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EXISTENCE OF UNSTABLE STATIONARY SOLUTIONS FOR NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS WITH ADDITIVE WHITE NOISE

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Abstract. This paper is concerned with the existence of unstable stationary solutions for nonlinear stochastic differential equations (SDEs) with additive white noise. Assume that the nonlinear term f is monotone (or anti-monotone) and the global Lipschitz constant of f is smaller than the positive real part of the principal eigenvalue of the competitive matrix A, the random dynamical system (RDS) generated by SDEs has an unstable $\mathscr{F}_+$-measurable random equilibrium, which produces a stationary solution for nonlinear SDEs. Here, $\mathscr{F}_+ = \sigma\{\omega \mapsto W_t(\omega) : t \geq 0\}$ is the future σ-algebra. In addition, we get that the α-limit set of all pull-back trajectories starting at the initial value $x(0) = x \in \mathbb{R}^n$ is a single point for all $\omega \in \Omega$, i.e., the unstable $\mathscr{F}_+$-measurable random equilibrium. Applications to stochastic neural network models are given.

1. **Introduction.** During the past decades, stochastic differential equations (SDEs) have been widely used to account the integrated effects of interior interactions and environmental fluctuations. A fundamental question in the study of SDEs is to consider the existence and global stability of stationary solutions under minimal conditions.

Various mathematical methods exist for verifying the stability of SDEs, including Lyapunov functions [7, 10, 11] and random dynamical systems (RDSs) [1, 2, 8]. If we consider the stability of the zero solution for SDEs, the former approach may be more effective, see [9, 12]. However, sometimes there are no trivial stationary solutions for SDEs. At the moment, the later method can be used to investigate the long-term behaviour of SDEs. For example, let us consider the following scalar SDE

$$dx = \mu x dt + \nu dW_t, \tag{1}$$

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where W_t is a Wiener process in $\mathbb{R}$ and μ, ν are constants. This equation generates an affine RDS (θ,ψ) with the cocycle

$$\psi(t,\omega,x) = e^{\mu t}x + \nu\int_0^t e^{\mu(t-\tau)}dW_\tau$$

for all $x \in \mathbb{R}$. It is easy to see that the RDS (θ,ψ) possesses an exponentially stable $\mathscr{F}_-$-measurable random equilibrium

$$u(\omega) = \int_{-\infty}^0 e^{-\mu\tau}dW_\tau$$

for $\mu < 0$, where $\mathscr{F}_- = \sigma\{\omega \mapsto W_t(\omega) : t \leq 0\}$ is the past σ-algebra. In the case that $\mu > 0$, it admits an unstable $\mathscr{F}_+$-measurable random equilibrium

$$v(\omega) = -\int_0^\infty e^{-\mu\tau}dW_\tau,$$

where $\mathscr{F}_+ = \sigma\{\omega \mapsto W_t(\omega) : t \geq 0\}$ is the future σ-algebra.

Motivated by our recent works [5, 6], this paper is devoted to the existence of unstable stationary solutions for nonlinear SDEs with additive white noise. To be specific, we will prove that under the condition that the nonlinear function f is monotone (or anti-monotone) and the global Lipschitz constant of f is moderately smaller than the positive real part of the principal eigenvalue of the competitive matrix A, the stochastic flow (θ,ψ) has an unstable $\mathscr{F}_+$-measurable random equilibrium, which yields a stationary solution for nonlinear SDEs. In addition, we conclude that the α-limit set of all pull-back trajectories starting from any initial value in $\mathbb{R}^n$ is a single point for all $\omega \in \Omega$, i.e., the unstable $\mathscr{F}_+$-measurable random equilibrium.

Our main result gives a criteria to guarantee the existence of unstable stationary solutions for nonlinear SDEs, which can be applied to stochastic neural network models, see Corollary 1, Examples 5.1 and 5.2. Our problem, assumptions and definitions of RDSs are stated in Section 2. Some useful lemmas and the long-term behavior of solutions for SDEs are presented in Section 3. Proofs of the existence of unstable stationary solutions are given in Section 4. Finally, we apply our results to several stochastic models from neural networks in Section 5.

2. **Preliminaries.**

In this section, we consider the following n-dimensional SDEs

$$dx(t) = [Ax(t) + f(x(t))]dt + \sigma dW(t) \tag{2}$$

with the initial value $x(0) = x \in \mathbb{R}^n$, where $W_t(\omega) = \big(W_t^1(\omega), \ldots, W_t^m(\omega)\big)^{\mathrm{T}}$ is an m-dimensional two-sided Brownian motion on the standard Wiener space $(\Omega, \mathscr{F}, \mathbb{P})$. Here, $\mathscr{F}$ is the Borel σ-algebra of $\Omega = C_0(\mathbb{R},\mathbb{R}^m) = \{\omega(t)$ is continuous, $\omega(0) = 0,\ t \in \mathbb{R}\}$. Moreover, $A = (a_{ij})_{n\times n}$ is an $n \times n$-dimensional matrix, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma = (\sigma_{ij})_{n\times m}$ is an $n \times m$-dimensional matrix. Throughout this paper, we use the maximum norm $|x| := \max\{|x_i| : i = 1, \ldots, n\}$ and $\|A\| := \max\{|a_{ij}| : i, j = 1, \ldots, n\}$, where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. In order to show our result, we will impose some conditions on A and f.

