Stochastics and Dynamics
Vol. 22, No. 8 (2022) 2240035 (23 pages)

DOI: 10.1142/S0219493722400354

Analysis of a microfluidic chemostat model with random dilution ratios

Jifa Jiang* and Xiang Lv†

Department of Mathematics, Shanghai Normal University, Shanghai 200234, P. R. China
**jiangjf@shnu.edu.cn*
†*lvxiang@shnu.edu.cn*

Received 2 September 2022
Accepted 14 October 2022
Published 25 January 2023

Dedicated to Prof. Jinqiao Duan on the occasion of his 60th birthday

In this paper, we first construct a microfluidic chemostat model for the growth of biofilms and planktonic populations with random dilution ratios and then investigate its dynamical behavior. Using the theory of monotone dynamical systems and the Multiplicative Ergodic Theorem, we show the existence of random attractors and stationary measures, and present Lyapunov exponents for the linearized cocycle with respect to the random model. Further on, if the top Lyapunov exponent is negative, we give the extinction of microbial populations, including the forward and pull-back trajectories.

Keywords: Chemostat; extinction; Lyapunov exponents; stability; random equilibrium; ergodicity.

AMS Subject Classification: 37N25, 37H15, 37H30

1. Introduction

The continuous culture of bacteria using the chemostat invented by Novick and Szilard in 1950 is an important experimental technique for microbiology [16]. Due to the excessive reagent consumption and difficulty to maintain bulk bioreactor, a group of physicists in Cal Tech use the microfluidic technology to invent the nanoliter scale micro-chemostat in 2005 [2]. Recently, Yang *et al.* also invented a nanoliter scale chemostat that utilizes two compartment design for long-term operation [12]. The device enables the automatic cell culture over hundreds of hours and with single cell resolution. The drastic reduction in working volume from milliliter scale to nanoliter scale has prompted the researchers to re-think the conventional chemostat theory [19]. Traditionally, there are two ways for bacteria culture, the

†Corresponding author.

batch growth mode and the chemostat. In the batch growth mode, the microbial population is allowed to grow exponentially in a closed system with a fixed amount of nutrient medium. Once a growth cycle is finished and a small fraction of the microbial population sample is removed and re-inoculate with fresh medium to continue another cycle. In the chemostat mode, the microbial population is diluted continuously with fresh medium to maintain the nearly steady microbial population [10]. These two modes of operation seem very distinct from each other in term hardware realization.

Mathematically, the batch mode can be considered as an extension of a chemostat with periodic dilution [18]. In [13], Hsu and Yang considered a mathematical model of the growth of planktonic and biofilm population in a nanoliter chemostat with discrete dilution steps at time $T_j = jT$, $j = 1, 2, \ldots$ and at each dilution step, a fraction F of the microbial population is removed and remaining fraction $\eta = 1-F$ is refilled or used to inoculate another batch of fresh medium. They derived criteria by stability analysis to determine the extinction and coexistence of the species and yielded the operation diagrams of dilution and removable ratio, which is useful for the experimenters to design the microfluidic devices for investigating microbial competition and biofilm growth under serial dilution conditions. Modeling of extinction and coexistence of planktonic and biofilm population under periodic dilution has significant implication in many areas of microbiology. For example, the biofilm infection has accounted for over 80 percent of clinical infection and the planktonic population detached from the biofilm infection site causes symptom for the patient.

Due to some errors created in the actual operation and measurement, it is clear that the above Fig. 1, i.e. [12, Fig. 3(c), p. 044126-6] showed that the dilution ratio cannot remain as a constant and jumped around 25%. From [12, Fig. 3(c), p. 044126-6], we can observe that the dilution ratio should be a random variable

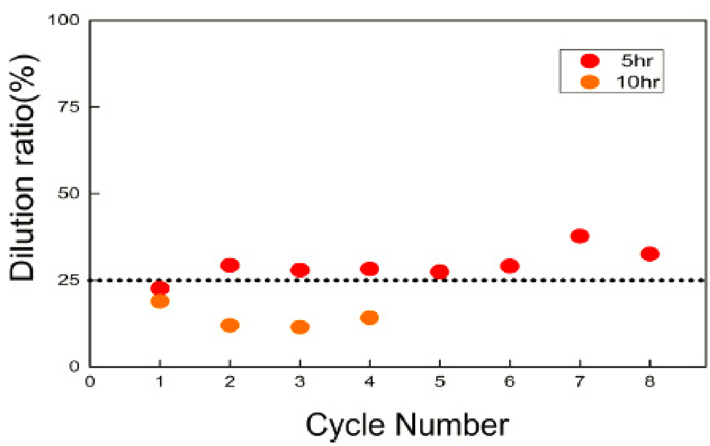

Fig. 1. Measured dilution ratios versus cycle number. Dilution ratio data are extracted from growth curves for $T = 10\,\text{h}$ and $T = 5\,\text{h}$. The ideal dilution ratio, $\eta = 1/4$, is displayed as a dot line for comparison. Note that the data points of the first cycle are from the inoculation cycle.

with uniform distribution in the interval $(0,1)$. From the experiment procedures in [12] the dilution ratio can be used as a random control variable as we did in plotting the operation diagrams in the deterministic model. This fact spurs us to consider the random version of the mathematical model of planktonic and biofilm populations in a nanoliter chemostat with periodic dilution and proper choice of plumbing protocol studied in [13]. In the meantime, some stochastic chemostat models have been investigated in the literature [3–6, 8, 9, 11, 14], which are based on birth and death processes (see e.g., [6, 8, 9]) or stochastic perturbations of growth functions (see e.g., [3–5, 14]).

In this work, we mainly focus on the way to model the random fluctuations of dilution ratios. Our result will show that some minor errors and vibrations of dilution ratios will not destroy the stability and global extinction in the deterministic model. The rest of the paper is organized as follows. In Sec. 2, we first construct a random model for the growth of biofilms and planktonic cells in a nanoliter chemostat. Then we will show the existence of random attractors and stationary measures, and calculate the top Lyapunov exponent for the linearized cocycle with respect to the random model. Finally, we present two main theorems for the extinction of microbial populations. The proofs are referred to Sec. 3 and Appendix A.

2. Model and Main Results

Hsu and Yang [13] have presented the theory of a periodic serial dilution bioreactor, which is described as follows. Let R denote the concentration of a nutrient, x denote the volume density of a planktonic population and y denote the surface density of a biofilm population in the bioreactor. Then the system they have considered is

$$\begin{aligned} \frac{dR}{dt} &= -\frac{1}{\gamma}f_x(R)x - \frac{\delta}{\gamma}f_y(R)y, \\ \frac{dx}{dt} &= (f_x(R)-\alpha)x + \delta\beta y, \\ \frac{dy}{dt} &= (f_y(R)-\beta)y + \frac{\alpha x}{\delta}, \end{aligned} \tag{2.1}$$

where γ is the yield constant, $f_x(R)$ and $f_y(R)$ are the uptake functions for planktonic and biofilm cells, which satisfy $f_x(0)=0$, $f_x'(R)>0$, $f_y(0)=0$ and $f_y'(R)>0$ for all $R>0$, α,β are the adsorption and the detachment rates, respectively, δ is the surface-to-volume ratio.

Let $z_0 \triangleq (R_0,x_0,y_0)$ be a given point in the non-negative orthant $\mathbb{R}^3_+$. Suppose that $\Phi_t(z_0) \triangleq (R(t,z_0),x(t,z_0),y(t,z_0))$ is the solution of (2.1) passing through z_0 at $t=0$.

Their dilution regulation at each time $t=T_k \triangleq kT$ is executed by

$$\begin{aligned} R(T_k^+) &= \eta R(T_k^-) + (1-\eta)R^{(0)}, \\ x(T_k^+) &= \eta x(T_k^-), \quad y(T_k^+) = \eta y(T_k^-), \end{aligned} \tag{2.2}$$

where $0 < \eta < 1$, $R(T_k^+) \triangleq \lim_{t \to T_k + 0} R(t)$, $R(T_k^-) \triangleq \lim_{t \to T_k - 0} R(t)$, similarly for others. Define

$$P^\eta(z_0) \triangleq \eta \Phi_T(z_0) + (1-\eta)E_0 \tag{2.3}$$

with $E_0 = (R^{(0)}, 0, 0)$. Thus, (2.2) is just the k-iteration:

$$(P^\eta)^k(z_0) = \underbrace{P^\eta \circ P^\eta \circ \cdots \circ P^\eta}_{k \text{ times}}(z_0). \tag{2.4}$$

They have proved that whether coexistence or extinction occurs totally depends on the radius of linearization matrix at E_0 for P^η being greater than one or not. It is clear that the dilution ratio η is taken to be the same at each dilution step. A natural question arises: what is the outcome if the dilution ratio is not always the same? For example, if we are given ratios $\{\eta_0, \eta_1\}$, at each dilution step, we choose the ratio η_0 or η_1 with the probability $\frac{1}{2}$ just like flipping a coin:

$$\underbrace{P^{\eta_{\xi_k}} \circ P^{\eta_{\xi_{k-1}}} \circ \cdots \circ P^{\eta_{\xi_1}}}_{k \text{ times}}(z_0), \quad \xi_i \in \{0,1\},\ 1 \le i \le k, \tag{2.5}$$

then what the evolution result is? Whether this mechanism can control the growth of microbial populations or not? Our study in this paper reveals that a suitable chosen probability distribution for ξ_j will strengthen the occurrence for extinction, which is important in the infection control. We will formulate the mathematical framework of this problem, the readers are referred to [1, 7].

