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A SMALL-GAIN THEOREM FOR NONLINEAR STOCHASTIC SYSTEMS WITH INPUTS AND OUTPUTS I: ADDITIVE WHITE NOISE*

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Abstract. This paper studies a small-gain theorem for nonlinear stochastic equations driven by additive white noise in both trajectories and stationary distribution. Motivated by the most recent work of Marcondes de Freitas and Sontag [*SIAM J. Control Optim.*, 53 (2015), pp. 2657–2695], we first define the *input-to-state characteristic operator* $\mathcal{K}(u)$ of the system in a suitably chosen input space via a backward Itô integral, and then for a given output function h we define the *gain operator* as the composition of output function h and the input-to-state characteristic operator $\mathcal{K}(u)$ on the input space. Suppose that the output function is either order-preserving or anti-order-preserving in the usual vector order and the global Lipschitz constant of the output function is less than the absolute of the negative principal eigenvalue of the linear matrix. Then we prove the so-called *small-gain theorem*: the gain operator has a unique fixed point; the image for the input-to-state characteristic operator at the fixed point is a globally attracting stochastic equilibrium for the random dynamical system generated by the stochastic system. Under the same assumption for the relation between the Lipschitz constant of the output function and the maximal real part of the stable linear matrix, we prove that the stochastic system has a unique stationary distribution, which is regarded as a stationary distribution version of the small-gain theorem. These results can be applied to stochastic cooperative, competitive, and predator-prey systems, or even others.

Key words. stochastic control systems, input and output, small-gain theorem, random dynamical systems, stochastic equilibrium, global stability

AMS subject classifications. 93E03, 93E15, 93C10, 60H10, 37H10

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1. Introduction.

The present paper is concerned with the classical small-gain theorem, which was first proved by Zames [37] in 1966. It has been widely used as a powerful tool to investigate the robust stability of interconnected control systems. More precisely, in [37], a sufficient condition for input-output stability of a feedback loop is that the system gain is smaller than one. It is noticed that in the original research, most of them discuss systems with linear finite gain from input to output. For the case of nonlinear gain functions, it was first considered by Hill [15] and Mareels and Hill [27], where the notions of monotone gain were proposed and a nonlinear version of the small-gain theorem was developed. Since then, more and more researchers have focused on extending the small-gain theorem to nonlinear feedback systems. In [33], Sontag introduced the concept of input-to-state stability, which was developed by Jiang, Teel, and Praly in [20], and a nonlinear ISS-type

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small-gain theorem was obtained. Inspired by the excellent works [20, 33], many other nonlinear small-gain theorems have been extensively and intensively studied (see [1, 2, 7, 11, 18, 19, 25, 27, 28, 33, 34, 35]), which led to new applications in the design and analysis of nonlinear control systems [23, 24] and various systems in mathematical biology for the robust stability of the feedback connection.

So far, there has been a well-developed theory of the feedback analysis for deterministic systems. However, several of the small-gain theorems mentioned above are highly undesirable in the application of interconnections and stabilization for more realistic models with noise disturbances, which may arise from its surrounding environmental perturbations, measurement errors, or intrinsic uncertainties of a coupling system due to high complexity. Due to the limitations of the deterministic control theory, it is natural to investigate the stochastic nonlinear control, which has been a research topic in recent years; see [8, 9, 10, 31]. Recently, Marcondes de Freitas and Sontag [12, 13, 14] have initiated the study of random dynamical systems (RDSs) with inputs and outputs whose parameters are perturbed by so-called *real noise*, which generates a metric dynamical system if it is stationary and helps one to solve random differential equations pathwisely (see [3, p. 57]). Their approach is divided into two steps: the first is for a given *stationary* input u, to define the so-called *input-to-state characteristic operator* $\mathcal{K}(u)$; and the second is for a given output function, to define *gain operator* $\mathcal{K}^h(u)$ by the composition of the output function and the input-to-state characteristic operator $\mathcal{K}(u)$ on the space of stationary inputs. Then they transformed the problem of obtaining a small-gain theorem into the existence of a fixed point for the gain operator. If it does exist in a manner of the Banach fixed point theorem, then the image for input-to-state characteristic operator $\mathcal{K}(\cdot)$ at the fixed point is a random equilibrium, which is globally asymptotically stable for interconnected random systems. Such a small-gain theorem can be applied to random competitive systems, which is new as far as the authors know.