(A1) A is competitive, i.e., $a_{ij} \leq 0$ for all $i, j \in \{1, \ldots, n\}$ and $i \neq j$. In addition, we suppose that all real parts of eigenvalues of A are positive. That is, $-A$ is cooperative and there exists a constant $\lambda > 0$ such that

$$\|\Psi(-t)\| := \max\{|\Psi_{ij}(-t)| : i, j = 1, \ldots, n\} \leq \mathrm{e}^{-\lambda t} \tag{3}$$

for all $t \geq 0$. Here, $\Psi(t) = \exp(At)$ is the fundamental matrix of the following linear ordinary differential equations (ODEs):

$$dx(t) = Ax(t)dt. \tag{4}$$

(A2) $f : \mathbb{R}^n \to \mathbb{R}^n$ is bounded and satisfies the Lipschitz condition

$$|f(x) - f(y)| \leq L|x - y| \tag{5}$$

for all $x, y \in \mathbb{R}^n$, where $L > 0$ is the Lipschitz constant. Moreover, we suppose that f is monotone, i.e.,

$$x \leq_{\mathbb{R}^n_+} y \quad \Rightarrow \quad f(x) \leq_{\mathbb{R}^n_+} f(y) \qquad \text{for all } x, y \in \mathbb{R}^n,$$

or anti-monotone, i.e.,

$$x \leq_{\mathbb{R}^n_+} y \quad \Rightarrow \quad f(x) \geq_{\mathbb{R}^n_+} f(y) \qquad \text{for all } x, y \in \mathbb{R}^n.$$

Here, $x \leq_{\mathbb{R}^n_+} y$ shows that $y - x \in \mathbb{R}^n_+$ for all $x, y \in \mathbb{R}^n$, where $\mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n | x_i \geq 0, i = 1, \ldots, n\}$.

The main thought in this paper is to consider the long-term behaviour of stochastic flows generated by (2) and prove that the α-limit set of (pull back) trajectories emanating from the initial value $x(0) = x \in \mathbb{R}^n$ is a single point for any $\omega \in \Omega$. For the convenience of the reader, we will recall some basic notations related to RDSs. For more details, we refer the reader to [1, 2].

Definition 2.1. A metric dynamical system is defined by $\theta \equiv (\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{R}\})$, where $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space and θ is a flow:

$$\theta : \mathbb{R} \times \Omega \mapsto \Omega, \qquad \theta_0 = \text{id}, \qquad \theta_{t_2} \circ \theta_{t_1} = \theta_{t_1 + t_2}$$

for all $t_1, t_2 \in \mathbb{R}$, which is $(\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}, \mathscr{F})$-measurable. In addition, we also assume that $\theta_t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Let X be a separable complete metric space, i.e., a Polish space, which is equipped with the Borel σ-algebra $\mathscr{B}(X)$ generated by open sets of X.

Definition 2.2. An RDS with two-sided time $\mathbb{R}$ and the phase space X is a couple (θ, ψ) consisting of a metric dynamical system $\theta \equiv (\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{R}\})$ and a cocycle ψ over θ, i.e., a $(\mathscr{B}(\mathbb{R}) \otimes \mathscr{F} \otimes \mathscr{B}(X), \mathscr{B}(X))$-measurable mapping

$$\psi : \mathbb{R} \times \Omega \times X \mapsto X, \quad (t, \omega, x) \mapsto \psi(t, \omega, x),$$

such that for any $\omega \in \Omega$,

(i) the mapping $\psi(t, \omega, \cdot) : X \to X$ is continuous for all $t \in \mathbb{R}$;
(i) $\psi(0, \omega, \cdot)$ is the identity on X;
(i) $\psi(t_1 + t_2, \omega, x) = \psi(t_2, \theta_{t_1}\omega, \psi(t_1, \omega, x))$ for all $t_1, t_2 \in \mathbb{R}$ and $x \in X$.

Definition 2.3. The multifunction $D : \Omega \to 2^X \backslash \{\varnothing\}$ is called a random set if the mapping $\omega \to d(x, D(\omega))$ is measurable for each $x \in X$. Here, $d(x, B)$ is the distance in X between the point x and the set $B \subset X$. A random set D is called a random closed (resp. compact) set if $D(\omega)$ is closed (resp. compact) in X for each $\omega \in \Omega$.

By the standard theory of SDEs [10, 14], it is easy to obtain the existence and uniqueness of solutions for (2). Let $\psi(t, \omega, x) = x(t, \omega, x)$ be the unique solution of (2) with the initial value $x(0) = x \in \mathbb{R}^n$, which generates a two-sided RDS (θ, ψ) in $\mathbb{R}^n$, see [1, Chap. 2]. Here, θ is the Wiener shift operator defined by $\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t)$ for all $t \in \mathbb{R}$, which is an ergodic metric dynamical system.