Let $(\Omega_0, \mathscr{F}_0, \mathbb{P}_0)$ be a probability space, $X \subset (0,1)$ a subset, whose σ-algebra, denoted by $\mathscr{B}$, is assigned to all subsets of X if X is countable; or to Borel σ-algebra if X is an interval, $\xi \triangleq \{\xi_i : \Omega_0 \mapsto X \,|\, i \in \mathbb{Z}\}$ is a sequence of independent and identically distributed (i.i.d.) random variables (for example, in the above described case, $X = \{\eta_0, \eta_1\}$). Denote $X^{\mathbb{Z}}$ by Ω, consisting of all two-sided infinite sequences $\omega = \{\omega_i \,|\, i \in \mathbb{Z}\}$ with all $\omega_i \in X$, $\mathscr{F} = \mathscr{B}^{\mathbb{Z}}$ the σ-algebra generated by all cylindrical sets:

$$A_{n_1,\ldots,n_m} \triangleq \{\omega \,|\, \omega_{n_j} \in A_{n_j}, j = 1,\ldots,m\} \triangleq \prod_{j \in \{n_1,\ldots,n_m\}} A_{n_j} \times \prod_{j \notin \{n_1,\ldots,n_m\}} X,$$

where $A_{n_j} \in \mathscr{B}$ and $\{n_1, \ldots, n_m\}$ is an arbitrary m-tuple of integers. Then

$$\xi : (\Omega_0, \mathscr{F}_0) \mapsto (\Omega, \mathscr{F}), \quad \tilde{\omega}_0 \mapsto \{\xi_i(\tilde{\omega}_0) \,|\, i \in \mathbb{Z}\}$$

is measurable and $\mathbb{P} \triangleq \xi\mathbb{P}_0$ is a probability measure on $(\Omega, \mathscr{F})$. Define the coordinate process by

$$\zeta_i : (\Omega, \mathscr{F}) \mapsto (X, \mathscr{B}), \quad \omega = \{\omega_i \,|\, i \in \mathbb{Z}\} \mapsto \omega_i.$$

Therefore, the stochastic processes $\xi \triangleq \{\xi_i : \Omega_0 \mapsto X \,|\, i \in \mathbb{Z}\}$ and $\zeta \triangleq \{\zeta_i : \Omega \mapsto X \,|\, i \in \mathbb{Z}\}$ are *equivalent* in the sense that

$$\begin{aligned} &\mathbb{P}(\zeta_{n_1} \in A_{n_1}, \zeta_{n_2} \in A_{n_2}, \ldots, \zeta_{n_m} \in A_{n_m}) \\ &\quad = \mathbb{P}_0(\xi_{n_1} \in A_{n_1}, \xi_{n_2} \in A_{n_2}, \ldots, \xi_{n_m} \in A_{n_m}) \end{aligned} \tag{2.6}$$

for any $m \in \mathbb{Z}^+$, $n_1, n_2, \ldots, n_m \in \mathbb{Z}$ and $A_{n_1}, A_{n_2}, \ldots, A_{n_m} \in \mathscr{B}$. It follows from (2.6) and the independence and identical distribution of $\{\xi_i : \Omega_0 \mapsto X \,|\, i \in \mathbb{Z}\}$ that

$$\begin{aligned} &\mathbb{P}(\zeta_{n_1} \in A_{n_1}, \zeta_{n_2} \in A_{n_2}, \ldots, \zeta_{n_m} \in A_{n_m}) \\ &\quad = \prod_{j=1}^{m} \mathbb{P}(\zeta_{n_j} \in A_{n_j}) = \prod_{j=1}^{m} \mathbb{P}(\zeta_0 \in A_{n_j}). \end{aligned} \tag{2.7}$$

This shows that the sequence of random variables $\{\zeta_i : \Omega \mapsto X \,|\, i \in \mathbb{Z}\}$ is i.i.d. and the coordinate process ζ is *stationary*, i.e.

$$\begin{aligned} &\mathbb{P}(\zeta_{n_1+n} \in A_{n_1}, \zeta_{n_2+n} \in A_{n_2}, \ldots, \zeta_{n_m+n} \in A_{n_m}) \\ &\quad = \mathbb{P}(\zeta_{n_1} \in A_{n_1}, \zeta_{n_2} \in A_{n_2}, \ldots, \zeta_{n_m} \in A_{n_m}) \end{aligned} \tag{2.8}$$

for any $m \in \mathbb{Z}^+, n, n_1, n_2, \ldots, n_m \in \mathbb{Z}$ and $A_{n_1}, A_{n_2}, \ldots, A_{n_m} \in \mathscr{B}$. In the subsequent context, the probability space $(\Omega_0, \mathscr{F}_0, \mathbb{P}_0)$ and the stochastic process $\xi \triangleq \{\xi_i : \Omega_0 \mapsto X \,|\, i \in \mathbb{Z}\}$ will be replaced by the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and the coordinate process $\zeta \triangleq \{\zeta_i : \Omega \mapsto X \,|\, i \in \mathbb{Z}\}$, respectively.

For each $n \in \mathbb{Z}$, we define the standard Bernoulli shift:

$$\theta_n : (\Omega, \mathscr{F}) \mapsto (\Omega, \mathscr{F}), \quad \omega = \{\omega_i \,|\, i \in \mathbb{Z}\} \mapsto \theta_n \omega = \{\omega_{i+n} \,|\, i \in \mathbb{Z}\}.$$

Thus if $n > 0$ (or $n < 0$), then every coordinate component of ω is shifted to the left (or right) $|n|$-position. By definition, we have the following lemma.

Lemma 2.1. *The coordinate process* $\zeta \triangleq \{\zeta_i : \Omega \mapsto X \,|\, i \in \mathbb{Z}\}$ *satisfies*

$$\zeta_n(\omega) = \zeta_0(\theta_n \omega) \tag{2.9}$$

for all $n \in \mathbb{Z}$ *and* $\omega \in \Omega$.

It is easy to see that $\{\theta_n : \Omega \mapsto \Omega \,|\, n \in \mathbb{Z}\}$ is a one-parameter group, i.e.

$$\theta_0 = \mathrm{id}, \quad \theta_n \circ \theta_m = \theta_{n+m} \quad \text{for all } n, m \in \mathbb{Z}.$$

By the definition of cylindrical sets and (2.8), θ_n is $\mathscr{F}$-measurable and preserves the probabilities on $\mathscr{F}$, i.e. $\theta_n \mathbb{P} = \mathbb{P}$ for all $n \in \mathbb{Z}$. Moreover, using Kolmogorov's zero-one law, we can conclude that the tail σ-algebra is trivial mod $\mathbb{P}$ (see [1, p. 547]). Therefore, θ (defined in Proposition 2.1) is *ergodic* under $\mathbb{P}$, that is, for any θ-invariant set $A \in \mathscr{F}$, one has either $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$. In summary, according to the definition of metric dynamical system (MDS), see [1, p. 537] or [7, p. 10], we have the following proposition.

Proposition 2.1. $\theta \equiv (\Omega, \mathscr{F}, \mathbb{P}, \{\theta_n, n \in \mathbb{Z}\})$ *is an ergodic MDS.*

Proposition 2.2. φ *is a cocycle over* θ, *that is*,

$$\varphi : \mathbb{Z}^+ \times \Omega \times \mathbb{R}^3_+ \mapsto \mathbb{R}^3_+$$

is measurable, which admits the properties

(i) *the mapping* $z_0 \mapsto \varphi(n, \omega, z_0) \equiv \varphi(n, \omega)z_0$ *is continuous for all* $n \in \mathbb{Z}^+$ *and* $\omega \in \Omega$;
(ii) *the mappings* $\varphi(n, \omega, \cdot)$ *satisfy the cocycle property*:

$$\varphi(0, \omega) = \mathrm{id}, \quad \varphi(n + m, \omega) = \varphi(n, \theta_m\omega) \circ \varphi(m, \omega)$$

for all $n, m \in \mathbb{Z}^+$ *and* $\omega \in \Omega$.

(θ, φ) *forms a random dynamical system* (*RDS*) *with one-sided time* $\mathbb{Z}^+$.

Here, $\varphi(n, \omega)z_0 \triangleq P^{\zeta_n(\omega)} \circ P^{\zeta_{n-1}(\omega)} \circ \cdots \circ P^{\zeta_1(\omega)}(z_0)$ for all $\omega \in \Omega$ and $z_0 \in \mathbb{R}^3_+$. The proof of this proposition is given in Appendix A.

Definition 2.1. A random variable $V : \Omega \to \mathbb{R}^3_+$ is called a random equilibrium of RDS (θ, φ) if

$$\varphi(n, \omega)V(\omega) = V(\theta_n\omega) \quad \text{for all } n \in \mathbb{Z}^+ \text{ and } \omega \in \Omega.$$

We are going to present the properties of the cocycle on the boundary $\partial\mathbb{R}^3_+$ of $\mathbb{R}^3_+$. Denote by $\mathbb{R}^+_R$, $\mathbb{R}^+_{xy}$, $\mathbb{R}^+_{Rx}$, $\mathbb{R}^+_{Ry}$ the non-negative R-axis, the non-negative orthant of x–y plane, the non-negative orthant of R–x plane and the non-negative orthant of R–y plane, respectively. Let $\mathrm{Int}\mathbb{R}^3_+$ denote the interior of $\mathbb{R}^3_+$, the other symbols are similar.

Proposition 2.3. *Let* φ *be defined in Proposition* 2.2, *we have*

(i) $\varphi(n, \omega)\mathbb{R}^+_R \subset \mathbb{R}^+_R$ *for all* $n \in \mathbb{Z}^+$ *and* $\omega \in \Omega$, *and* $E_0 = (R^{(0)}, 0, 0)$ *is a random equilibrium, admitting the property*: *if the expectation* $\mathbb{E}\zeta_0 < 1$, *then for all* $z^R_0 \triangleq (R_0, 0, 0) \in \mathbb{R}^+_R$,

$$\lim_{n\to\infty} \varphi(n, \omega)z^R_0 = \lim_{n\to\infty} \varphi(n, \theta_{-n}\omega)z^R_0 = E_0$$

on a θ-*invariant set of full measure*;
(ii) $\Phi_t(\mathrm{Int}\mathbb{R}^3_+) \subset \mathrm{Int}\mathbb{R}^3_+$ *for all* $t > 0$, *and* $\varphi(n, \omega)(\mathrm{Int}\mathbb{R}^3_+) \subset \mathrm{Int}\mathbb{R}^3_+$ *for all* $n \in \mathbb{Z}^+$ *and* $\omega \in \Omega$;
(iii) $\varphi(n, \omega)(\mathbb{R}^+_{xy}\backslash\{O\}) \subset \mathrm{Int}\mathbb{R}^3_+$ *for all* $n \in \mathbb{Z}^+\backslash\{0\}$ *and* $\omega \in \Omega$;
(iv) $\varphi(n, \omega)(\mathrm{Int}\mathbb{R}^+_{Rx}) \subset \mathrm{Int}\mathbb{R}^3_+$ *for all* $n \in \mathbb{Z}^+\backslash\{0\}$ *and* $\omega \in \Omega$;
(v) $\varphi(n, \omega)(\mathrm{Int}\mathbb{R}^+_{Ry}) \subset \mathrm{Int}\mathbb{R}^3_+$ *for all* $n \in \mathbb{Z}^+\backslash\{0\}$ *and* $\omega \in \Omega$.