It is well known that control of stochastic differential equations, which cannot be pathwise differential equations, is a classical field (see [3, p. 68]). Motivated by the work of Marcondes de Freitas and Sontag [14], the main purpose of this paper is to carry out Marcondes de Freitas and Sontag's idea developed in RDSs with inputs and outputs accompanying real noise parameters in stochastic control of interconnected control systems so that we can provide small-gain results for nonlinear stochastic systems driven by additive white noise in both trajectories and stationary distribution. First, we consider a linear stochastic system driven by additive white noise with inputs and define the *input-to-state characteristic operator* $\mathcal{K}(u)$ of the system in a suitably chosen input space via a backward Itô integral. For a given output function h, we then define the *gain operator* as the composition of output function h and the input-to-state characteristic operator $\mathcal{K}(u)$ on the input space. Suppose that the output function is either order-preserving or anti-order-preserving in the usual vector order and the global Lipschitz constant of the output function is less than the absolute of the negative principal eigenvalue of the linear matrix. Then we prove the so-called *small-gain theorem*: the gain operator has a unique fixed point u by the Banach fixed point theorem, and the input-to-state characteristic operator $\mathcal{K}(u)$ at the fixed point u is a unique, globally attracting stochastic equilibrium for the RDS generated by the stochastic system. Under the same assumption for the relation between the Lipschitz constant of the output function and the maximal real part of the stable linear matrix, we prove that the stochastic system has a unique stationary distribution by the Khasminskii theorem [22], which may be regarded as a stationary distribution version of the small-gain theorem. Our result can be applied to stochastic cooperative, competitive, and predator-prey systems, or even others with additive white noise to obtain a globally stable stochastic equilibrium, which, illustrated in Examples 5.1–5.3, are new to us. What we will process is much more concise than that of Marcondes de Freitas and Sontag in [14].

The rest of this paper is organized as follows. In section 2, we will formulate the discussed problem, review some preliminary concepts and definitions, introduce some notation and known results needed in the subsequent content, and define the input-to-state characteristic operator. Section 3 describes the asymptotic behavior of stochastic solutions, establishes some auxiliary lemmas, and presents the definition of gain operator and its properties. In section 4, a stochastic small-gain theorem is proved and the global convergence to a unique stochastic equilibrium is presented. Section 5 presents three examples utilizing our results for stochastic cooperative, competitive, and predator-prey systems, respectively. In section 6, we summarize the small-gain theorem in trajectories and discuss proving a small-gain theorem in stationary solution or measure.

2. Formulation of the problem and preliminaries.

We start with a biochemical model which contains three chemical species X_1, X_2, and X_3 interacting with each other, as shown in Figure 1. In the study of molecular biology, biochemical reaction systems are usually modeled by ordinary or partial differential equations. Furthermore, we notice that the nonlinear dynamics of biochemical reaction networks on the order of a cell is stochastic. Let us consider a nonlinear stochastic control system as shown in Figure 1, which consists of three elements in a feedback loop as follows:

$$dx_i = [a_i x_i + h_i(x_{i-1})]dt + \sigma_i dW_t^i, \qquad i = 1, 2, 3,$$

where $W_t(\omega) = \big(W_t^1(\omega), W_t^2(\omega), W_t^3(\omega)\big)$ is a three-dimensional standard Brownian motion with $W_0^i(\omega) = 0,\ i = 1, 2, 3, \omega \in \Omega$. Here and below, the indices of x are taken modulo 3. This type of system can be modeled in the study of finance, statistics, engineering, and multiagent systems, where a_1, a_2, and a_3 are negative constants and h_1, h_2, and h_3 are decreasing functions of feedback.

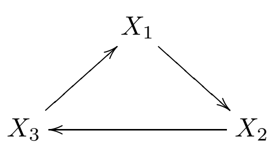

FIG. 1. *Biochemical circuit. The symbol "$X_i \to X_j$" means that species X_i represses the production of species X_j, $i, j = 1, 2, 3$.*

$$\xrightarrow{u_1} X_1 \xrightarrow{h_1(x_1)} \qquad \xrightarrow{u_2} X_2 \xrightarrow{h_2(x_2)} \qquad \xrightarrow{u_3} X_3 \xrightarrow{h_3(x_3)}$$

FIG. 2. *Decomposition of the feedback loop into input-output modules. In each partition, u_i indicates the input into the element X_i and $h_i(x_i)$ indicates the subsequent output—feedback of the current state.*

Motivated by [14], we can open up the closed loop as shown in Figure 2, rewriting the model as a stochastic system with inputs

$$dx_i = [a_i x_i + u_i]dt + \sigma_i dW_t^i, \qquad i = 1, 2, 3,$$

together with outputs

$$u_i(t) = h_i(x_{i-1}(t)), \qquad i = 1, 2, 3.$$

In view of the above analysis, we can rewrite the form of stochastic solutions by the variation-of-constants formula of stochastic differential equations (SDEs), which is very important in the present work.

