{\sqrt{l(s)}}\big(|\dot{u}(s)|^2 + l(s)\,|u(s)|^2\big) ds \\
&\leq \frac{1}{2}\big(|u(t_k)|^2 + |u(t_{-k})|^2\big) + \frac{1}{2}\int_{t_{-k}}^{t_k} \frac{1}{\sqrt{l(s)}}\left[|\dot{u}(s)|^2 + \big(L(s)u(s), u(s)\big)\right] ds \\
&\leq \frac{1}{2}\big(|u(t_k)|^2 + |u(t_{-k})|^2\big) + \frac{1}{2\sqrt{l_*}}\int_{t_{-k}}^{t_k} \left[|\dot{u}(s)|^2 + \big(L(s)u(s), u(s)\big)\right] ds, \quad k \in N.
\end{aligned}$$

Let $k \to \infty$ in the above, we can get

$$\left|u(t^*)\right|^2 \leq \frac{1}{2\sqrt{l_*}}\int_{-T}^{T} \left[|\dot{u}(s)|^2 + \big(L(s)u(s), u(s)\big)\right] ds,$$

which implies that (2.7) holds. The proof is complete. ■

Lemma 2.4. *Under the conditions of Theorem 1.1, problem (2.1) possesses a nontrivial solution for all $T \geq T_0$.*

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Proof. Step 1. I satisfies the Palais–Smale condition, i.e., for every sequence $\{u_k\} \subset E_T$, $\{u_k\}$ has a convergent subsequence if $I(u_k)$ is bounded and $I'(u_k) \to 0$ as $k \to \infty$.

Assume that $\{u_k\} \subset E_T$, $I(u_k)$ is bounded and $I'(u_k) \to 0$ as $k \to \infty$, then there exists a constant $M_T > 0$ such that

$$I(u_k) \leq M_T, \qquad \left\|I'(u_k)\right\|^*_{E_T} \leq M_T \tag{2.11}$$

for every $k \in N$. We first proved that $\{u_k\}_{k\in N}$ is bounded. By (2.11) and (H2),

$$\|u_k\|^2 \leq 2I(u_k) + \frac{2}{\mu}\int_{-T}^{T}\left[\left(\nabla W\big(t, u_k(t)\big), u_k(t)\right) + \nu\,(L(t)u_k(t), u_k(t)) + \beta(t)\right]\mathrm{d}t. \tag{2.12}$$

From (2.12) and (2.3), we obtain

$$\left(1 - \frac{2}{\mu}\right)\|u_k\|^2 \leq 2I(u_k) - \frac{2}{\mu}I'(u_k)u_k + \frac{2\nu}{\mu}\int_{-T}^{T}(L(t)u_k(t), u_k(t))\,\mathrm{d}t + \frac{2}{\mu}\int_{-T}^{T}\beta(t)\mathrm{d}t. \tag{2.13}$$

Combining (2.13) with (H2) and (2.11), we have

$$\begin{aligned}&\left(1 - \frac{2}{\mu}\right)\int_{-T}^{T}|\dot{u}_k(t)|^2\,\mathrm{d}t + \left(1 - \frac{2}{\mu} - \frac{2\nu}{\mu}\right)\int_{-T}^{T}(L(t)u_k(t), u_k(t))\,\mathrm{d}t\\ &\quad - \frac{2}{\mu}M_T\|u_k\| - 2M_T - \frac{2}{\mu}\|\beta\|_{L^1(R,R^+)} \leq 0.\end{aligned} \tag{2.14}$$

Since $\mu > 2$ and $\nu \in [0, \frac{\mu}{2} - 1)$, (2.14) shows that $\{u_k\}_{k\in N}$ is bounded in E_T. Going if necessary to a subsequence, we can assume that there exists $u_T \in E_T$ such that $u_k \rightharpoonup u_T$ as $k \to \infty$ in E_T, which implies $u_k \to u_T$ uniformly on $[-T, T]$. Hence $\big(I'(u_k) - I'(u_T)\big)(u_k - u_T) \to 0$, $\|u_k - u_T\|_{L^2_{[-T,T]}} \to 0$ and

$$\int_{-T}^{T}\left(\nabla W\big(t, u_k(t)\big) - \nabla W\big(t, u_T(t)\big), u_k(t) - u_T(t)\right)\mathrm{d}t \to 0, \quad \text{as } k \to \infty. \tag{2.15}$$