Next, using the variation-of-constants formula [10, Theorem 3.1] and the backward Itô integral (see Arnold [1, p.97]), it follows that for all $t \geq 0$,

$$\begin{aligned}\psi(-t,\omega,x) &= \Psi(-t)x + \Psi(-t)\int_0^{-t}\Psi^{-1}(s)f\big(\psi(s,\omega,x)\big)ds \\ &\quad + \Psi(-t)\int_0^{-t}\Psi^{-1}(s)\sigma dW_s \\ &= \Psi(-t)x + \int_0^{-t}\Psi(-t-s)f\big(\psi(s,\omega,x)\big)ds \\ &\quad + \int_0^{-t}\Psi(-t-s)\sigma dW_s, \end{aligned} \tag{6}$$

which together with the definition of θ implies that

$$\begin{aligned}\psi(-t,\theta_t\omega,x) &= \Psi(-t)x + \int_0^{-t}\Psi(-t-s)f\big(\psi(s,\theta_t\omega,x)\big)ds \\ &\quad + \int_0^{-t}\Psi(-t-s)\sigma dW_s(\theta_t\omega) \\ &= \Psi(-t)x + \int_t^{0}\Psi(-\tau)f\big(\psi(\tau-t,\theta_t\omega,x)\big)d\tau \\ &\quad + \int_t^{0}\Psi(-\tau)\sigma dW_\tau \\ &= \Psi(-t)x - \int_0^{t}\Psi(-\tau)f\big(\psi(\tau-t,\theta_t\omega,x)\big)d\tau \\ &\quad - \int_0^{t}\Psi(-\tau)\sigma dW_\tau, \qquad t \geq 0. \end{aligned} \tag{7}$$

In the remainder of this section, motivated by the recent work [5], we need to give a key operator $\mathcal{L}$, which is defined by

$$[\mathcal{L}(g)](\omega) = -\int_0^{\infty}\Psi(-\tau)g(\theta_\tau\omega)d\tau - \int_0^{\infty}\Psi(-\tau)\sigma dW_\tau \tag{8}$$

for all $\omega \in \Omega$. Here, the random variable $g : \Omega \longmapsto \mathbb{R}^n$ is tempered with respect to the measure preserving flow θ, see Chueshov [2, p.23].

Remark 1. By a similar argument in [5], it is clear that the operator $\mathcal{L}$ is well defined and the pull back trajectories starting at the initial point $x \in \mathbb{R}^n$ $\{\psi(-t,\theta_t\omega, x) : t \geq 0\}$ is a bounded set in $\mathbb{R}^n$ for all $\omega \in \Omega$.

3. **Some inequalities with respect to the stochastic flow ψ.**

In this section, we shall consider the dynamical behavior of stochastic flow ψ and show some helpful lemmas to prove our main results. First, we start with a lemma for convenience.

Lemma 3.1 ([13, Lemma A.2])**.** *Assume that $(x_t)_{t\in\Lambda}$ is a net in a normed space X endowed with a solid, normal cone $X_+ \subseteq X$. Moreover, assume that the net converges to a single point $p_\infty \in X$, and that*

$$x_t^- := \inf\{x_\tau : \tau \geq t\} \quad and \quad x_t^+ := \sup\{x_\tau : \tau \geq t\}$$

exist for all $t \in \Lambda$. Then the nets $(x_t^-)_{t\in\Lambda}$ and $(x_t^+)_{t\in\Lambda}$ also converge to the point p_∞.

Lemma 3.2. *For all $t \geq 0$, define*

$$\xi_t^f(\omega) = \inf \overline{\{f(\psi(-\tau, \theta_\tau \omega, x)) : \tau \geq t\}} = \inf\{f(\psi(-\tau, \theta_\tau \omega, x)) : \tau \geq t\}$$

and

$$\eta_t^f(\omega) = \sup \overline{\{f(\psi(-\tau, \theta_\tau \omega, x)) : \tau \geq t\}} = \sup\{f(\psi(-\tau, \theta_\tau \omega, x)) : \tau \geq t\},$$

where $x \in \mathbb{R}^n$ and $\omega \in \Omega$. Here, the supremum (sup) *and infimum* (inf) *are the least upper bound and the greatest lower bound in $\mathbb{R}^n$, respectively. Then ξ_t^f and η_t^f are two $\mathscr{F}_+$-measurable random variables for all $t \geq 0$, where $\mathscr{F}_+ = \sigma\{\omega \mapsto W_t(\omega) : t \geq 0\}$ is the future σ-algebra.*

Proof. By definitions of the metric dynamical system θ and the future σ-algebra $\mathscr{F}_+$, it is obvious that the random variable $\psi(-\tau, \theta_\tau \cdot, x)$ is $\mathscr{F}_+$-measurable for all $\tau \geq 0$ and $x \in \mathbb{R}^n$. The rest of the proof can be followed by the same argument in [5, Proposition 3.2], we omit it here. The proof is complete. □

Lemma 3.3. *Assume that* (A1) *and* (A2) *hold. It follows that*

$$[\mathcal{L}(\overline{\lim}^\theta f(\psi))](\omega) \leq [\underline{\lim}_\theta \psi](\omega) \leq [\overline{\lim}^\theta \psi](\omega) \leq [\mathcal{L}(\underline{\lim}_\theta f(\psi))](\omega) \tag{9}$$

for all $\omega \in \Omega$. Here, $\leq$ represents $\leq_{\mathbb{R}^n_+}$,

$$[\underline{\lim}_\theta \psi](\omega) := \lim_{t\to\infty} \inf\{\psi(-\tau, \theta_\tau \omega, x) : \tau \geq t\}, \quad x \in \mathbb{R}^n,$$

$$[\overline{\lim}^\theta \psi](\omega) := \lim_{t\to\infty} \sup\{\psi(-\tau, \theta_\tau \omega, x) : \tau \geq t\}, \quad x \in \mathbb{R}^n,$$

$$[\underline{\lim}_\theta f(\psi)](\omega) := \lim_{t\to\infty} \inf\{f\big(\psi(-\tau, \theta_\tau \omega, x)\big) : \tau \geq t\}, \quad x \in \mathbb{R}^n,$$

and

$$[\overline{\lim}^\theta f(\psi)](\omega) := \lim_{t\to\infty} \sup\{f\big(\psi(-\tau, \theta_\tau \omega, x)\big) : \tau \geq t\}, \quad x \in \mathbb{R}^n.$$