In general, we consider the following nonlinear stochastic system in this paper:

$$dX_t = [AX_t + h(X_t)]dt + \sigma dW_t, \tag{2.1}$$

where $W_t(\omega) = \big(W_t^1(\omega), \dots, W_t^m(\omega)\big)$ is a two-sided time Wiener process with values in $\mathbb{R}^m$ on the canonical Wiener space $(\Omega, \mathscr{F}, \mathbb{P})$, i.e., $\mathscr{F}$ is the Borel σ-algebra of $\Omega = C_0(\mathbb{R}, \mathbb{R}^m) = \{\omega(t) \text{ continuous}, \ \omega(0) = 0, \ t \in \mathbb{R}\}$; $\mathbb{P}$ is the Wiener measure, $A = (a_{ij})_{d\times d}$ is a $(d \times d)$-dimensional matrix, $h : \mathbb{R}^d \to \mathbb{R}_+^d$, and $\sigma = (\sigma_{ij})_{d\times m}$ is a $(d \times m)$-dimensional matrix.

Let us first present the assumption for linear matrix A. Consider the corresponding linear ordinary differential equations:

$$dX_t = AX_t dt. \tag{2.2}$$

Denote by $\Phi_j(t) = (\Phi_{1j}(t), \dots, \Phi_{dj}(t))^T$ the solution of equations (2.2) with initial value $x(0) = e_j$, $j = 1, \dots, d$. Define the $d \times d$ matrix

$$\Phi(t) = (\Phi_1(t), \dots, \Phi_d(t)) = (\Phi_{ij}(t))_{d\times d}.$$

Then $\Phi(t)$ is the fundamental matrix of equations (2.2) and $\Phi(t + s) = \Phi(t) \circ \Phi(s)$ for all $t, s \geq 0$. Assume now that A is stable in the sense that all real parts of its eigenvalues are negative:

$$\text{Re}\ \mu \leq \lambda < 0 \quad \text{for all eigenvalues } \mu \text{ of } A. \tag{2.3}$$

It follows from [30, Chapter 2, Proposition 2.10] that there is a basis of $\mathbb{R}^d$ such that the norm $|\cdot|$ satisfies that

$$\ll Ax, x \gg \leq \ \lambda \mid x \mid^2 \quad \text{ for all } x \in \mathbb{R}^d \text{ and}$$
$$\|\Phi(t)\| \leq e^{\lambda t}, \ t \geq 0.$$

Now we propose the assumptions on A as follows.

(A) A is *cooperative* in the sense that $a_{ij} \geq 0$ for all $i, j \in \{1, \dots, d\}$ and $i \neq j$, which is stable such that (2.3) and

$$\|\Phi(t)\| := \max\{|\Phi_{ij}(t)| : i, j = 1, \dots, d\} \leq e^{\lambda t}, \ t \geq 0, \tag{2.4}$$

hold. In this paper, we use the norm $|x| := \max\{|x_i| : i = 1, \dots, d\}$.

Concerning the existence and uniqueness of stochastic solutions and the stability of SDEs (2.1), we make the following hypotheses:

(H_1) $h \in C_b^1(\mathbb{R}^d, \mathbb{R}_+^d)$, i.e., the function h and its derivatives are both bounded, and h is order-preserving, i.e.,

$$h(x_1) \leq_{\mathbb{R}_+^d} h(x_2) \text{ whenever } x_1 \leq_{\mathbb{R}_+^d} x_2,$$

or anti-order-preserving, i.e.,

$$h(x_1) \geq_{\mathbb{R}_+^d} h(x_2) \text{ whenever } x_1 \leq_{\mathbb{R}_+^d} x_2.$$

Here, $x \leq_{\mathbb{R}_+^d} y$ means that $y - x \in \mathbb{R}_+^d$ for all $x, y \in \mathbb{R}^d$.