Moreover, from (2.3), an easy computation shows that

$$\big(I'(u_k) - I'(u_T)\big)(u_k - u_T) = \|u_k - u_T\|^2 - \int_{-T}^{T}\left(\nabla W\big(t, u_k(t)\big) - \nabla W\big(t, u_T(t)\big), u_k(t) - u_T(t)\right)\mathrm{d}t. \tag{2.16}$$

This shows that $u_k \to u_T$ in E_T. Hence I satisfies the Palais–Smale condition.

From (H1), it follows that there exist $0 < \varepsilon_0 < \frac{l_*}{4}$, $\rho_0 > 0$ such that

$$W(t, x) \leq \varepsilon_0|x|^2 \tag{2.17}$$

for all $|x| \leq \rho_0$ and $t \in R$.

Step 2. There are $\rho > 0$, $\alpha > 0$ such that $I|_S \geq \alpha$, where $S = \{u \in E_T \,|\, \|u\| = \rho\}$. Choose $\rho = \rho_0 \cdot \sqrt{2\sqrt{l_*}}$, by Lemma 2.3 it is easy to prove that for all $u \in S$, $\|u\|_\infty \leq \rho_0$, that is $|u(t)| \leq \rho_0$ for all $t \in [-T, T]$, which together with (2.17) implies that

$$\begin{aligned}I(u) &= \frac{1}{2}\|u\|^2 - \int_{-T}^{T}W(t, u)\mathrm{d}t\\ &\geq \frac{1}{2}\int_{-T}^{T}|\dot{u}(t)|^2\,\mathrm{d}t + \frac{1}{2}\int_{-T}^{T}\big(L(t)u(t), u(t)\big)\mathrm{d}t - \int_{-T}^{T}\varepsilon_0|u(t)\,|^2\,\mathrm{d}t\\ &\geq \frac{1}{2}\int_{-T}^{T}|\dot{u}(t)|^2\,\mathrm{d}t + \frac{1}{4}\int_{-T}^{T}\big(L(t)u(t), u(t)\big)\mathrm{d}t\\ &\geq \frac{1}{4}\|u\|^2 = \frac{1}{4}\rho^2 := \alpha\end{aligned}$$

for every $u \in S$.

Step 3. There exists $e \in E_T \setminus \overline{B}_\rho$ such that $I(e) \leq 0$. By (H3), there exist $\varepsilon_1 > 0$ and $r > 0$ such that

$$\frac{W(t, x)}{|x|^2} \geq \frac{\pi^2 + \varepsilon_1}{2T_0^2} + \frac{l_1}{2} \quad \text{for all } |x| > r \text{ and } t \in [-T_0, T_0].$$

Let $\delta = \max_{t\in[-T_0,T_0],|x|\leq r}|W(t.x)|$, hence we can have

$$W(t, x) \geq \left(\frac{\pi^2 + \varepsilon_1}{2T_0^2} + \frac{l_1}{2}\right)(|x|^2 - r^2) - \delta \quad \text{for all } x \in R \text{ and } t \in [-T_0, T_0]. \tag{2.18}$$

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Let

$$e(t)=\begin{cases}\xi\,|\sin(\omega t)|\,e_1, & t\in[-T_0,T_0],\\ 0, & t\in[-T,T]\setminus[-T_0,T_0],\end{cases}$$

where $\omega=\frac{\pi}{T_0}$, $e_1=(1,0,\ldots,0)$. Then by (H1) and (2.18) we obtain