Proof. For convenience, we only show that

$$[\mathcal{L}(\overline{\lim}^\theta f(\psi))](\omega) \leq [\underline{\lim}_\theta \psi](\omega) \tag{10}$$

is correct, and other inequalities in (9) can be proceeded analogously. By (A2), Remark 1 and Lemma 3.2, we can easily have that $\underline{\lim}_\theta \psi$, $\overline{\lim}^\theta \psi$, $\underline{\lim}_\theta f(\psi)$ and $\overline{\lim}^\theta f(\psi)$ are well defined, which are all $\mathscr{F}_+$-measurable random variables. This together with Proposition 3.3 in [6] and Fubini's theorem gives that the same conclusion is true for $[\mathcal{L}(\overline{\lim}^\theta f(\psi))]$ and $[\mathcal{L}(\underline{\lim}_\theta f(\psi))]$. In addition, using Lebesgue's dominated convergence theorem, it is evident that

$$[\mathcal{L}(\overline{\lim}^\theta f(\psi))](\omega) = [\mathcal{L}(\lim_{t\to\infty} \eta_t^f)](\omega) = \lim_{t\to\infty} [\mathcal{L}(\eta_t^f)](\omega).$$

Consequently, in order to prove the inequality (10), it only remains to verify

$$[\mathcal{L}(\eta_t^f)](\omega) \leq [\underline{\lim}_\theta \psi](\omega) \tag{11}$$

for all $t \geq 0$ and $\omega \in \Omega$. To see this, we first observe that f is bounded, it follows that there exists a positive vector $b = (b_1, \ldots, b_n)^{\mathrm{T}} \in \mathrm{int}\mathbb{R}^n_+$ such that $f(x) \in [-b, b]$

for all $x \in \mathbb{R}^n$, where $[-b, b]$ is a conic interval. Furthermore, by the definition of the operator $\mathcal{L}$, for all $t \geq 0$ and $\omega \in \Omega$, we have that

$$
\begin{aligned}
&[\mathcal{L}(\eta_t^f)](\omega)\\
&= -\int_0^\infty \Psi(-s)\sup\{f\big(\psi(-\tau,\theta_\tau \bullet, x)\big) + b : \tau \geq t\}(\theta_s\omega) ds\\
&\quad + \int_0^\infty \Psi(-s) b ds - \int_0^\infty \Psi(-s)\sigma dW_s\\
&= -\int_0^\infty \Psi(-s)\sup\{f\big(\psi(-\tau,\theta_{\tau+s}\omega, x)\big) + b : \tau \geq t\} ds\\
&\quad + \int_0^\infty \Psi(-s) b ds - \int_0^\infty \Psi(-s)\sigma dW_s\\
&= \lim_{\substack{\tilde{t}\to\infty\\ \tilde{t}\geq t}} \Bigg\{ -\int_0^{\tilde{t}-t} \Psi(-s)\sup\{f\big(\psi(-\tau,\theta_{\tau+s}\omega, x)\big) + b : \tau \geq t\} ds\\
&\quad + \int_0^{\tilde{t}} \Psi(-s) b ds - \int_0^{\tilde{t}} \Psi(-s)\sigma dW_s + \Psi(-\tilde{t})x \Bigg\}\\
&= \lim_{\substack{\tilde{\tilde{t}}\to\infty\\ \tilde{\tilde{t}}\geq t}} \inf \Bigg\{ -\int_0^{\tilde{t}-t} \Psi(-s)\sup\{f\big(\psi(-\tau,\theta_{\tau+s}\omega, x)\big) + b : \tau \geq t\} ds\\
&\quad + \int_0^{\tilde{t}} \Psi(-s) b ds - \int_0^{\tilde{t}} \Psi(-s)\sigma dW_s + \Psi(-\tilde{t})x : \tilde{t} \geq \tilde{\tilde{t}} \Bigg\} \text{ (Lemma 3.1)}\\
&\leq \lim_{\substack{\tilde{\tilde{t}}\to\infty\\ \tilde{\tilde{t}}\geq t}} \inf \Bigg\{ -\int_0^{\tilde{t}-t} \Psi(-s)[f\big(\psi(-\tilde{t}+s,\theta_{\tilde{t}}\omega, x)\big) + b] ds\\
&\quad + \int_0^{\tilde{t}} \Psi(-s) b ds - \int_0^{\tilde{t}} \Psi(-s)\sigma dW_s + \Psi(-\tilde{t})x : \tilde{t} \geq \tilde{\tilde{t}} \Bigg\} \qquad (12)\\
&\leq \lim_{\substack{\tilde{\tilde{t}}\to\infty\\ \tilde{\tilde{t}}\geq t}} \inf \Bigg\{ -\int_0^{\tilde{t}} \Psi(-s)[f\big(\psi(-\tilde{t}+s,\theta_{\tilde{t}}\omega, x)\big) + b] ds\\
&\quad + \int_0^{\tilde{t}} \Psi(-s) b ds - \int_0^{\tilde{t}} \Psi(-s)\sigma dW_s + \Psi(-\tilde{t})x : \tilde{t} \geq \tilde{\tilde{t}} \Bigg\}\\
&\quad + \underbrace{\lim_{\substack{\tilde{\tilde{t}}\to\infty\\ \tilde{\tilde{t}}\geq t}} \sup \Bigg\{ \int_{\tilde{t}-t}^{\tilde{t}} \Psi(-s)[f\big(\psi(-\tilde{t}+s,\theta_{\tilde{t}}\omega, x)\big) + b] ds : \tilde{t} \geq \tilde{\tilde{t}} \Bigg\}}_{\text{Lemma 3.1 and Remark 2.1 imply that this equation is equal to 0}}\\
&= \lim_{\tilde{\tilde{t}}\to\infty} \inf \Big\{ \psi(-\tilde{t},\theta_{\tilde{t}}\omega, x) : \tilde{t} \geq \tilde{\tilde{t}} \Big\}\\
&= [\underline{\lim}_\theta \psi](\omega),
\end{aligned}
$$