Proof. From the uniqueness of the initial problem for (2.1), it easily follows that $\Phi_t(R_0, 0, 0) \equiv (R_0, 0, 0)$. By the dilution regulation (2.2), we have $\varphi(n, \omega)(R_0, 0, 0) = (R(n, \omega, R_0), 0, 0)$ for all $n \in \mathbb{Z}^+$ and $\omega \in \Omega$, where

$$R(n, \omega, R_0) = R_0 \prod_{i=1}^{n} \zeta_i(\omega) + R^{(0)} \left(1 - \prod_{i=1}^{n} \zeta_i(\omega)\right) \tag{2.10}$$

can be proved by induction and the regulation (2.2). Applying the Birkhoff–Khinchin Theorem (see [1, p. 539]), we can see that

$$\lim_{n\to\infty}\frac{1}{n}\ln\left(\prod_{i=1}^{n}\zeta_i(\omega)\right)=\lim_{n\to\infty}\frac{1}{n}\ln\left(\prod_{i=1}^{n}\zeta_0(\theta_i\omega)\right)=\mathbb{E}(\ln(\zeta_0)) \tag{2.11}$$

on a θ-invariant set Ω^+ of full measure. Moreover, by the Jensen inequality, it is clear that $\mathbb{E}(\ln(\zeta_0))\le\ln(\mathbb{E}\zeta_0)<0$, which shows that

$$\lim_{n\to\infty}\prod_{i=1}^{n}\zeta_i(\omega)=0.$$

Hence, $\lim_{n\to\infty}\varphi(n,\omega)z_0^R=E_0$ on the set Ω^+.

In addition, combining (2.10) and (2.9), it is easy to obtain that

$$\begin{aligned}R(n,\theta_{-n}\omega,R_0)&=R_0\prod_{i=1}^{n}\zeta_i(\theta_{-n}\omega)+R^{(0)}\left(1-\prod_{i=1}^{n}\zeta_i(\theta_{-n}\omega)\right)\\&=R_0\prod_{i=0}^{n-1}\zeta_0(\theta_{-i}\omega)+R^{(0)}\left(1-\prod_{i=0}^{n-1}\zeta_0(\theta_{-i}\omega)\right).\end{aligned}$$

Therefore, the same result holds for the pull-back trajectories. This proves the assertion of (i).

Let $\Phi_t(z_0)\triangleq(R(t,z_0),x(t,z_0),y(t,z_0))$ be the solution of (2.1) with $z_0\triangleq(R_0,x_0,y_0)\in\mathrm{Int}\mathbb{R}^3_+$. We can assert that $R(t,z_0)>0$ for all $t>0$, otherwise, $R(t,z_0)\equiv 0$ by the uniqueness of solution for the initial problem of (2.1). Take $R(t,z_0)$ into the second and third equations of (2.1). Then the second and third equations form a non-autonomous cooperative and irreducible system. Employing the Kamke theorem (see [17]), we obtain that $(x(t,z_0),y(t,z_0))\in\mathrm{Int}\mathbb{R}_{xy}$ for all $t>0$. That is, $\Phi_t(\mathrm{Int}\mathbb{R}^3_+)\subset\mathrm{Int}\mathbb{R}^3_+$ for all $t>0$. The assertion of (ii) follows from this fact and the dilution regulation (2.2) inductively.

Take a point $z_0=(0,x_0,y_0)\in\mathbb{R}^+_{xy}\backslash\{O\}$. The last paragraph proves that the second and third components of $\Phi_t(z_0)$ are positive for all $t>0$. By the dilution regulation (2.2), the first component of $\varphi(n,\omega)z_0$ is always positive, so $\varphi(1,\omega)z_0\in\mathrm{Int}\mathbb{R}^3_+$. The assertion of (iii) can be inductively verified by the dilution regulation (2.2).

Let $z_0=(R_0,x_0,y_0)\in\mathbb{R}^3_+$ with $x_0^2+y_0^2>0$. Then the second and third components of $\Phi_t(z_0)$ are positive for all $t>0$, which is proved in (ii). Therefore, by (2.2), $\varphi(1,\omega)z_0\in\mathrm{Int}\mathbb{R}^3_+$. The conclusions of (iv) and (v) follow from the inductive proof. □

In this paper, we will calculate the top Lyapunov exponent for the linearized cocycle for $\varphi(n,\omega)$ at the random equilibrium E_0, which is $\lambda_{\mathrm{top}}\triangleq\mathbb{E}(\ln(\zeta_0))+(\lambda\vee 0)T$. Here, λ will be given in Proposition 3.1. The global behavior for the cocycle $\varphi(n,\omega)$ can be partially decided by the sign of λ_{top}. More precisely, if $\lambda_{\mathrm{top}}<0$, then all forward and pull-back trajectories will converge to E_0 on a θ-invariant set of full measure.

In what follows, we always suppose that $\mathbb{E}(\zeta_i) = \alpha < 1$ for any $i \in \mathbb{Z}$.

Lemma 2.2. *For any* $z_0 = (R_0, x_0, y_0) \in \mathbb{R}^3_+$, *set* $K_0 \triangleq (\gamma R_0 + x_0 + \delta y_0)$. *Then we have*

$$\gamma R(n,\omega,z_0) + x(n,\omega,z_0) + \delta y(n,\omega,z_0) = K_0 \prod_{i=1}^{n} \zeta_i(\omega) + \gamma R^{(0)} \left(1 - \prod_{i=1}^{n} \zeta_i(\omega)\right). \tag{2.12}$$

As a result, both the forward trajectory $\varphi(n,\omega,z_0)$ *and the pull-back trajectory* $\varphi(n,\theta_{-n}\omega,z_0)$ *are bounded, where*

$$\varphi(n,\omega,z_0) = (R(n,\omega,z_0), x(n,\omega,z_0), y(n,\omega,z_0))$$

and

$$\varphi(n,\theta_{-n}\omega,z_0) = (R(n,\theta_{-n}\omega,z_0), x(n,\theta_{-n}\omega,z_0), y(n,\theta_{-n}\omega,z_0)).$$

Proof. Define $K(R,x,y) \triangleq (\gamma R + x + \delta y)$. Then it is easy to check that K is an invariant function for (2.1). We will prove (2.12) by induction.

For $n = 1$, by the dilution regulation (2.2) and the invariance of K, we get that

$$\begin{aligned}
&\gamma R(1,\omega,z_0) + x(1,\omega,z_0) + \delta y(1,\omega,z_0)\\
&\quad = \zeta_1(\omega)[\gamma R(T,z_0) + x(T,z_0) + \delta y(T,z_0)] + \gamma R^{(0)}(1 - \zeta_1(\omega))\\
&\quad = K_0\zeta_1(\omega) + \gamma R^{(0)}(1 - \zeta_1(\omega)),
\end{aligned}$$

i.e. (2.12) is true for $n = 1$.

Suppose inductively that (2.12) is true for $n = m$. Then we consider the case of $n = m + 1$:

$$\begin{aligned}
&\gamma R(m+1,\omega,z_0) + x(m+1,\omega,z_0) + \delta y(m+1,\omega,z_0)\\
&\quad = \zeta_{m+1}(\omega)[\gamma R(T,\varphi(m,\omega,z_0)) + x(T,\varphi(m,\omega,z_0)) + \delta y(T,\varphi(m,\omega,z_0))]\\
&\qquad + \gamma R^{(0)}(1 - \zeta_{m+1}(\omega))\\
&\quad = \zeta_{m+1}(\omega)[\gamma R(m,\omega,z_0) + x(m,\omega,z_0) + \delta y(m,\omega,z_0)] + \gamma R^{(0)}(1 - \zeta_{m+1}(\omega))\\
&\quad = \zeta_{m+1}(\omega)\left[K_0 \prod_{i=1}^{m} \zeta_i(\omega) + \gamma R^{(0)}\left(1 - \prod_{i=1}^{m} \zeta_i(\omega)\right)\right] + \gamma R^{(0)}(1 - \zeta_{m+1}(\omega))\\
&\quad = K_0 \prod_{i=1}^{m+1} \zeta_i(\omega) + \gamma R^{(0)}\left(1 - \prod_{i=1}^{m+1} \zeta_i(\omega)\right),
\end{aligned}$$

where the first, second and third equalities have used the dilution regulation (2.2), the invariance of K and the inductive assumption, respectively. This yields that (2.12) holds for $n = m + 1$ and completes the proof. $\square$

Moreover, (2.12) implies that this RDS possesses a unique random compact pull-back attractor and the existence of stationary measure, which are described as follows. We shall use the norm $\|z\| \triangleq \gamma|R| + |x| + \delta|y|$ in Corollaries 2.1 and 2.2.

For any $\epsilon > 0$, define

$$B(\epsilon) \triangleq \{(R,x,y) \in \mathbb{R}^3_+ \mid \gamma R^{(0)} - \epsilon \leq K(R,x,y) \leq \gamma R^{(0)} + \epsilon\}.$$

Corollary 2.1. *Let $\mathcal{D} \triangleq \{D(\omega)\}$ be the universe of all tempered random closed set from $\mathbb{R}^3_+$. Then this RDS possesses a unique random compact pull-back attractor $\{A(\omega)\}$ in the universe $\mathcal{D}$ with $A(\omega) \subset B(0)$ on a θ-invariant set of full measure.*

Proof. We first claim that all concepts about random attractors are adopted from Chueshov [7]. By [7, Theorem 1.8.1], it suffices to prove that $B(\epsilon)$ is absorbing in the universe $\mathcal{D}$ for all $\epsilon > 0$.

For any given $D \in \mathcal{D}$, by the temperedness, there exists a tempered random variable $r(\omega)$ such that $D(\omega) \subset \{(R,x,y) \in \mathbb{R}^3_+ \mid K(R,x,y) \leq r(\omega)\}$. We shall prove that there is a θ-invariant set $\Omega^- \subset \Omega$ of full measure such that for any $\omega \in \Omega^-$, there exists $N_0(\omega) \in \mathbb{N}$ such that

$$\varphi(n, \theta_{-n}\omega)D(\theta_{-n}\omega) \subset B(\epsilon) \quad \text{for all } n \geq N_0(\omega). \tag{2.13}$$

Replacing ω with $\theta_{-n}\omega$ in (2.12), taking $(R_0, x_0, y_0) = v(\theta_{-n}\omega) \in D(\theta_{-n}\omega)$ and using (2.9), we conclude that

$$\|\varphi(n,\theta_{-n}\omega)v(\theta_{-n}\omega)\| = \|v(\theta_{-n}\omega)\| \prod_{i=0}^{n-1} \zeta_{-i}(\omega) + \gamma R^{(0)} \left(1 - \prod_{i=0}^{n-1} \zeta_{-i}(\omega)\right). \tag{2.14}$$

By the Birkhoff–Khinchin Theorem (see [1, p. 539]), there is a θ-invariant set $\Omega^- \subset \Omega$ of full measure such that for any $\omega \in \Omega^-$,

$$\lim_{n\to\infty} \frac{1}{n} \ln \left(\prod_{i=0}^{n-1} \zeta_{-i}(\omega) \right) = \lim_{n\to\infty} \frac{1}{n} \ln \left(\prod_{i=0}^{n-1} \zeta_0(\theta_{-i}\omega) \right) = \mathbb{E}(\ln(\zeta_0)). \tag{2.15}$$

By the Jensen inequality, it is evident that $\mathbb{E}(\ln(\zeta_0)) \leq \ln\alpha < 0$ and there exists an $\epsilon_0 > 0$ such that $\beta_0 \triangleq \ln\alpha + \epsilon_0 < 0$. This together with (2.15) implies that for any $\omega \in \Omega^-$, there is an $N_1(\omega) > 0$ such that

$$\prod_{i=0}^{n-1} \zeta_{-i}(\omega) \leq \exp(\beta_0 n) \quad \text{for all } n \geq N_1(\omega).$$

From the temperedness of $r(\omega)$, it follows that

$$\lim_{n\to\infty} r(\theta_{-n}\omega)\exp(\beta_0 n) = 0 \quad \text{for all } \omega \in \Omega^-.$$

Combining this fact, $\|v(\theta_{-n}\omega)\| \leq \|r(\theta_{-n}\omega)\|$ and (2.15), we can use (2.14) to check the definition (2.13) of absorbing sets.