(H_2) Let $L = \max\{\sup_{x\in\mathbb{R}^d} |\frac{\partial h_i(x)}{\partial x_j}|, i, j = 1, \dots, d\}$ such that $-\frac{Ld^2}{\lambda} < 1$.

In cellular neural networks, h can be regarded as input-output sigmoid characteristics with identical neurons; see [4, 16]. By the fundamental theory of SDEs [26, 29], we can obtain the existence and uniqueness of solutions for equations (2.1) with the initial value $x(0) = x_0 \in \mathbb{R}^d$.

Let $\varphi(t,\omega)x = x(t,\omega,x)$ be the unique solution of equations (2.1). Then, using the variation-of-constants formula [26, Theorem 3.1], we have

$$\begin{aligned}\varphi(t,\omega)x &= \Phi(t)x + \Phi(t)\int_0^t \Phi^{-1}(s)h(\varphi(s,\omega)x)ds + \Phi(t)\int_0^t \Phi^{-1}(s)\sigma dW_s \\ &= \Phi(t)x + \int_0^t \Phi(t-s)h(\varphi(s,\omega)x)ds + \int_0^t \Phi(t-s)\sigma dW_s, \quad \mathbb{P}\text{-a.s.},\ t \geq 0. \qquad (2.5)\end{aligned}$$

Before stating our main results, we will introduce some basic concepts and notation related to the theory of random dynamical systems (RDSs), which can be found in [3, 5] for more details.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and (X, ϱ) be a separable complete metric space equipped with the Borel σ-algebra $\mathscr{B}(X)$.

Definition 2.1. *A family of transformations $\{\theta_t : \Omega \mapsto \Omega, t \in \mathbb{R}\}$ with a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called a metric dynamical system if*

1. *it is a one-parameter group, i.e.,*

$$\theta_0 = id, \quad \theta_t \circ \theta_s = \theta_{t+s} \quad \text{for all } t, s \in \mathbb{R};$$

2. *$(t,\omega) \mapsto \theta_t\omega$ is $(\mathscr{B}(\mathbb{R}) \otimes \mathscr{F}, \mathscr{F})$-measurable;*
3. *$\theta_t\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$, i.e., $\mathbb{P}(\theta_t B) = \mathbb{P}(B)$ for all $B \in \mathscr{F}$ and all $t \in \mathbb{R}$.*

Definition 2.2. *An RDS with the state space X over a metric dynamical system $\theta \equiv (\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t, t \in \mathbb{R}\})$ is a $(\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F} \otimes \mathscr{B}(X), \mathscr{B}(X))$-measurable mapping*

$$\varphi : \mathbb{R}_+ \times \Omega \times X \mapsto X, \quad (t,\omega,x) \mapsto \varphi(t,\omega,x)$$

such that

1. *$\varphi(t,\omega,\cdot) : X \to X$ is continuous for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$;*
2. *the mappings $\varphi(t,\omega) := \varphi(t,\omega,\cdot)$ form a cocycle over θ*

$$\varphi(0,\omega) = \text{id}, \quad \varphi(t+s,\omega) = \varphi(t,\theta_s\omega) \circ \varphi(s,\omega)$$

for all $t, s \in \mathbb{R}_+$ and $\omega \in \Omega$. Here $\circ$ means composition of mappings.

Definition 2.3. *A set-valued mapping $D : \Omega \to 2^X\backslash\{\varnothing\}$ is called a random set if, for every $x \in X$, the mapping $\omega \to \text{dist}_X(x, D(\omega))$ is measurable, where $\text{dist}_X(x, B)$ is the distance in X between the element x and the set $B \subset X$. A random set D is called a random closed (resp., compact) set if $D(\omega)$ is closed (resp., compact) in X for each $\omega \in \Omega$. A random set D is said to be bounded if there exist $x_0 \in X$ and a random variable $r(\omega) > 0$ such that*

$$D(\omega) \subset \{x \in X : \varrho(x, x_0) \leq r(\omega)\} \text{ for all } \omega \in \Omega.$$

Combining the theory of SDEs and RDSs, we can see that the solution of equations (2.1) generates an RDS (θ, φ) in $\mathbb{R}^d$; see [3, Chapter 2], [5, Chapter 2], where θ is connected with the Wiener process, i.e., $\theta_t\omega(\cdot) := \omega(t+\cdot) - \omega(t)$. Let $B^+ :=$

$\{B_t^+ | B_t^+ = W_t : t \geq 0\}$ and $B^- := \{B_t^- | B_t^- = W_{-t} : t \geq 0\}$. Then B^+ and B^- are two independent Brownian motions with one-sided time $\mathbb{R}_+$, which can be used to define the forward and backward Itô integrals (see Arnold [3, p. 97]) as follows.