$$\begin{aligned}I(e) &= \frac{1}{2}\int_{-T}^{T}|\dot{e}(t)|^2\,dt+\frac{1}{2}\int_{-T}^{T}\big(L(t)e(t),e(t)\big)dt-\int_{-T}^{T}W(t,e)dt\\
&= \frac{1}{2}\xi^2\omega^2\int_{-T_0}^{T_0}|\cos(\omega t)|^2\,dt+\frac{1}{2}\int_{-T_0}^{T_0}\big(L(t)e(t),e(t)\big)dt-\int_{-T_0}^{T_0}W(t,\xi\,|\sin(\omega t)|\,e_1)dt\\
&\le \frac{1}{2}\xi^2\omega^2\int_{-T_0}^{T_0}|\cos(\omega t)|^2\,dt+\frac{l_1}{2}\xi^2\int_{-T_0}^{T_0}|\sin(\omega t)|^2\,dt\\
&\quad -\left(\frac{\pi^2+\varepsilon_1}{2T_0^2}+\frac{l_1}{2}\right)\xi^2\int_{-T_0}^{T_0}|\sin(\omega t)|^2\,dt+2T_0\left(\left(\frac{\pi^2+\varepsilon_1}{2T_0^2}+\frac{l_1}{2}\right)r^2+\delta\right)\\
&= -\frac{\varepsilon_1}{2T_0}\xi^2+2T_0\left(\left(\frac{\pi^2+\varepsilon_1}{2T_0^2}+\frac{l_1}{2}\right)r^2+\delta\right)\\
&\to -\infty \quad \text{as } \xi\to\infty.\end{aligned}$$

So for all $T\ge T_0$, we can choose a large enough ξ such that $\|e\|\ge\rho$ and moreover $I(e)<0$. And from (H1), we can easily have $I(0)=0$. Thus by the Lemma 2.1, there exists a critical point $u_T\in E_T$ of I such that $I(u_T)\ge\alpha>0$ for all $T\ge T_0$. This shows that problem (2.1) has at least one nontrivial solution for all $T\ge T_0$. ■

Remark 2.1. In Lemma 2.4, Step 1 and Step 2 still hold true for any $T>0$.

Furthermore, if we define the set of paths

$$\Gamma_T=\{g(t):[0,1]\longrightarrow E_T\,|\,g(0)=0,g(1)=e\},\tag{2.19}$$

then there exists a solution u_T of (2.1) at which

$$\inf_{g\in\Gamma_T}\max_{s\in[0,1]}I\big(g(s)\big)=N_T\tag{2.20}$$

is achieved. Let now $\widetilde{T}>T$. Then $\Gamma_T\subset\Gamma_{\widetilde{T}}$, since any function in E_T can be regarded as belonging to $E_{\widetilde{T}}$ if one extends it by zero in $[-\widetilde{T},\widetilde{T}]\setminus[-T,T]$. Hence for $\widetilde{T}$ the set of competing paths in (2.19) is greater than that for T, which implies that

$$N_{\widetilde{T}}\le N_T\le N_{T_0},\quad \text{for all } \widetilde{T}\ge T\ge T_0.$$

Hence for the solution of (2.1)

$$I(u_T)=\frac{1}{2}\|u_T\|^2-\int_{-T}^{T}W\big(t,u_T(t)\big)dt\le N_{T_0},\quad \text{uniformly in } T\ge T_0.\tag{2.21}$$

Lemma 2.5. *u_T is bounded uniformly in $T\ge T_0$.*

Proof. It is clear that

$$I'(u_T)=0,\quad \text{for } T\ge T_0.\tag{2.22}$$

By (2.21) and (2.22), we obtain

$$I(u_T)\le N_{T_0},\qquad \|I'(u_T)\|=0.$$

The following proof is the same as the Step 1 in Lemma 2.4, then we can obtain that $\|u_T\|$ is bounded uniformly in $T\ge T_0$. ■

Proof of Theorem 1.1. Take a sequence $T_n\to\infty$ and consider the problem (2.1) on the interval $[-T_n,T_n]$. By the conclusions of Lemmas 2.4 and 2.5, it has a nontrivial solution u_n, and $\|u_n\|$ is bounded uniformly in n.