Consequently, [7, Theorem 1.8.1] implies that there is a random compact attractor $A(\omega) \subset B(\epsilon)$ for all $\epsilon > 0$. So, $A(\omega) \subset B(0)$ because of ϵ being arbitrary. The proof is complete. □

Let $\mathscr{B}(\mathbb{R}^3_+)$ denote the Borel σ-algebra on $\mathbb{R}^3_+$. For any given $z_0 \in \mathbb{R}^3_+$ and $A \in \mathscr{B}(\mathbb{R}^3_+)$, we can define the transition probability function:

$$P_n(z_0, A) = P(n, z_0, A) \triangleq \mathbb{P}(\varphi(n, \omega, z_0) \in A). \tag{2.16}$$

Define the average of the transition probability by

$$Q(n, z_0, A) \triangleq \frac{1}{n} \sum_{k=0}^{n-1} P(k, z_0, A). \tag{2.17}$$

From (2.12) and the independence of $\{\zeta_i : \Omega \to X \,|\, i \in \mathbb{Z}\}$, we can calculate

$$\mathbb{E}(\|\varphi(k, \omega, z_0)\|) = K_0\alpha^k + \gamma R^{(0)}(1 - \alpha^k).$$

Let $B_r \triangleq \{(R_0, x_0, y_0) \in \mathbb{R}^3_+ \,|\, \|(R_0, x_0, y_0)\| \le r\}$ and B_r^c denote its complement. By Chebyshev's inequality, it is obvious that $P(k, z_0, B_r^c) \le \frac{1}{r}\mathbb{E}(\|\varphi(k, \omega, z_0)\|)$. Thus,

$$Q(n, z_0, B_r^c) \le \frac{1}{nr} \sum_{k=0}^{n-1} ((K_0 - \gamma R^{(0)})\alpha^k + \gamma R^{(0)}) \le \frac{\gamma R^{(0)} + K_0}{r}.$$

This inequality implies that for any $z_0 \in \mathbb{R}^3_+$, the collection of measures $\{Q(n, z_0, \cdot) \,|\, n \in \mathbb{Z}^+\}$ is tight. Applying the Krylov–Bogoliubov procedure, we conclude the following corollary.

Corollary 2.2. *For any given $z_0 \in \mathbb{R}^3_+$, the cocycle φ admits at least one stationary measure μ:*

$$\mu(B) = \int_{\mathbb{R}^3_+} P_n(z, B)\mu(dz) \quad \text{for all } n \in \mathbb{Z}^+ \text{ and } B \in \mathscr{B}(\mathbb{R}^3_+),$$

which is the limit of a subsequence of $\{Q(n, z_0, \cdot) \,|\, n \in \mathbb{Z}^+\}$ in the weak sense.

Note that $\delta_{E_0}(\cdot)$ is a stationary measure, which is corresponding to the extinction equilibrium E_0 in Theorems 2.1 and 2.2. Our main results on the extinction for the planktonic and biofilm populations are the following theorems. The details of proofs are given in Sec. 3.

Theorem 2.1. *If $\mathbb{E}(\ln(\zeta_0)) + \lambda T < 0$, then the extinction equilibrium $E_0 = (R^{(0)}, 0, 0)$ is globally attractive with respect to the forward trajectory. That is, for any non-negative initial value $z_0 = (R_0, x_0, y_0) \in \mathbb{R}^3_+$, we have that $\varphi(n, \omega, z_0) = (R(n, \omega, z_0), x(n, \omega, z_0), y(n, \omega, z_0)) \to (R^{(0)}, 0, 0)$ as $n \to \infty$ for all $\omega \in \Omega^+$. Here, Ω^+ is a θ-invariant set of full measure given in* (2.11) *and $\lambda = \lambda_1 \vee \lambda_2$ with λ_1 and λ_2 being two eigenvalues of the matrix $A_1(R^{(0)})$, which is defined in* (3.3).

Theorem 2.2. *Assume that there exists $b \in (0,1)$ such that $\zeta_n(\omega) \le b$ for all $\omega \in \Omega$ and $n \in \mathbb{Z}$. If $\mathbb{E}(\ln(\zeta_0)) + \lambda T < 0$, then the extinction equilibrium $E_0 = (R^{(0)}, 0, 0)$ is globally attractive with respect to the pull-back trajectory. In particular, for any non-negative initial value $z_0 = (R_0, x_0, y_0) \in \mathbb{R}^3_+$, we have that $\varphi(n, \theta_{-n}\omega)z_0 = (R(n, \theta_{-n}\omega, z_0), x(n, \theta_{-n}\omega, z_0), y(n, \theta_{-n}\omega, z_0)) \to (R^{(0)}, 0, 0)$ as $n \to \infty$ for all $\omega \in \Omega^-$. Here, Ω^- is a θ-invariant set of full measure given in* (2.15) *and $\lambda = \lambda_1 \vee \lambda_2$, where λ_1 and λ_2 are two eigenvalues of the matrix $A_1(R^{(0)})$, which is defined in* (3.3).

3. Extinction

We are going to compute the Lyapunov exponents of the linearized cocycle for $\varphi(n, \omega, z_0)$ at E_0 by the Birkhoff–Khinchin Theorem and the Multiplicative Ergodic Theorem (see [1, 15]).

It is easy to see that

$$D_{z_0}\varphi(1, \omega, E_0) = \zeta_1(\omega) D_{z_0}\Phi_T(E_0) = \zeta_1(\omega)\exp(AT),$$

where

$$A = \begin{bmatrix} 0 & -\dfrac{1}{\gamma} f_x(R^{(0)}) & -\dfrac{\delta}{\gamma} f_y(R^{(0)}) \\ 0 & f_x(R^{(0)}) - \alpha & \beta\delta \\ 0 & \alpha\delta^{-1} & f_y(R^{(0)}) - \beta \end{bmatrix}$$

is the Jacobian matrix of (2.1) at E_0. Note that $\varphi(n, \omega, E_0) = E_0$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$, using the cocycle property, the chain rule and (2.9), we get that

$$D_{z_0}\varphi(n, \omega, E_0) = \prod_{i=1}^{n} \zeta_i(\omega)\exp(nAT) = \prod_{i=1}^{n} \zeta_0(\theta_i\omega)\exp(nAT), \tag{3.1}$$

which is an invertible linear cocycle with two-sided time over the ergodic MDS θ.

Proposition 3.1. *Suppose that $-\infty < \mathbb{E}(\ln(\zeta_0)) < 0$. Then the linear cocycle* (3.1) *has Lyapunov exponents*

$$\mathbb{E}(\ln(\zeta_0)), \quad \mathbb{E}(\ln(\zeta_0)) + \lambda_i T \ \ (i = 1, 2). \tag{3.2}$$

Here, λ_i $(i = 1, 2)$ are the eigenvalues of the matrix $A_1(R^{(0)})$ and $A_1(R)$ is defined by

$$A_1 = \begin{bmatrix} f_x(R) - \alpha & \beta\delta \\ \alpha\delta^{-1} & f_y(R) - \beta \end{bmatrix}. \tag{3.3}$$

Proof. Define

$$M(\omega) \triangleq \zeta_1(\omega)\exp(AT),$$

we check that

$$\ln^+ \|M(\cdot)\|, \quad \ln^+ \|M^{-1}(\cdot)\| \in L^1(\Omega, \mathscr{F}, \mathbb{P}),$$

where $M^{-1}(\theta_{-1}\omega) = \frac{1}{\zeta_0(\omega)}\exp(-AT)$. In fact, it is evident that

$$\mathbb{E}(\ln^+ \|M(\cdot)\|) \leq \ln^+ \|\exp(AT)\| < \infty$$

and

$$\begin{aligned}\mathbb{E}(\ln^+ \|M^{-1}(\cdot)\|) &\leq \ln^+ \|\exp(-AT)\| - \mathbb{E}(\ln \zeta_0(\theta_1\omega)) \\ &= \ln^+ \|\exp(-AT)\| - \mathbb{E}(\ln \zeta_0(\omega)) < \infty,\end{aligned}$$

where the equality is due to the measure-preserving property of θ. Let Φ be the linear cocycle with two-sided time over θ, which is generated by M. That is,

$$\Phi(n,\omega) = \begin{cases} \prod_{i=1}^{n} \zeta_i(\omega)\exp(nAT), & n > 0, \\ \mathrm{id}, & n = 0, \\ \left(\prod_{i=0}^{-n-1} \zeta_{-i}(\omega)\right)^{-1} \exp(nAT), & n < 0. \end{cases}$$

Let $\widetilde{\Omega} \triangleq \Omega^+ \cap \Omega^-$, where Ω^+ and Ω^- are given in (2.11) and (2.15). Then by the Multiplicative Ergodic Theorem for two-sided time (see [1, pp. 153–154]), we only have to calculate the limits

$$\lim_{n\to\pm\infty} \frac{1}{n} \ln \|\Phi(n,\omega)z\|,$$

that is,

$$\lim_{n\to\infty} \frac{1}{n}\left[\ln\left(\prod_{i=1}^{n} \zeta_i(\omega)\right) + \ln \|\exp(nAT)z\|\right] \tag{3.4}$$

and

$$\lim_{n\to\infty} \frac{1}{n}\left[\ln\left(\prod_{i=0}^{n-1} \zeta_{-i}(\omega)\right) - \ln \|\exp(-nAT)z\|\right] \tag{3.5}$$

for z being in its Oseledets spaces.