For a continuous adapted process f of finite variation and $t > 0$, the *forward Itô integral* is defined by

$$\int_0^t f(s) d^+ W_s := \lim_{\Delta \to 0} \text{ in Pr.} \sum_{k=0}^{n-1} f(t_k)(B_{t_{k+1}}^+ - B_{t_k}^+)$$

for any partition $\Delta = \{0 = t_0 < t_1 < \cdots < t_n = t\}$; and for $t < 0$, the *backward Itô integral* is defined by

$$\int_t^0 f(s) d^- W_s := \lim_{\Delta \to 0} \text{ in Pr.} \sum_{k=0}^{n-1} f(t_{k+1})(B_{-t_{k+1}}^- - B_{-t_k}^-)$$

for any partition $\Delta = \{t = t_0 < t_1 < \cdots < t_n = 0\}$.

By definitions, we can prove that

$$\int_t^0 f(s) d^- W_s = -\int_0^{-t} f(-s) dB_s^- \quad \text{for any } t < 0, \tag{2.6}$$

$$\int_{-t}^0 f(-s) d^- W_s = \int_0^t f(t-s) d^+ W_s(\theta_{-t}\omega) \quad \text{for any } t > 0. \tag{2.7}$$

Now we are in position to introduce the concept of *input-to-state characteristic operator*. By the definition of θ, we can obtain the pull-back trajectories of solutions of (2.1) as follows:

(2.8)

$$\varphi(t, \theta_{-t}\omega)x = \Phi(t)x + \int_{-t}^0 \Phi(-s) h(\varphi(t+s, \theta_{-t}\omega)x) ds + \int_{-t}^0 \Phi(-s)\sigma dW_s, \ \mathbb{P}\text{-a.s.}, \ x \in \mathbb{R}^d,$$

where we have used (2.7) for the stochastic integral term. Returning the output function term to input function, we define the *input-to-state characteristic operator* $\mathcal{K}$:

$$[\mathcal{K}(u)](\omega) = \int_{-\infty}^0 \Phi(-s) u(\theta_s \omega) ds + \int_{-\infty}^0 \Phi(-s)\sigma dW_s, \quad \omega \in \Omega, \tag{2.9}$$

where u is a tempered random variable with respect to θ, i.e.,

$$\sup_{t \in \mathbb{R}} \left\{ e^{-\gamma |t|} \left| u(\theta_t \omega) \right|_2 \right\} < \infty \quad \text{for all } \omega \in \Omega \text{ and } \gamma > 0,$$

where $|x|_2 := (\sum_{i=1}^d |x_i|^2)^{\frac{1}{2}}$, $x \in \mathbb{R}^d$. We denote $\|\Phi\|_2 := (\sum_{i,j=1}^d |\Phi_{ij}|^2)^{\frac{1}{2}}$ in what follows, where Φ is a $(d \times d)$-dimensional matrix.

Remark 1. It is noticed that the operator $\mathcal{K}$ is well defined. In fact, for any tempered random variable u, since $\|\Phi(t)\| \leq e^{\lambda t}$, $t \geq 0$, we have $\|\Phi(t)\|_2 \leq d\|\Phi(t)\| \leq de^{\lambda t}$, and so

$$\begin{aligned}\left|\int_{-\infty}^{0}\Phi(-s)u(\theta_s\omega)ds\right|_2 &\leq \int_{-\infty}^{0}|\Phi(-s)u(\theta_s\omega)|_2\,ds \leq d\int_{-\infty}^{0}\mathrm{e}^{\lambda|s|}|u(\theta_s\omega)|_2 ds\\ &\leq d\sup_{t\in\mathbb{R}}\left\{\mathrm{e}^{\frac{\lambda}{2}|t|}\,|u(\theta_t\omega)|_2\right\}\int_{-\infty}^{0}e^{\frac{\lambda}{2}|s|}ds<\infty,\ \omega\in\Omega,\end{aligned}$$