where the inequality (3.4) is due to that A is a competitive matrix, which yields that $\Psi(-t)x = \exp(-At)x \geq_{\mathbb{R}^n_+} \Psi(-t)y$ for all $x \geq_{\mathbb{R}^n_+} y$ and $t \geq 0$, i.e., $\Psi(-t)$ is order preserving. The proof is complete. □

Remark 2. In this lemma, we do not assume that f is positive, i.e., $f : \mathbb{R}^n \to \mathbb{R}^n_+$, which is weaker than that in [5].

Lemma 3.4. *Assume that* (A1) *and* (A2) *hold. It follows that*

(i) *If f is monotone, we have that for all $\omega \in \Omega$,*

$$[f(\underline{\lim}_\theta \psi)](\omega) \le [\underline{\lim}_\theta f(\psi)](\omega) \le [\overline{\lim}^\theta f(\psi)](\omega) \le [f(\overline{\lim}^\theta \psi)](\omega). \tag{13}$$

(i) *If f is anti-monotone, we have that for all $\omega \in \Omega$,*

$$[f(\overline{\lim}^\theta \psi)](\omega) \le [\underline{\lim}_\theta f(\psi)](\omega) \le [\overline{\lim}^\theta f(\psi)](\omega) \le [f(\underline{\lim}_\theta \psi)](\omega). \tag{14}$$

Proof. The proof is similar in spirit to Lemma 3.4 in [5], we omit it here. The proof is complete. □

Lemma 3.5. *Assume that* (A1) *and* (A2) *hold. Let $\mathcal{L}^f = f \circ \mathcal{L}$, it follows that*

(i) *If f is monotone, then for all $t \ge 0$, $\omega \in \Omega$ and $k \in \mathbb{N}$,*

$$[(\mathcal{L}^f)^{2k}(\xi_t^f)](\omega) \le [\underline{\lim}_\theta f(\psi)](\omega) \le [\overline{\lim}^\theta f(\psi)](\omega) \le [(\mathcal{L}^f)^{2k}(\eta_t^f)](\omega). \tag{15}$$

(i) *If f is anti-monotone, then for all $t \ge 0$, $\omega \in \Omega$ and $k \in \mathbb{N}$,*

$$[(\mathcal{L}^f)^{k}(\xi_t^f)](\omega) \le [\underline{\lim}_\theta f(\psi)](\omega) \le [\overline{\lim}^\theta f(\psi)](\omega) \le [(\mathcal{L}^f)^{k}(\eta_t^f)](\omega). \tag{16}$$

Proof. By Lemma 3.2, it is immediate that

$$\xi_t^f(\omega) \le [\underline{\lim}_\theta f(\psi)](\omega) \le [\overline{\lim}^\theta f(\psi)](\omega) \le \eta_t^f(\omega)$$

for all $t \ge 0$ and $\omega \in \Omega$. In addition, since A is a competitive matrix, which together with (8) shows that $\mathcal{L}$ is anti-monotone with respect to the tempered random variable g. Therefore, we can easily see that

$$[\mathcal{L}(\xi_t^f)](\omega) \ge [\mathcal{L}(\underline{\lim}_\theta f(\psi))](\omega) \ge [\mathcal{L}(\overline{\lim}^\theta f(\psi))](\omega) \ge [\mathcal{L}(\eta_t^f)](\omega)$$

This together with Lemma 3.3 yields that

$$[\mathcal{L}(\xi_t^f)](\omega) \ge [\overline{\lim}^\theta \psi](\omega) \ge [\underline{\lim}_\theta \psi](\omega) \ge [\mathcal{L}(\eta_t^f)](\omega).$$

The rest of the proof can be followed in much the same way as Lemma 3.5 in our previous paper [5], so we omit it here. The proof is complete. □