Note that the characteristic polynomial of $A_1(R^{(0)})$ is

$$\begin{aligned}p(\lambda) = \lambda^2 &- [(f_x(R^{(0)}) - \alpha) + (f_y(R^{(0)}) - \beta)]\lambda \\ &+ (f_x(R^{(0)}) - \alpha)(f_y(R^{(0)}) - \beta) - \alpha\beta\end{aligned}$$

and the discriminant Δ of $p(\lambda)$ is

$$\Delta = [(f_x(R^{(0)}) - \alpha) - (f_y(R^{(0)}) - \beta)]^2 + 4\alpha\beta > 0.$$

Consequently, we can divide the proof of (3.2) into two cases.

Case 1. A has three different eigenvalues $\lambda_0 = 0$, $\lambda_i (i = 1, 2)$.

Let E_i be the eigenspace corresponding to λ_i $(i = 0, 1, 2)$. Then

$$\mathbb{R}^3 = E_0 \oplus E_1 \oplus E_2$$

is the Oseledets splitting. In fact, for any $z \in E_i$, $\exp(nAT)z = \exp(n\lambda_i T)z$ for all $n \in \mathbb{Z}$. So, the limits in (3.4) and (3.5) are both $\mathbb{E}(\ln(\zeta_0)) + \lambda_i T$ $(i = 0, 1, 2)$.

Case 2. Either λ_1 or λ_2 is zero.

Without loss of generality, we may assume that $\lambda_1 = 0$. We agree that $E_0 \oplus E_1$ is the generalized eigenspace corresponding to the zero eigenvalue. Then for any $z \in E_0 \oplus E_1$, $\| \exp(nAT)z\| = O(|n|)$ as $n \to \pm\infty$ and the limits in (3.4) and (3.5) are both $\mathbb{E}(\ln(\zeta_0))$. When $z \in E_2$, the limits in (3.4) and (3.5) are both $\mathbb{E}(\ln(\zeta_0)) + \lambda_2 T$, which is given in Case 1. This completes the proof. □

Proof of Theorem 2.1. Let us introduce an auxiliary system of (2.1) as the following:

$$\begin{aligned} \frac{dR}{dt} &= 0, \\ \frac{dx}{dt} &= (f_x(R) - \alpha)x + \delta\beta y, \\ \frac{dy}{dt} &= (f_y(R) - \beta)y + \frac{\alpha x}{\delta}, \end{aligned} \tag{3.6}$$

and execute the same dilution regulation at $t = T_k^+$ as (2.2), $k \in \mathbb{N}$.

Denote by $\Psi_t(z_0)$ the semi-flow generated by the system (3.6) and set $\psi(n, \omega)z_0 \equiv \psi(n, \omega, z_0)$ to denote the RDS generated by the system (3.6) and the dilution regulation (2.2) for all $t \geq 0$, $n \in \mathbb{N}$ and $\omega \in \Omega$, where $z_0 = (R_0, x_0, y_0) \in \mathbb{R}^3_+$. Let $G(z)$ and $\widehat{G}(z)$ denote the vector fields in (2.1) and (3.6), respectively. It is obvious that $G(z) \leq \widehat{G}(z)$ for all $z = (R, x, y) \in \mathbb{R}^3_+$ and (3.6) is a cooperative system in $\mathbb{R}^3_+$. In what follows, we define $(\hat{R}(n, \omega, z_0), \hat{x}(n, \omega, z_0), \hat{y}(n, \omega, z_0)) = \psi(n, \omega)z_0$ for all $z_0 = (R_0, x_0, y_0) \in \mathbb{R}^3_+$. Combining the Kamke theorem [17] and the dilution regulation (2.2), it is immediate that

$$(R(n, \omega, z_0), x(n, \omega, z_0), y(n, \omega, z_0)) \leq (\hat{R}(n, \omega, z_0), \hat{x}(n, \omega, z_0), \hat{y}(n, \omega, z_0)) \tag{3.7}$$

for all $\omega \in \Omega$ and $n \in \mathbb{N}$. We now turn to show that $(x(n, \omega, z_0), y(n, \omega, z_0)) \to (0, 0)$ as $n \to \infty$, $\omega \in \Omega^+$. By (3.7), it is sufficient to prove that $(\hat{x}(n, \omega, z_0), \hat{y}(n, \omega, z_0)) \to (0, 0)$ as $n \to \infty$, $\omega \in \Omega^+$. Applying the same procedure in (2.10), we have that

$$\hat{R}(n, \omega, z_0) = R_0 \prod_{i=1}^{n} \zeta_i(\omega) + R^{(0)} \left(1 - \prod_{i=1}^{n} \zeta_i(\omega)\right) \triangleq \hat{R}_n(\omega) \tag{3.8}$$

for all $\omega \in \Omega$ and $n \in \mathbb{N}$. Using (2.11), it is easy to see that $\hat{R}(n, \omega, z_0) \to R^{(0)}$ as $n \to \infty$ on Ω^+. By induction, we can verify that

$$\begin{pmatrix} \hat{x}(n, \omega, z_0) \\ \hat{y}(n, \omega, z_0) \end{pmatrix} = \prod_{i=1}^{n} \zeta_i(\omega) \cdot \prod_{i=0}^{n-1} \exp(A_1(\hat{R}_i(\omega))T) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \tag{3.9}$$

where $A_1(R)$ is given in Proposition 3.1. Note the eigenvalues λ_1 and λ_2 of $A_1(R_0)$ are continuous with respect to the parameters $R^{(0)}, \alpha, \beta$ and δ. Thus, there is an $\epsilon_0 > 0$ sufficiently small such that

$$\mathbb{E}(\ln(\zeta_0)) + T\lambda_i(R^{(0)} + \epsilon_0, \alpha, \beta, \delta) < 0, \quad i = 1, 2.$$

Now fix an $\omega \in \Omega^+$, there exists an $N_0 = N_0(\epsilon_0, \omega, R^{(0)})$ such that when $n \geq N_0$, $\hat{R}(n, \omega, z_0) < R^{(0)} + \epsilon_0$. From the monotonicity of f_x and f_y, Kamke's theorem and (3.9), it follows that

$$\begin{aligned} \begin{pmatrix} \hat{x}(n, \omega, z_0) \\ \hat{y}(n, \omega, z_0) \end{pmatrix} &\leq \prod_{i=1}^{n} \zeta_i(\omega) \underbrace{e^{A_1(R^{(0)}+\epsilon_0)T} e^{A_1(R^{(0)}+\epsilon_0)T} \cdots e^{A_1(R^{(0)}+\epsilon_0)T}}_{n-N_0} X_0 \\ &= \prod_{i=1}^{n} \zeta_i(\omega) \exp\{(n - N_0)A_1(R^{(0)} + \epsilon_0)T\} X_0, \end{aligned}$$

where

$$X_0 = \prod_{i=0}^{N_0-1} \exp(A_1(\hat{R}_i(\omega))T) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Since the non-negative orthant of x–y plane is a normal cone, it is a simple matter to check that for all $\omega \in \Omega^+$,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \ln \left\| \begin{pmatrix} \hat{x}(n, \omega, z_0) \\ \hat{y}(n, \omega, z_0) \end{pmatrix} \right\| \\ &\quad \leq \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln(\zeta_i(\omega)) + \lim_{n\to\infty} \frac{1}{n} \ln \| \exp\{(n - N_0)A_1(R^{(0)} + \epsilon_0)T\} X_0\| \\ &\quad \leq \mathbb{E} \ln(\zeta_0) + T \max\{\lambda_1(R^{(0)} + \epsilon_0), \lambda_2(R^{(0)} + \epsilon_0)\} \\ &\quad < 0. \end{aligned}$$

This implies that

$$\begin{pmatrix} \hat{x}(n, \omega, z_0) \\ \hat{y}(n, \omega, z_0) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x(n, \omega, z_0) \\ y(n, \omega, z_0) \end{pmatrix}$$

exponentially converge to zero as $n \to \infty$. Finally, $R(n, \omega, z_0) \to R^{(0)}$ as $n \to \infty$ follows from Lemma 2.2. The proof is complete. □

Proof of Theorem 2.2. In order to prove

$$\varphi(n,\theta_{-n}\omega)z_0 = (R(n,\theta_{-n}\omega,z_0), x(n,\theta_{-n}\omega,z_0), y(n,\theta_{-n}\omega,z_0)) \to (R^{(0)},0,0)$$

as $n \to \infty$ for all $\omega \in \Omega^-$ and $z_0 \in \mathbb{R}^3_+$, we first consider the auxiliary system (3.6) and carry out the same dilution regulation at $t = T_k^+$ as (2.2), $k \in \mathbb{N}$. Analysis similar to that in the proof of Theorem 2.1, we can easily have that

$$(R(n,\omega,z_0), x(n,\omega,z_0), y(n,\omega,z_0)) \le (\hat{R}(n,\omega,z_0), \hat{x}(n,\omega,z_0), \hat{y}(n,\omega,z_0)) \tag{3.10}$$

for all $\omega \in \Omega$, $n \in \mathbb{N}$ and $z_0 \in \mathbb{R}^3_+$, which yields that

$$\begin{aligned}&(R(n,\theta_{-n}\omega,z_0), x(n,\theta_{-n}\omega,z_0), y(n,\theta_{-n}\omega,z_0))\\ &\quad \le (\hat{R}(n,\theta_{-n}\omega,z_0), \hat{x}(n,\theta_{-n}\omega,z_0), \hat{y}(n,\theta_{-n}\omega,z_0))\end{aligned} \tag{3.11}$$

for all $\omega \in \Omega$, $n \in \mathbb{N}$ and $z_0 \in \mathbb{R}^3_+$. Now, we turn to show that

$$(x(n,\theta_{-n}\omega,z_0), y(n,\theta_{-n}\omega,z_0)) \to (0,0)$$

as $n \to \infty$ for all $\omega \in \Omega^-$ and $z_0 \in \mathbb{R}^3_+$. By (3.11), we only need to show that $(\hat{x}(n,\theta_{-n}\omega,z_0), \hat{y}(n,\theta_{-n}\omega,z_0)) \to (0,0)$ as $n \to \infty$ for all $\omega \in \Omega^-$ and $z_0 \in \mathbb{R}^3_+$. By (3.8), it is obvious that

$$\hat{R}(n,\omega,z_0) = R_0 \prod_{i=1}^{n} \zeta_i(\omega) + R^{(0)}\left(1 - \prod_{i=1}^{n}\zeta_i(\omega)\right), \quad \omega \in \Omega \text{ and } n \in \mathbb{N}. \tag{3.12}$$

Next, we will verify that $\hat{R}(n,\omega,z_0) \to R^{(0)}$ as $n \to \infty$ uniformly with respect to $\omega \in \Omega$ for any $z_0 \in \mathbb{R}^3_+$. Observe that

$$|\hat{R}(n,\omega,z_0) - R^{(0)}| \le \prod_{i=1}^{n} \zeta_i(\omega)(R_0 + R^{(0)}) \le b^n(R_0 + R^{(0)}), \quad \omega \in \Omega \text{ and } n \in \mathbb{N}.$$