which implies that $\lim_{t\to\infty}\int_{-t}^{0}\Phi(-s)u(\theta_s\omega)ds$ exists for all $\omega\in\Omega$. For any $t_1>t_2>0$,

$$\begin{aligned}\mathbb{E}\left|\int_{-t_1}^{0}\Phi(-s)\sigma dW_s-\int_{-t_2}^{0}\Phi(-s)\sigma dW_s\right|_2^2 &= \mathbb{E}\int_{-t_1}^{-t_2}\|\Phi(-s)\sigma\|_2^2\,ds\\ &\leq \sum_{i=1}^{d}\sum_{j=1}^{m}\sigma_{ij}^2\int_{-t_1}^{-t_2}\|\Phi(-s)\|_2^2 ds,\end{aligned}$$

which together with assumption (A) shows that $\int_{-t}^{0}\Phi(-s)\sigma dW_s$ converges in L^2 as $t\to\infty$. By (2.6), $\int_{-t}^{0}\Phi(-s)\sigma dW_s$ is a continuous martingale. Hence, it follows from [21, Problem 3.20 in Chapter 1] that $\int_{-t}^{0}\Phi(-s)\sigma dW_s$ converges $\mathbb{P}$-a.s. to an integrable random variable $X_\infty:=\int_{-\infty}^{0}\Phi(-s)\sigma dW_s$ as $t\to\infty$. Furthermore, by the boundedness of h (see hypothesis (H_1)), we obtain that $\{\varphi(t,\theta_{-t}\omega)x : t\geq 0\}$ is a bounded set for $\mathbb{P}$-a.s. $\omega\in\Omega$ and $x\in\mathbb{R}^d$.

3. Asymptotic behavior of RDS generated by Itô SDEs.

In this section, we will give some preliminary propositions and lemmas to describe the dynamics of the pull-back trajectory which will be used in the proof of our main result. To make the paper self-contained, we begin with a known result in [14] which provides convenience for reading.

LEMMA 3.1 (see [14, Lemma A.2]). *Suppose that $(x_\alpha)_{\alpha\in A}$ is a net in a normed space X, partially ordered by a solid, normal cone $X_+\subseteq X$. Suppose, in addition, that the net converges to an element $x_\infty\in X$, and that the infima and suprema*

$$x_\alpha^- := \inf\{x_{\alpha'} : \alpha'\geq\alpha\}\quad \textit{and}\quad x_\alpha^+ := \sup\{x_{\alpha'} : \alpha'\geq\alpha\}$$

exist for every $\alpha\in A$. Then the nets $(x_\alpha^-)_{\alpha\in A}$ and $(x_\alpha^+)_{\alpha\in A}$ so defined also converge to x_∞.

PROPOSITION 3.2. *For each $\tau>0$, let*

$$a_\tau^h(\omega)=\inf\overline{\{h(\varphi(t,\theta_{-t}\omega)x) : t\geq\tau\}}$$

and

$$b_\tau^h(\omega)=\sup\overline{\{h(\varphi(t,\theta_{-t}\omega)x) : t\geq\tau\}},\quad x\in\mathbb{R}^d,\ \omega\in\Omega,$$

where inf *and* sup *mean the greatest lower bound and the least upper bound, respectively. Then $a_\tau^h(\omega)$ and $b_\tau^h(\omega)$ are random variables with respect to the σ-algebra $\mathscr{F}$. When $h=\mathrm{id}$, we use the notations a_τ^{id} and b_τ^{id}.*

Proof. First, we show that $a_\tau^h(\omega)$ and $b_\tau^h(\omega)$ are well defined. By Remark 1, it is clear that $D_\tau^h(\omega):=\overline{\{h(\varphi(t,\theta_{-t}\omega)x) : t\geq\tau\}}$ and $D_\tau^{\mathrm{id}}(\omega):=\overline{\{\varphi(t,\theta_{-t}\omega)x : t\geq\tau\}}$ are two bounded sets for fixed $\tau\geq 0$, $\omega\in\Omega$, and $x\in\mathbb{R}^d$, which implies that $D_\tau^h(\omega)$ and $D_\tau^{\mathrm{id}}(\omega)$ are order-bounded. Since $\mathbb{R}_+^d$ is strongly minihedral [5, Definition 3.1.7],