Lemma 3.6. *Assume that $\frac{nL}{\lambda} < 1$,* (A1) *and* (A2) *hold. In addition, we define $\mathcal{M}_{\mathscr{F}_+} := \mathcal{M}_{\mathscr{F}_+}(\Omega; [-b, b])$ to be the space of all $\mathscr{F}_+$-measurable functions $g : \Omega \to [-b, b]$. Here, $b = (b_1, \ldots, b_n)^{\mathrm{T}}$ is a positive vector in $\mathrm{int}\mathbb{R}^n_+$ such that $f(x) \in [-b, b]$ for all $x \in \mathbb{R}^n$, where $[-b, b]$ is a conic interval. Moreover, we consider a metric on $\mathcal{M}_{\mathscr{F}_+}(\Omega; [-b, b])$ as follows:*

$$d(g_1, g_2) := |g_1 - g_2|_\infty = \sup_{\omega \in \Omega} |g_1(\omega) - g_2(\omega)|,$$

where $g_1, g_2 \in \mathcal{M}_{\mathscr{F}_+}(\Omega; [-b, b])$. Then, we conclude that $(\mathcal{M}_{\mathscr{F}_+}, d)$ is a complete metric space and the operator $\mathcal{L}^f := f \circ \mathcal{L} : \mathcal{M}_{\mathscr{F}_+} \to \mathcal{M}_{\mathscr{F}_+}$ is a contraction mapping.

Proof. First, it is easy to check that $(\mathcal{M}_{\mathscr{F}_+}, d)$ is a complete metric space. In order to get the result, it is necessary to verify the well-posedness of the operator $\mathcal{L}^f$. For any given $g \in \mathcal{M}_{\mathscr{F}_+}$, using Proposition 3.3 in [6], we see at once that $g(\theta_\tau \omega)$ is $(\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F}_+, \mathscr{B}(\mathbb{R}^n))$-measurable. Combining this and Fubini's theorem, we immediately have that $\mathcal{L}^f$ is well defined.

Now, we proceed to show that the operator $\mathcal{L}^f$ is contracted. Given any $g_1, g_2 \in \mathcal{M}_{\mathscr{F}_+}$, note that $|\Psi x| \leq n|\Psi| \cdot |x|$ for all $x \in \mathbb{R}^n$ and $\Psi \in \mathbb{R}^{n\times n}$, it follows that

$$\begin{aligned}
&\left|\mathcal{L}^f(g_1) - \mathcal{L}^f(g_2)\right|_\infty \\
&= \sup_{\omega\in\Omega} \left|\left[f\left(\mathcal{L}(g_1)\right)\right](\omega) - \left[f\left(\mathcal{L}(g_2)\right)\right](\omega)\right| \\
&\leq L \sup_{\omega\in\Omega} \left|\left[\mathcal{L}(g_1)\right](\omega) - \left[\mathcal{L}(g_2)\right](\omega)\right| \\
&= L \sup_{\omega\in\Omega} \left|\int_0^\infty \Psi(-\tau) g_2(\theta_\tau\omega) d\tau - \int_0^\infty \Psi(-\tau) g_1(\theta_\tau\omega) d\tau\right| \\
&\leq nL \int_0^\infty \|\Psi(-\tau)\| \cdot |g_1 - g_2|_\infty d\tau \\
&\leq nL \int_0^\infty e^{-\lambda\tau} d\tau \cdot |g_1 - g_2|_\infty \\
&= \frac{nL}{\lambda}|g_1 - g_2|_\infty, \quad \text{where } \frac{nL}{\lambda} < 1.
\end{aligned}$$

The proof is complete. □

4. Main results.

In this section, we will state our main results as the following theorem.

Theorem 4.1. *Assume that all assumptions in Lemma 3.6 hold, then we have that there exists a unique fixed point $g \in \mathcal{M}_{\mathscr{F}_+}$ for the contraction mapping $\mathcal{L}^f : \mathcal{M}_{\mathscr{F}_+} \to \mathcal{M}_{\mathscr{F}_+}$ satisfying*

$$\lim_{t\to\infty} \psi(-t, \theta_t\omega, x) = [\mathcal{L}(g)](\omega) \tag{17}$$

for all $x \in \mathbb{R}^n$ and $\omega \in \Omega$. In addition, $\psi(t, \omega, [\mathcal{L}(g)](\omega)) = [\mathcal{L}(g)](\theta_t\omega)$ for all $t \in \mathbb{R}$. This gives that $[\mathcal{L}(g)](\omega)$ is an unstable $\mathscr{F}_+$-measurable random equilibrium for the stochastic flow ψ, which generates an unstable stationary solution $[\mathcal{L}(g)](\theta_t\omega)$ for (2).