This implies that for any $\epsilon > 0$, there exists $N = N(\epsilon, R_0, R^{(0)}) \in \mathbb{N}$ independent of $\omega \in \Omega$ such that for $n \ge N$,

$$R^{(0)} - \epsilon < \hat{R}(n,\omega,z_0) < R^{(0)} + \epsilon \quad \text{for all } \omega \in \Omega \text{ and } z_0 \in \mathbb{R}^3_+ \tag{3.13}$$

and so

$$R^{(0)} - \epsilon < \hat{R}(n,\theta_{-n}\omega,z_0) < R^{(0)} + \epsilon \quad \text{for all } \omega \in \Omega \text{ and } z_0 \in \mathbb{R}^3_+. \tag{3.14}$$

Given any small enough $\epsilon > 0$, we can construct an auxiliary system as follows:

$$\frac{dz}{dt} \triangleq \widehat{G}^{\epsilon}_+(z) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & f_x(R^{(0)}+\epsilon) - \alpha & \beta\delta \\ 0 & \alpha\delta^{-1} & f_y(R^{(0)}+\epsilon) - \beta \end{bmatrix} \begin{pmatrix} R \\ x \\ y \end{pmatrix}. \tag{3.15}$$

Meanwhile, we will use the symbol $\Psi_t^{+,\epsilon}$ to denote the semi-flow generated by (3.15) and let $\psi^{\epsilon}_+(n,\omega)z_0$ denote the RDS generated by (3.15) and the dilution regulation (2.2), where $t \ge 0$, $\epsilon > 0$, $n \in \mathbb{N}$ and $\omega \in \Omega$. For any small $\epsilon > 0$

and $z_0 = (R_0, x_0, y_0) \in \mathbb{R}^3_+$, by the above analysis, we can choose an integer $N = N(\epsilon, R_0, R^{(0)})$ such that (3.13) holds for $n \geq N$. Define

$$(R^\epsilon_+(n,\omega,z_0), x^\epsilon_+(n,\omega,z_0), y^\epsilon_+(n,\omega,z_0)) \triangleq \psi^\epsilon_+(n-N,\theta_N\omega)\circ\psi(N,\omega)z_0, \quad (3.16)$$

where $\psi(n,\omega)z_0$ is the RDS generated by (3.6) and the dilution regulation (2.2), $n \geq N$, $\omega \in \Omega$ and $z_0 \in \mathbb{R}^3_+$. We claim that

$$(\hat{R}(n,\omega,z_0), \hat{x}(n,\omega,z_0), \hat{y}(n,\omega,z_0)) \leq (R^\epsilon_+(n,\omega,z_0), x^\epsilon_+(n,\omega,z_0), y^\epsilon_+(n,\omega,z_0)) \quad (3.17)$$

for all $n > N$, $\omega \in \Omega$ and $z_0 \in \mathbb{R}^3_+$. Now, we proceed by induction on n. If $n = N+1$, we have

$$\begin{aligned}
&(\hat{R}(N+1,\omega,z_0), \hat{x}(N+1,\omega,z_0), \hat{y}(N+1,\omega,z_0))\\
&\quad = \psi(N+1,\omega)z_0\\
&\quad = \psi(1,\theta_N\omega)\circ\psi(N,\omega)z_0\\
&\quad = \zeta_1(\theta_N\omega)\Psi_T(\psi(N,\omega)z_0) + (1-\zeta_1(\theta_N\omega))R^{(0)}\\
&\quad \leq \zeta_1(\theta_N\omega)\Psi^{+,\epsilon}_T(\psi(N,\omega)z_0) + (1-\zeta_1(\theta_N\omega))R^{(0)}\\
&\quad = \psi^\epsilon_+(1,\theta_N\omega)\circ\psi(N,\omega)z_0\\
&\quad = (R^\epsilon_+(N+1,\omega,z_0), x^\epsilon_+(N+1,\omega,z_0), y^\epsilon_+(N+1,\omega,z_0)) \qquad (3.18)
\end{aligned}$$

for all $\omega \in \Omega$ and $z_0 \in \mathbb{R}^3_+$, where the forth inequality holds because of the Kamke theorem and (3.13). In addition, by (3.15), it is evident that $R^\epsilon_+(N+1,\omega,z_0) = \hat{R}(N+1,\omega,z_0)$, which induces that

$$R^{(0)} - \epsilon < R^\epsilon_+(N+1,\omega,z_0) < R^{(0)} + \epsilon \quad \text{for all } \omega \in \Omega \text{ and } z_0 \in \mathbb{R}^3_+.$$

Using the cocycle property, for $n = N+2$, we can similarly obtain

$$\begin{aligned}
&(\hat{R}(N+2,\omega,z_0), \hat{x}(N+2,\omega,z_0), \hat{y}(N+2,\omega,z_0))\\
&\quad = \psi(N+2,\omega)z_0\\
&\quad = \psi(1,\theta_{N+1}\omega)\circ\psi(N+1,\omega)z_0\\
&\quad = \zeta_1(\theta_{N+1}\omega)\Psi_T[(\hat{R}(N+1,\omega,z_0), \hat{x}(N+1,\omega,z_0), \hat{y}(N+1,\omega,z_0))]\\
&\qquad + (1-\zeta_1(\theta_{N+1}\omega))R^{(0)}\\
&\quad \leq \zeta_1(\theta_{N+1}\omega)\Psi_T[(R^\epsilon_+(N+1,\omega,z_0), x^\epsilon_+(N+1,\omega,z_0), y^\epsilon_+(N+1,\omega,z_0)]\\
&\qquad + (1-\zeta_1(\theta_{N+1}\omega))R^{(0)}\\
&\quad \leq \zeta_1(\theta_{N+1}\omega)\Psi^{+,\epsilon}_T[(R^\epsilon_+(N+1,\omega,z_0), x^\epsilon_+(N+1,\omega,z_0), y^\epsilon_+(N+1,\omega,z_0))]\\
&\qquad + (1-\zeta_1(\theta_{N+1}\omega))R^{(0)}
\end{aligned}$$

$$
\begin{aligned}
&= \psi_+^\epsilon(1, \theta_{N+1}\omega) \circ [\psi_+^\epsilon(1, \theta_N\omega) \circ \psi(N, \omega) z_0] \\
&= \psi_+^\epsilon(2, \theta_N\omega) \circ \psi(N, \omega) z_0 \\
&= (R_+^\epsilon(N+2, \omega, z_0), x_+^\epsilon(N+2, \omega, z_0), y_+^\epsilon(N+2, \omega, z_0))
\end{aligned} \tag{3.19}
$$

for all $\omega \in \Omega$ and $z_0 \in \mathbb{R}_+^3$. Here, the above inequalities follow by (3.18), the Kamke theorem and the monotone property of Ψ_T. Therefore, (3.17) can be obtained by induction on n. From (3.17), it is immediate that

$$
\begin{aligned}
&(\hat{R}(n, \theta_{-n}\omega, z_0), \hat{x}(n, \theta_{-n}\omega, z_0), \hat{y}(n, \theta_{-n}\omega, z_0)) \\
&\quad \le (R_+^\epsilon(n, \theta_{-n}\omega, z_0), x_+^\epsilon(n, \theta_{-n}\omega, z_0), y_+^\epsilon(n, \theta_{-n}\omega, z_0))
\end{aligned} \tag{3.20}
$$

for all $n > N$, $\omega \in \Omega$ and $z_0 \in \mathbb{R}_+^3$.

In what follows, we shall prove that $\psi_+^\epsilon(n, \theta_{-n}\omega) z_0 \to (R^{(0)}, 0, 0)$ as $n \to \infty$ for all $\omega \in \Omega^-$ and $z_0 \in \mathbb{R}_+^3$. Since $\mathbb{E}(\ln(\zeta_0)) + \lambda T < 0$, then there exists $\epsilon > 0$ such that $\mathbb{E}(\ln(\zeta_0)) + \lambda_\epsilon T < 0$, where $\lambda_\epsilon = \lambda_1^\epsilon \vee \lambda_2^\epsilon$, which are two eigenvalues of the following matrix:

$$
A_1^\epsilon = \begin{bmatrix} f_x(R^{(0)} + \epsilon) - \alpha & \beta\delta \\ \alpha\delta^{-1} & f_y(R^{(0)} + \epsilon) - \beta \end{bmatrix}. \tag{3.21}
$$

Here, it is easy to show that λ_1^ϵ and λ_2^ϵ are two real constants when ϵ is sufficiently small. For any $z_0 = (R_0, x_0, y_0) \in \mathbb{R}_+^3$, a direct computation shows that

$$
\begin{aligned}
\psi_+^\epsilon(n, \theta_{-n}\omega) z_0 = \Bigg(& R_0 \prod_{i=0}^{n-1} \zeta_{-i}(\omega) + R^{(0)} \left(1 - \prod_{i=0}^{n-1} \zeta_{-i}(\omega)\right), \\
& \left(\prod_{i=0}^{n-1} \zeta_{-i}(\omega) \exp(n A_1^\epsilon T) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right)^{\mathrm{T}} \Bigg)
\end{aligned} \tag{3.22}
$$

for all $n > N$, $\omega \in \Omega$ and $z_0 \in \mathbb{R}_+^3$. According to the Multiplicative Ergodic Theorem (see [1, pp. 153–154]), we see at once that the Lyapunov exponent

$$
\lambda(\omega, x_0, y_0) = \lim_{n \to \infty} \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} \zeta_{-i}(\omega) \exp(n A_1^\epsilon T) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right| \tag{3.23}
$$

exists for all $\omega \in \Omega^-$, where Ω^- is a θ-invariant set of full measure given by (2.15). Moreover, since θ is an ergodic MDS, by the Birkhoff–Khinchin Theorem (see [1, p. 539]), we have that

$$
\begin{aligned}
\lambda(\omega, x_0, y_0) &\equiv \lambda(x_0, y_0) \\
&= \lim_{n \to \infty} \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} \zeta_{-i}(\omega) \right| + \lim_{n \to \infty} \frac{1}{n} \ln \left| \exp(n A_1^\epsilon T) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \right|
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\ln(\zeta_{-i}(\omega)) + \lim_{n\to\infty}\frac{1}{n}\ln\left|\exp(nA_1^{\epsilon}T)\begin{pmatrix}x_0\\ y_0\end{pmatrix}\right| \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\ln(\zeta_0(\theta_{-i}\omega)) + \lim_{n\to\infty}\frac{1}{n}\ln\left|\exp(nA_1^{\epsilon}T)\begin{pmatrix}x_0\\ y_0\end{pmatrix}\right| \\
&= \mathbb{E}(\ln(\zeta_0)) + \lim_{n\to\infty}\frac{1}{n}\ln\left|\exp(nA_1^{\epsilon}T)\begin{pmatrix}x_0\\ y_0\end{pmatrix}\right| \\
&\le \mathbb{E}(\ln(\zeta_0)) + \lambda_\epsilon T < 0
\end{aligned}
\tag{3.24}
$$

for all $\omega \in \Omega^-$. This together with (3.14) and (3.22) implies that given any $z_0 = (R_0, x_0, y_0) \in \mathbb{R}_+^3$,

$$
\psi_+^{\epsilon}(n, \theta_{-n}\omega)z_0 \to (R^{(0)}, 0, 0) \tag{3.25}
$$

as $n \to \infty$ for all $\omega \in \Omega^-$.