Arguing like for Theorem 2.1 in [8], from the fact that

$$|u_n(t_1)-u_n(t_2)|\le\int_{t_1}^{t_2}|\dot{u}_n(t)|\,dt\le\sqrt{t_2-t_1}\left(\int_{t_1}^{t_2}|\dot{u}_n(t)|^2\,dt\right)^{\frac{1}{2}}$$

we conclude that the sequence $\{u_n\}$ is equicontinuous and uniformly bounded on every interval $[-T_n,T_n]$ and we can select a subsequence $\{u_{n_k}\}$ such that it converges uniformly on any bounded interval to a function u. And since $\|u_n\|$ is bounded uniformly in n, we can conclude that $u\in W^{1,2}(R,R^n)$ and

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$$\int_R \left[|\dot{u}(t)|^2 + \big(L(t)u(t), u(t)\big)\right] \mathrm{d}t < +\infty. \tag{2.23}$$

Expressing $\ddot{u}_{n_k}$ using Eq. (2.1), we conclude that the sequence $\ddot{u}_{n_k}$, and then also $\dot{u}_{n_k}$, converges uniformly on bounded intervals. Writing

$$u_{n_k}(t) = \int_a^t (t-s)\ddot{u}_{n_k}(s)\mathrm{d}s, \quad \text{with } a = -T_{n_k} - 1,$$

we conclude that $u \in C^2(R, R^n)$, and $\ddot{u}_{n_k} \to \ddot{u}$ uniformly on bounded intervals. Hence we can pass to the limit in Eq. (2.1), and we conclude that u satisfies (1.1), i.e., u is a classical solution of (1.1). Note that, by the proof of Corollary 2.1 we can similarly have

$$|u(t)| \le \left[\int_{t-1}^{t+1} \left(|\dot{u}(s)|^2 + |u(s)|^2\right) \mathrm{d}s\right]^{\frac{1}{2}}, \quad \text{for every } t \in R.$$

From (2.23), we conclude that the limits of $u(t)$ exist as $|t| \to \infty$. The only possibility is $u(\pm\infty) = 0$.

We now prove that $\dot{u}(t) \to 0$ as $|t| \to \infty$. By Corollary 2.1, we get that

$$|\dot{u}(t)|^2 \le \int_{t-1}^{t+1} \left(|u(s)|^2 + |\dot{u}(s)|^2\right) \mathrm{d}s + \int_{t-1}^{t+1} |\ddot{u}(s)|^2 \mathrm{d}s, \quad \text{for every } t \in R.$$

From (2.23), we can easily have

$$\int_{t-1}^{t+1} \left(|u(s)|^2 + |\dot{u}(s)|^2\right) \mathrm{d}s \to 0, \quad \text{as } |t| \to \infty.$$

Hence we only need to prove that

$$\int_{t-1}^{t+1} |\ddot{u}(s)|^2 \mathrm{d}s \to 0, \quad \text{as } |t| \to \infty. \tag{2.24}$$

It follows from (1.1)

$$\begin{aligned}\int_{t-1}^{t+1} |\ddot{u}(s)|^2 \mathrm{d}s &= \int_{t-1}^{t+1} \left[\nabla W\big(s, u(s)\big) - \big(L(s)u(s), u(s)\big)\right]^2 \mathrm{d}s \\ &\le 2\int_{t-1}^{t+1} \left(\left|\nabla W\big(s, u(s)\big)\right|^2 + \left|\big(L(s)u(s), u(s)\big)\right|^2\right) \mathrm{d}s.\end{aligned}$$

By (H1) with (2.23) and the fact that $u(t) \to 0$ as $|t| \to \infty$, (2.24) is proved.

In the end, we have to show that $u(t) \not\equiv 0$. Since $u_{n_k} \to u$ uniformly in E_A for all $A < \infty$, where u_{n_k} can be extended by zero in $[-A, A]\backslash[-T_{n_k}, T_{n_k}]$ or restricted to the interval $[-A, A]$. Hence it suffices to show there is an $A > 0$ such that $u_{n_k} \not\to 0$ in E_A. For this purpose, at first we introduce a function Y. Let $Y : [0, +\infty) \to [0, +\infty)$ be given as follows: $Y(0) = 0$ and

$$Y(s) = \sup_{t\in R, 0<|x|\le s} \frac{\left|\big(\nabla W(t, x), x\big)\right|}{|x|^2}, \quad \text{for } s > 0. \tag{2.25}$$