Proof. By Lemma 3.5 and Lemma 3.6, analysis similar to that in the proof of Theorem 4.2 in [5] shows that (17) holds, which together with the definition of the cocycle ψ yields that $[\mathcal{L}(g)](\omega)$ is an unstable $\mathscr{F}_+$-measurable random equilibrium. The proof is complete. □

5. Applications to stochastic neural networks.

In this section, we will present some applications of Theorem 4.1. First, we consider the following stochastic model with additive white noise

$$dx_i(t) = \left[(Ax)_i(t) + \sum_{j=1}^n T_{ij} f_{ij}(x_j(t))\right] dt + \sigma_i dW_i(t), \quad i = 1, 2, \ldots, n, \tag{18}$$

which can describe the dynamical behavior of a neural network with n neurons under stochastic noise perturbations. Here, the matrix $T = (T_{ij})_{n\times n}$ shows the connection strengths between neurons, the transfer function f_{ij} is assumed to be sigmoid and $\sigma_i W_i$ is the turbulent noise in the external environment. If we ignore the noise in (18), which has been investigated by Hopfield [3, 4]. Next, define $g_{ij} = T_{ij} \cdot f_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$, we assume that g_{ij} satisfies the condition that

(B1) $g_{ij}: \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous with the Lipschitz constant $L_{ij} \geq 0$, monotone (or anti-monotone). In addition, there exist some positive constants b_{ij} such that $|g_{ij}(x)| \leq b_{ij}$ for all $x \in \mathbb{R}$ and $i, j \in \{1, 2, \dots, n\}$.

(B2)
$$\frac{n \max_{1\leq i\leq n}(\sum_{j=1}^{n} L_{ij})}{\lambda} < 1,$$
where λ is defined in the assumption (A1).

It is clear that there are some functions satisfying (B1) and (B2), such as $\arctan(x)$ and $\tanh(x)$, see [16, 17]. In order to use Lemma 3.6 and Theorem 4.1, let $b_i = \sum_{j=1}^{n} b_{ij}$ for all $i \in \{1, 2, \dots, n\}$ and $L = \max_{1\leq i\leq n}(\sum_{j=1}^{n} L_{ij})$, it follows that

Corollary 1. *Assume that* (A1), (B1) *and* (B2) *hold, then we have that there exists an unstable stationary solution* (*random equilibrium*) *for* (18).

From now on, for simplicity of notation, we only discuss the case that $n = 3$.

Example 5.1. First, we consider the following stochastic model
$$dx_i(t) = \left[(Ax)_i(t) + \frac{1}{3}\arctan(x_{i-1}(t))\right]dt + \sigma_i dW_i(t), \quad i = 1, 2, 3, \tag{19}$$
with the initial value $x(0) = x \in \mathbb{R}^3$, where
$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & -2 & 4 \end{bmatrix}, \tag{20}$$
$x = (x_1, x_2, x_3)^{\mathrm{T}} \in \mathbb{R}^3$ and x_0 represents x_3. By (20), it is easy to see that the eigenvalues of $-A$ are $\lambda_1 = -4, \lambda_{2,3} = \frac{-5\pm\sqrt{5}}{2}$, then we have that
$$\Psi(-t) = \exp(-At) = \begin{bmatrix} \frac{5+\sqrt{5}}{10}\mathrm{e}^{\frac{-5+\sqrt{5}}{2}t} + \frac{5-\sqrt{5}}{10}\mathrm{e}^{\frac{-5-\sqrt{5}}{2}t} & \frac{\sqrt{5}}{5}\mathrm{e}^{\frac{-5+\sqrt{5}}{2}t} - \frac{\sqrt{5}}{5}\mathrm{e}^{\frac{-5-\sqrt{5}}{2}t} & 0 \\ \frac{\sqrt{5}}{5}\mathrm{e}^{\frac{-5+\sqrt{5}}{2}t} - \frac{\sqrt{5}}{5}\mathrm{e}^{\frac{-5-\sqrt{5}}{2}t} & \frac{5-\sqrt{5}}{10}\mathrm{e}^{\frac{-5+\sqrt{5}}{2}t} + \frac{5+\sqrt{5}}{10}\mathrm{e}^{\frac{-5-\sqrt{5}}{2}t} & 0 \\ 2\mathrm{e}^{-4t} + \frac{-5+3\sqrt{5}}{5}\mathrm{e}^{\frac{-5+\sqrt{5}}{2}t} - \frac{5+3\sqrt{5}}{5}\mathrm{e}^{\frac{-5-\sqrt{5}}{2}t} & -4\mathrm{e}^{-4t} + \frac{10-4\sqrt{5}}{5}\mathrm{e}^{\frac{-5+\sqrt{5}}{2}t} + \frac{10+4\sqrt{5}}{5}\mathrm{e}^{\frac{-5-\sqrt{5}}{2}t} & \mathrm{e}^{-4t} \end{bmatrix},$$
where $t \geq 0$. In order to get that
$$\|\Psi(-t)\| := \max\{|\Psi_{ij}(-t)| : i, j = 1, 2, 3\} \leq \mathrm{e}^{\frac{-5+\sqrt{5}}{2}t} = \mathrm{e}^{\lambda_2 t} \tag{21}$$
for all $t \geq 0$, it is sufficient to show that $|\Psi_{31}(-t)| \leq \mathrm{e}^{\frac{-5+\sqrt{5}}{2}t}$ and $|\Psi_{32}(-t)| \leq \mathrm{e}^{\frac{-5+\sqrt{5}}{2}t}$ for all $t \geq 0$. Proofs of the remainder components of $\Psi(-t)$ are obvious. Since the matrix A is competitive, which implies that $\Psi_{ij}(-t) \geq 0$ for all $t \geq 0$ and $i, j = 1, 2, 3$, see Proposition 3.1.1 in [15]. This gives that we only need to show that
$$\Gamma_{31}(t) := \frac{\Psi_{31}(-t)}{\mathrm{e}^{\frac{-5+\sqrt{5}}{2}t}} \leq 1 \quad \text{and} \quad \Gamma_{32}(t) := \frac{\Psi_{32}(-t)}{\mathrm{e}^{\frac{-5+\sqrt{5}}{2}t}} \leq 1 \tag{22}$$
for all $t \geq 0$. It is easily seen that
$$\Gamma_{31}(t) = 2\mathrm{e}^{\frac{-3-\sqrt{5}}{2}t} + \frac{-5+3\sqrt{5}}{5} - \frac{5+3\sqrt{5}}{5}\mathrm{e}^{-\sqrt{5}t},$$