Now, we turn to verify that $(x_+^{\epsilon}(n, \theta_{-n}\omega, z_0), y_+^{\epsilon}(n, \theta_{-n}\omega, z_0)) \to (0,0)$ as $n \to \infty$ for small enough $\epsilon > 0$, $\omega \in \Omega^-$ and $z_0 \in \mathbb{R}_+^3$. To prove this, it suffices to check that for fixed initial condition $z_0 = (R_0, x_0, y_0) \in \mathbb{R}_+^3$ and the integer $N \in \mathbb{N}$ given in (3.13), there exists a positive vector $C_N = C(z_0, N) = C(z_0, \epsilon) \in \mathbb{R}_+^3$ such that $\psi(N, \theta_{-n}\omega)z_0 \le C_N$ for all $\omega \in \Omega^-$ and $n \in \mathbb{N}$. From (3.12), it is easily seen that $\hat{R}(n, \omega, z_0) \le R_0 + R^{(0)}$ for all $n \in \mathbb{N}$, $\omega \in \Omega^-$ and $z_0 \in \mathbb{R}_+^3$. Note that (3.6) is a cooperative system in $\mathbb{R}_+^3$. Using the same method in the proof of (3.17), the Kamke theorem and the regulation (2.2), we conclude that

$$
\begin{aligned}
\psi(N,\omega)z_0 &\le \left(R_0 + R^{(0)}, \left(\prod_{i=1}^{N}\zeta_i(\omega)\exp(NA_{R_0}T)\begin{pmatrix}x_0\\ y_0\end{pmatrix}\right)^{\mathrm{T}}\right) \\
&\le \left(R_0 + R^{(0)}, \left(\exp(NA_{R_0}T)\begin{pmatrix}x_0\\ y_0\end{pmatrix}\right)^{\mathrm{T}}\right) \triangleq C_N
\end{aligned}
\tag{3.26}
$$

and so $\psi(N, \theta_{-n}\omega)z_0 \le C_N$ for all $\omega \in \Omega^-$, $n \in \mathbb{N}$ and $z_0 \in \mathbb{R}_+^3$, where

$$
A_{R_0} = \begin{bmatrix} f_x(R_0 + R^{(0)}) - \alpha & \beta\delta \\ \alpha\delta^{-1} & f_y(R_0 + R^{(0)}) - \beta \end{bmatrix}.
$$

Since (3.15) is also a cooperative system, it is evident that when $n > N$, for all $\omega \in \Omega^-$ and $z_0 \in \mathbb{R}_+^3$,

$$
\begin{aligned}
&(R_+^{\epsilon}(n, \theta_{-n}\omega, z_0), x_+^{\epsilon}(n, \theta_{-n}\omega, z_0), y_+^{\epsilon}(n, \theta_{-n}\omega, z_0)) \\
&\quad = \psi_+^{\epsilon}(n - N, \theta_{N-n}\omega) \circ \psi(N, \theta_{-n}\omega)z_0 \\
&\quad \le \psi_+^{\epsilon}(n - N, \theta_{N-n}\omega) \circ C_N \to (R^{(0)}, 0, 0) \quad \text{as } n \to \infty.
\end{aligned}
\tag{3.27}
$$

Here, we have used (3.16) and (3.25). That is,

$$(x_+^{\epsilon}(n, \theta_{-n}\omega, z_0), y_+^{\epsilon}(n, \theta_{-n}\omega, z_0)) \to (0, 0)$$

as $n \to \infty$ for all $\omega \in \Omega^-$ and $z_0 \in \mathbb{R}_+^3$. By (3.20), it follows that

$$(\hat{x}(n, \theta_{-n}\omega, z_0), \hat{y}(n, \theta_{-n}\omega, z_0)) \to (0, 0)$$

as $n \to \infty$ for all $\omega \in \Omega^-$ and $z_0 \in \mathbb{R}_+^3$. This together with (3.11) yields that

$$(x(n, \theta_{-n}\omega, z_0), y(n, \theta_{-n}\omega, z_0)) \to (0, 0)$$

as $n \to \infty$ for all $\omega \in \Omega^-$ and $z_0 \in \mathbb{R}_+^3$.

Finally, it remains to verify that $R(n, \theta_{-n}\omega, z_0) \to R^{(0)}$ as $n \to \infty$ for all $\omega \in \Omega^-$ and $z_0 \in \mathbb{R}_+^3$. From (2.12), it is immediate that

$$\begin{aligned} K(n, \omega, z_0) &\triangleq \gamma R(n, \omega, z_0) + x(n, \omega, z_0) + \delta y(n, \omega, z_0) \\ &= K_0 \prod_{i=1}^{n} \zeta_i(\omega) + \gamma R^{(0)} \left[1 - \prod_{i=1}^{n} \zeta_i(\omega)\right] \end{aligned} \tag{3.28}$$

for all $\omega \in \Omega$, $n \in \mathbb{N}$ and $z_0 = (R_0, x_0, y_0) \in \mathbb{R}_+^3$, where $K_0 = \gamma R_0 + x_0 + \delta y_0$. Consequently, by (3.28), we can easily obtain that

$$K(n, \theta_{-n}\omega, z_0) = K_0 \prod_{i=0}^{n-1} \zeta_{-i}(\omega) + \gamma R^{(0)} \left[1 - \prod_{i=0}^{n-1} \zeta_{-i}(\omega)\right] \tag{3.29}$$

for all $\omega \in \Omega$, $n \in \mathbb{N}$ and $z_0 \in \mathbb{R}_+^3$. Using (2.15) and (3.29), it is easy to see that

$$K(n, \theta_{-n}\omega, z_0) \to \gamma R^{(0)} \quad \text{as } n \to \infty$$

for all $\omega \in \Omega^-$ and $z_0 \in \mathbb{R}_+^3$. This gives that $R(n, \theta_{-n}\omega, z_0) \to R^{(0)}$ as $n \to \infty$ for all $\omega \in \Omega^-$ and $z_0 \in \mathbb{R}_+^3$. The proof is complete. □

4. Discussion

Nanochemostat is recently invented by a group of physicists at Cal Tech in 2005 [2]. Besides its nanoliter size compared with milliliter size of conventional chemostat, nanochemostat can efficiently deal with many biological complexities for instance enzyme yield optimization, system biology, bioenergy generation, etc. The deterministic model for nanochemostat is quite different from that for conventional chemostat. In nanochemostat, Hsu and Yang considered a continuous model for the growth of microbes in a fixed period T. Right after the period T, they imposed resetting operations: a portion η microbes is deleted and a fresh medium is injected to start another growth cycle. Thus, they constructed a T-periodic map P and studied the stability property of the fixed points for the discrete map P [13]. However from the biological reality, some of the parameters in the deterministic model (2.1) and (2.2) are in fact random or stochastic, for instance the parameters η, α and β in (2.1) and (2.2).

In this paper, we study a random model corresponding to (2.1) and (2.2) with the dilution ratio $\eta := \eta(\omega)$ as a random variable with uniform distribution as the experimental data observed in different cycles [12]. Our main results are stated in Theorems 2.1 and 2.2. First, we note that in these theorems, $z_0 = (R_0, x_0, y_0)$ is the initial value, $\varphi(n,\omega)z_0$ and $\varphi(n,\theta_{-n}\omega)z_0$ correspond to forward and pull-back trajectories of the nth iteration for the T-periodic map P. Theorems 2.1 and 2.2 concern the extinction of planktonic and biofilm populations in the nanochemostat. We state that if $\mathbb{E}(\ln(\zeta_0)) + \lambda T < 0$, then

$$\varphi(n,\omega)z_0 = (R(n,\omega,z_0), x(n,\omega,z_0), y(n,\omega,z_0)) \to (R^{(0)}, 0, 0)$$

as $n \to \infty$ (Theorem 2.1) and

$$\varphi(n,\theta_{-n}\omega)z_0 = (R(n,\theta_{-n}\omega,z_0), x(n,\theta_{-n}\omega,z_0), y(n,\theta_{-n}\omega,z_0)) \to (R^{(0)}, 0, 0)$$

as $n \to \infty$ (Theorem 2.2). The condition $\mathbb{E}(\ln(\zeta_0)) + \lambda T < 0$ corresponds to the extinction condition $\eta < e^{-f(R^{(0)})T}$ (when $f_x(R) = f_y(R) \triangleq f(R)$) in the deterministic model (2.1) and (2.2), such as [13]. We close this paper by pointing out some open problems: In [13], the authors also investigated the uniform persistence and the global stability of positive fixed points, which is corresponding to the case $\mathbb{E}(\ln(\zeta_0)) + \lambda T > 0$ and will encounter the problem of non-uniform hyperbolicity. This will be the subject of future work.

Acknowledgments

We would like to thank Prof. Sze-Bi Hsu for encouraging and discussing to propose the model (2.1) and (2.2) and the warm hospitality during our visit to the Department of Mathematics, National Tsing-Hua University, Taiwan. This work was partially supported by the National Natural Science Foundation of China (NSFC) under Grant Nos. 12171321, 11971316 and 11771295; the NSF of Shanghai Grants under Nos. 19ZR1437100 and 20JC1413800; Chen Guang Project (14CG43) of Shanghai Municipal Education Commission, Shanghai Education Development Foundation; Yangfan Program of Shanghai (14YF1409100) and Shanghai Gaofeng Project for University Academic Program Development.

Appendix A

Proof of Proposition 2.2. Let us go back to dilution steps and describe the evolution (2.5) with the time k. Recall that the coordinate process $\zeta_n : \Omega \mapsto X$,

$n \in \mathbb{Z}^+$ is the dilution ratio of the nth step.