Then Y is a continuous, nondecreasing, $Y(s) \ge 0$ for $s \ge 0$. As Rabinowitz, we use the properties of Y given by (2.25). Then by (L), the definition of Y implies

$$\int_{-T_n}^{T_n} \big(\nabla W\big(t, u_n(t)\big), u_n(t)\big)\mathrm{d}t \le \frac{1}{l_*} Y\left(\|u_n\|_{L^\infty_{[-T_n,T_n]}}\right) \|u_n\|^2_{E_{T_n}}, \quad \text{for every } n \in N. \tag{2.26}$$

Since $I'(u_n)u_n = 0$, (2.3) gives

$$\int_{-T_n}^{T_n} \big(\nabla W\big(t, u_n(t)\big), u_n(t)\big)\mathrm{d}t = \int_{-T_n}^{T_n} |\dot{u}_n(t)|^2 \mathrm{d}t + \int_{-T_n}^{T_n} \big(L(t)u_n(t), u_n(t)\big)\mathrm{d}t = \|u_n\|^2_{E_{T_n}}. \tag{2.27}$$

Substituting (2.27) into (2.26), we obtain

$$Y\left(\|u_n\|_{L^\infty_{[-T_n,T_n]}}\right) \ge l_* > 0. \tag{2.28}$$

The remainder of the proof is the same as in [16]. If $\|u_n\|_{L^\infty_{[-T_n,T_n]}} \to 0$, as $n \to +\infty$, we would have $Y(0) \ge l_* > 0$, a contradiction. Thus there is $\gamma > 0$ such that

$$\|u_n\|_{L^\infty_{[-T_n,T_n]}} \ge \gamma, \quad \text{for every } n \in N. \tag{2.29}$$

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If $u_{n_k} \to 0$ in E_A for every $A \in R^+$, then by Lemma 2.3, we have

$$\begin{aligned}
\|u_{n_k}\|^2_{L^\infty_{[-T_{n_k},T_{n_k}]}} &\le \frac{1}{2\sqrt{l_*}} \int_{-T_{n_k}}^{T_{n_k}} \left[\left|\dot{u}_{n_k}(s)\right|^2 + \big(L(s)u_{n_k}(s), u_{n_k}(s)\big) \right] ds \\
&= \frac{1}{2\sqrt{l_*}} \int_{R} \left[\left|\dot{u}_{n_k}(s)\right|^2 + \big(L(s)u_{n_k}(s), u_{n_k}(s)\big) \right] ds \\
&= \frac{1}{2\sqrt{l_*}} \int_{-A}^{A} \left[\left|\dot{u}_{n_k}(s)\right|^2 + \big(L(s)u_{n_k}(s), u_{n_k}(s)\big) \right] ds \\
&\quad + \frac{1}{2\sqrt{l_*}} \int_{R\setminus[-A,A]} \left[\left|\dot{u}_{n_k}(s)\right|^2 + \big(L(s)u_{n_k}(s), u_{n_k}(s)\big) \right] ds \\
&\to 0, \quad \text{as } A, k \to \infty
\end{aligned}$$

which contradicts (2.29), where u_{n_k} can be extended by zero in $R \setminus [-T_{n_k}, T_{n_k}]$. Hence there is one nontrivial homoclinic orbit of problem (1.1). ■

3. Example and remark

As an application, we consider the following example for the case of $n = 1$:

$$\ddot{u}(t) - (t^2 + 1)u(t) + \nabla W\big(t, u(t)\big) = 0 \tag{3.1}$$

where

$$W(t,x) = \begin{cases} \frac{3}{4}x^4 + \frac{1}{2}|x|^3, & |x| < 1, \\ x^4 + \frac{1}{4}x^2, & |x| \ge 1. \end{cases}$$

Let $\mu = 4$ and $\nu = \frac{3}{4}$ in Theorem 1.1, by the direct calculation, we can easily see that the assumptions [H1]–[H3] hold. So by applying Theorem 1.1, we know that Eq. (3.1) possesses a nontrivial homoclinic solution.

Remark 3.1. In this example, we can have

$$\big(\nabla W(t,x), x\big) - 4W(t,x) = \begin{cases} -\frac{1}{2}|x|^3 \ge -\frac{3}{4}(t^2+1)x^2, & |x| < 1, \\ -\frac{1}{2}x^2 \ge -\frac{3}{4}(t^2+1)x^2, & |x| \ge 1, \end{cases}$$

which does not satisfy the condition of Theorem A.

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