and

$$\frac{\mathrm{d}\Gamma_{31}(t)}{\mathrm{d}t} = -(3+\sqrt{5})\mathrm{e}^{\frac{-3-\sqrt{5}}{2}t} + (3+\sqrt{5})\mathrm{e}^{-\sqrt{5}t} \geq 0$$

for all $t \geq 0$. In addition, we note that $\Gamma_{31}(0) = 0$ and $\lim_{t\to\infty} \Gamma_{31}(t) = \frac{-5+3\sqrt{5}}{5} < 1$, it is immediate that $\Gamma_{31}(t) \leq 1$ for all $t \geq 0$. By a similar argument, we get that

$$\Gamma_{32}(t) = -4\mathrm{e}^{\frac{-3-\sqrt{5}}{2}t} + \frac{10-4\sqrt{5}}{5} + \frac{10+4\sqrt{5}}{5}\mathrm{e}^{-\sqrt{5}t},$$

and

$$\frac{\mathrm{d}\Gamma_{32}(t)}{\mathrm{d}t} = 2(3+\sqrt{5})\mathrm{e}^{\frac{-3-\sqrt{5}}{2}t} - (4+2\sqrt{5})\mathrm{e}^{-\sqrt{5}t}$$

for all $t \geq 0$. Note that $\Gamma'_{32}(0) = 2 > 0$, it is obvious that there exists a critical point $t_0 > 0$ such that $\Gamma_{32}(t_0)$ is the biggest value of $\Gamma_{32}(t)$ for all $t \geq 0$. Furthermore, direct computation shows that $\frac{2}{5} < t_0 < \frac{3}{5}$, which implies that

$$\Gamma_{32}(t_0) \leq -4\mathrm{e}^{\frac{-3-\sqrt{5}}{2}\cdot\frac{3}{5}} + \frac{10-4\sqrt{5}}{5} + \frac{10+4\sqrt{5}}{5}\mathrm{e}^{-\sqrt{5}\cdot\frac{2}{5}} \approx 0.928689 < 1.$$

That is, (22) holds and then (21) holds. Finally, set $L = \frac{1}{3}$, $\lambda = \frac{5-\sqrt{5}}{2}$, it is evident that

$$\frac{3\max_{1\leq i\leq 3}(\sum_{j=1}^{3} L_{ij})}{\lambda} = \frac{3L}{\lambda} = \frac{2}{5-\sqrt{5}} < 1.$$

Therefore, by Corollary 1, we get that there exists an unstable stationary solution (random equilibrium) for stochastic system (19), which is the α-limit set of all pull-back trajectories starting at any initial value $x(0) = x \in \mathbb{R}^3$.

Example 5.2. Next, we consider the following stochastic model

$$dx_i(t) = \left[(Ax)_i(t) + \frac{1}{4+\tanh(x_{i-1}(t))}\right]dt + \sigma_i dW_i(t), \quad i = 1,2,3, \tag{23}$$

with the initial value $x(0) = x \in \mathbb{R}^3$, where $\tanh(u) = \frac{e^u - e^{-u}}{e^u + e^{-u}}$ for all $u \in \mathbb{R}$,

$$A = \begin{bmatrix} 1 & -\sqrt[3]{2} & 0 \\ 0 & 2 & -\sqrt[3]{2} \\ -\sqrt[3]{2} & 0 & 4 \end{bmatrix}, \tag{24}$$

$x = (x_1, x_2, x_3)^{\mathrm{T}} \in \mathbb{R}^3$ and x_0 represents x_3. Following the same method as in [5], we can easily get that the eigenvalues of $-A$ are $\lambda_1 = -3$, $\lambda_{2,3} = -2 \pm \sqrt{2}$ and

$$\|\Psi(-t)\| := \max\{|\Psi_{ij}(-t)| : i,j = 1,2,3\} \leq \mathrm{e}^{(-2+\sqrt{2})t} = \mathrm{e}^{\lambda_2 t}.$$

Moreover, note that

$$-1 \leq \tanh(y) \leq 1 \quad \text{and} \quad 0 \leq \frac{\mathrm{d}}{\mathrm{d}y}(\tanh(y)) \leq 1$$

for all $y \in \mathbb{R}$, set $L = \frac{1}{9}$, $\lambda = 2 - \sqrt{2}$, we can easily see that

$$\frac{3\max_{1\leq i\leq 3}(\sum_{j=1}^{3} L_{ij})}{\lambda} = \frac{3L}{\lambda} = \frac{1}{3(2-\sqrt{2})} < 1.$$

Hence, using Corollary 1, (23) admits an unstable stationary solution (random equilibrium).

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