(i) *The first step to dilute*

For any given initial point $z_0 \in \mathbb{R}^3_+$, solve the solution of (2.1) passing through z_0 at $t = 0$ on the interval $[0, T]$. Then we get $\Phi_T(z_0)$. Replacing the dilution ratio η by $\zeta_1(\omega)$ in (2.3), we obtain the result of the first dilution step:

$$\varphi(1, \omega)z_0 = \zeta_1(\omega)\Phi_T(z_0) + (1 - \zeta_1(\omega))E_0. \tag{A.1}$$

Substituting $\theta_{-1}\omega$ for ω and using (2.9), we have the pull-back result at the time $n = 1$:

$$\varphi(1, \theta_{-1}\omega)z_0 = \zeta_0(\omega)\Phi_T(z_0) + (1 - \zeta_0(\omega))E_0. \tag{A.2}$$

(ii) *The second step to dilute*

Let us solve the solution of equations (2.1) passing through $\varphi(1, \omega)z_0$ at $t = T$ on $[T, 2T]$, which is $\Phi_{t-T}(\varphi(1, \omega)z_0)$. Then the value at $t = 2T$ of this solution is $\Phi_T(\varphi(1, \omega)z_0)$. Substituting the dilution ratio $\zeta_2(\omega)$ for η in (2.3), one has

$$\varphi(2, \omega)z_0 = \zeta_2(\omega)\Phi_T(\varphi(1, \omega)z_0) + (1 - \zeta_2(\omega))E_0 \tag{A.3}$$

which satisfies

$$\varphi(2, \omega)z_0 = \varphi(1, \theta_1\omega) \circ \varphi(1, \omega)z_0. \tag{A.4}$$

Here, $\circ$ means the composition of mappings. In fact, by definition,

$$\begin{aligned} \varphi(1, \theta_1\omega) \circ \varphi(1, \omega)z_0 &= \zeta_1(\theta_1\omega)\Phi_T(\varphi(1, \omega)z_0) + (1 - \zeta_1(\theta_1\omega))E_0 \\ &= \zeta_2(\omega)\Phi_T(\varphi(1, \omega)z_0) + (1 - \zeta_2(\omega))E_0 \\ &= \varphi(2, \omega)z_0, \end{aligned}$$

where we have use (2.9) in the second equality.

Replacing ω by $\theta_{-2}\omega$ and using (2.9), we have the pull-back result at the time $n = 2$:

$$\varphi(2, \theta_{-2}\omega)z_0 = \zeta_0(\omega)\Phi_T(\varphi(1, \theta_{-2}\omega)z_0) + (1 - \zeta_0(\omega))E_0, \tag{A.5}$$

where

$$\varphi(1, \theta_{-2}\omega)z_0 = \zeta_{-1}(\omega)\Phi_T(z_0) + (1 - \zeta_{-1}(\omega))E_0. \tag{A.6}$$

(iii) *The third step to dilute*

Similarly, $\Phi_{t-2T}(\varphi(2, \omega)z_0)$ is the solution of (2.1) on $[2T, 3T]$ passing through the initial point $\varphi(2, \omega)z_0$ at $t = 2T$. Hence, we arrive at the value of this solution at $t = 3T$, which is $\Phi_T(\varphi(2, \omega)z_0)$. Diluting it by the ratio $\zeta_3(\omega)$ according to the regulation (2.2), we reach the result of the third dilution step:

$$\varphi(3, \omega)z_0 = \zeta_3(\omega)\Phi_T(\varphi(2, \omega)z_0) + (1 - \zeta_3(\omega))E_0, \tag{A.7}$$

which possesses the property

$$\varphi(3,\omega)z_0 = \varphi(1,\theta_2\omega)\circ\varphi(2,\omega)z_0 = \varphi(2,\theta_1\omega)\circ\varphi(1,\omega)z_0. \tag{A.8}$$

We only prove the second equality, the first one is similar. Together with (2.9), (A.3) and (A.4), we have

$$\begin{aligned}\varphi(2,\theta_1\omega)\circ\varphi(1,\omega)z_0 &= \zeta_2(\theta_1\omega)\Phi_T(\varphi(1,\theta_1\omega)\circ\varphi(1,\omega)z_0) + (1-\zeta_2(\theta_1\omega))E_0\\ &= \zeta_3(\omega)\Phi_T(\varphi(2,\omega)z_0) + (1-\zeta_3(\omega))E_0\\ &= \varphi(3,\omega)z_0.\end{aligned}$$

Replacing ω by $\theta_{-3}\omega$ in (A.7) and using (2.9), we have the pull-back result at the time $n=3$:

$$\varphi(3,\theta_{-3}\omega)z_0 = \zeta_0(\omega)\Phi_T(\varphi(2,\theta_{-3}\omega)z_0) + (1-\zeta_0(\omega))E_0, \tag{A.9}$$

where

$$\varphi(2,\theta_{-3}\omega)z_0 = \zeta_{-1}(\omega)\Phi_T(\varphi(1,\theta_{-3}\omega)z_0) + (1-\zeta_{-1}(\omega))E_0 \tag{A.10}$$

and

$$\varphi(1,\theta_{-3}\omega)z_0 = \zeta_{-2}(\omega)\Phi_T(z_0) + (1-\zeta_{-2}(\omega))E_0. \tag{A.11}$$

(nth) *The nth step to dilute.*

We inductively assume that we have finished the $(n-1)$th step dilution and obtained the result of $\varphi(n-1,\omega)z_0$, which possesses the cocycle property:

$$\varphi(n-1,\omega) = \varphi(i,\theta_j\omega)\circ\varphi(j,\omega) \quad \text{for all } i,j \text{ with } i+j=n-1. \tag{A.12}$$

Analogously, $\Phi_{t-(n-1)T}(\varphi(n-1,\omega)z_0)$ is the solution of (2.1) passing through $\varphi(n-1,\omega)z_0$ at $t=(n-1)T$. According to the regulation (2.2), the nth step dilution result is

$$\varphi(n,\omega)z_0 = \zeta_n(\omega)\Phi_T(\varphi(n-1,\omega)z_0) + (1-\zeta_n(\omega))E_0. \tag{A.13}$$

We claim that it admits the cocycle property:

$$\varphi(n,\omega) = \varphi(i,\theta_j\omega)\circ\varphi(j,\omega) \quad \text{for all } i,j \text{ with } i+j=n. \tag{A.14}$$

In fact, from the definition of each dilution step, (2.9) and (A.12), it follows that

$$\begin{aligned}\varphi(i,\theta_j\omega)\circ\varphi(j,\omega)z_0 &= \zeta_i(\theta_j\omega)\Phi_T(\varphi(i-1,\theta_j\omega)\circ\varphi(j,\omega)z_0) + (1-\zeta_i(\theta_j\omega))E_0\\ &= \zeta_n(\omega)\Phi_T(\varphi(n-1,\omega)z_0) + (1-\zeta_n(\omega))E_0\\ &= \varphi(n,\omega)z_0.\end{aligned}$$

In order to get the pull-back results of $\{\varphi(j,\theta_{-n}\omega)z_0 \,|\, j=1,2,\ldots,n\}$, observing (A.2), (A.5), (A.6) and (A.9)–(A.11), we just process each dilution step as $\varphi(j,\omega)z_0$, but the dilution ratios $\{\zeta_1,\zeta_2,\ldots,\zeta_n\}$ are orderly replaced by $\{\zeta_{1-n},\zeta_{2-n},\ldots,\zeta_0\}$.

Together with the measurability of $\{\zeta_i : \Omega\to X \,|\, i\in\mathbb{Z}\}$ and the continuity of $\Phi_T(z_0)$, the proof is complete. $\square$

References

1. L. Arnold, *Random Dynamical Systems* (Springer-Verlag, 1998).
2. F. K. Balagadde, L. C. You, C. L. Hansen, F. H. Arnold and S. R. Quake, Long-term monitoring of bacteria undergoing programmed population control in a microchemostat, *Science* **309** (2005) 137–140.
3. F. Campillo, M. Joannides and I. Larramendy-Valverde, Stochastic modeling of the chemostat, *Ecol. Model.* **222** (2011) 2676–2689.
4. F. Campillo, M. Joannides and I. Larramendy-Valverde, Approximation of the Fokker–Planck equation of the stochastic chemostat, *Math. Comput. Simulation* **99** (2014) 37–53.
5. T. Caraballo, M. J. Garrido-Atienza and J. López-de-la-Cruz, Modeling and analysis of random and stochastic input flows in the chemostat model, *Discrete Contin. Dyn. Syst. Ser. B* **24** (2019) 3591–3614.
6. N. Champagnat, P.-E. Jabin and S. Méléard, Adaptation in a stochastic multi-resources chemostat model, *J. Math. Pures Appl.* **101** (2014) 755–788.
7. I. Chueshov, *Monotone Random Systems Theory and Applications*, Lecture Notes in Mathematics (Springer-Verlag, 2002).
8. P. Collet, S. Martínez, S. Méléard and J. San Martín, Stochastic models for a chemostat and long-time behavior, *Adv. in Appl. Probab.* **45** (2013) 822–837.
9. K. Crump and W. O'Young, Some stochastic features of bacterial constant growth apparatus, *Bull. Math. Biol.* **41** (1979) 53–66.
10. P. De Leenheer and H. L. Smith, Feedback control for chemostat models, *J. Math. Biol.* **46** (2003) 48–70.
11. J. Grasman, M. D. Gee and O. A. V. Herwaarden, Breakdown of a chemostat exposed to stochastic noise, *J. Engrg. Math.* **53** (2005) 291–300.
12. G. Y. Gu, W. Y. Lee, C. C. Chiang and Y. T. Yang, A nanoliter microfluidic serial dilution bioreactor, *Biomicrofluidics* **9** (2015) 044126.
13. S. B. Hsu and Y. T. Yang, Theory of a microfluidic serial dilution bioreactor for growth of planktonic and biofilm populations, *J. Math. Biol.* **72** (2016) 1401–1427.
14. L. Imhof and S. Walcher, Exclusion and persistence in deterministic and stochastic chemostat models, *J. Differential Equations* **217** (2005) 26–53.
15. Z. Lian and K. N. Lu, Lyapunov exponents and invariant manifolds for random dynamical systems in a Banach space, *Mem. Amer. Math. Soc.* **206** (2010) 1–106.
16. A. Novick and L. Szilard, Description of the chemostat, *Science* **112** (1950) 715–716.
17. H. L. Smith, *Monotone Dynamical Systems, An Introduction to the Theory of Competitive and Cooperative Systems*, Mathematical Surveys and Monographs, Vol. 41 (Amer. Math. Soc., 1995).
18. H. L. Smith, Bacteria competition in serial transfer culture, *Math. Biosci.* **229** (2011) 149–159.
19. H. L. Smith and P. L. Waltman, *The Theory of the Chemostat* (Cambridge Univ. Press, 1995).