$a_\tau^h(\omega)$ and $b_\tau^h(\omega)$ exist. Let $\gamma_x^\tau(\omega) := \bigcup_{t\geq\tau}\{h(\varphi(t,\theta_{-t}\omega)x)\}$ be the tail of the pull-back trajectory emanating from x. It is noticed from Remark 1.5.1 in [5] that

$$(t,x) \mapsto \varphi(t,\theta_{-t}\omega)x \text{ is a continuous mapping}$$

from $\mathbb{R}_+ \times \mathbb{R}^d$ into $\mathbb{R}^d$; then, by Proposition 1.3.5 in [5], we have that

$$\omega \mapsto \overline{\gamma_x^\tau(\omega)} := \overline{\bigcup_{t\geq\tau}\{h(\varphi(t,\theta_{-t}\omega)x)\}}$$

is a random compact set with respect to $\mathscr{F}$. Together with Theorem 3.2.1 in [5], we can conclude that $a_\tau^h(\omega)$ and $b_\tau^h(\omega)$ are $\mathscr{F}$-measurable random variables in $\mathbb{R}_+^d$. When $h = \mathrm{id}$, we can obtain the same result from Remark 1. The proof is complete. ∎

Lemma 3.3. *Assume that conditions* (H$_1$) *and* (A) *hold. Let $\varphi(t,\omega)x$ be a solution of stochastic system* (2.1) *with initial value $x \in \mathbb{R}^d$. Then we have*

$$\mathcal{K}(\theta - \underline{\lim}\, h(\varphi)) \leq \theta - \underline{\lim}\, \varphi \leq \theta - \overline{\lim}\, \varphi \leq \mathcal{K}(\theta - \overline{\lim}\, h(\varphi)), \quad \mathbb{P}\text{-a.s.,} \tag{3.1}$$

where

$$[\theta - \underline{\lim}\, \varphi](\omega) := \lim_{\tau\to\infty} \inf\{\varphi(t,\theta_{-t}\omega)x : t \geq \tau\}, \quad x \in \mathbb{R}^d,\ \omega \in \Omega,$$

and

$$[\theta - \overline{\lim}\, \varphi](\omega) := \lim_{\tau\to\infty} \sup\{\varphi(t,\theta_{-t}\omega)x : t \geq \tau\}, \quad x \in \mathbb{R}^d,\ \omega \in \Omega.$$

Analogously, we can define $\theta - \underline{\lim}\, h(\varphi)$ and $\theta - \overline{\lim}\, h(\varphi)$.

Proof. Here, we only prove the first inequality for the sake of convenience and the rest of the inequalities can be proved analogously. First, in view of the fact that

$$\inf\{\varphi(t,\theta_{-t}\omega)x : t \geq \tau\} = \inf \overline{\{\varphi(t,\theta_{-t}\omega)x : t \geq \tau\}}$$

and

$$\inf\{h(\varphi(t,\theta_{-t}\omega)x) : t \geq \tau\} = \inf \overline{\{h(\varphi(t,\theta_{-t}\omega)x) : t \geq \tau\}}$$

by Lemma A.1 in [14], similar to Proposition 3.2, we can easily get that $\theta - \underline{\lim}\, h(\varphi)$ and $\theta - \underline{\lim}\, \varphi$ exist, which are also two $(\mathscr{F}, \mathscr{B}(\mathbb{R}^d))$-measurable random variables. Then $\mathcal{K}(\theta - \underline{\lim}\, h(\varphi))$ is well defined and $(\mathscr{F}, \mathscr{B}(\mathbb{R}^d))$-measurable by (2.9) and the Fubini theorem. It is noticed that

$$[\theta - \underline{\lim}\, h(\varphi)](\omega) := \lim_{\tau\to\infty} \inf\{h(\varphi(t,\theta_{-t}\omega)x) : t \geq \tau\}, \quad x \in \mathbb{R}^d,\ \omega \in \Omega,$$

and by Lebesgue's dominated convergence theorem, we have $\mathcal{K}(\theta - \underline{\lim}\, h(\varphi)) = \lim_{\tau\to\infty} \mathcal{K}(a_\tau)$, where $a_\tau(\omega) := \inf\{h(\varphi(t,\theta_{-t}\omega)x) : t \geq \tau\}$. Therefore, we can choose an increasing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \uparrow \infty$ and it is enough to prove that

$$\mathcal{K}(a_{\tau_n}) \leq \theta - \underline{\lim}\, \varphi, \quad \mathbb{P}\text{-a.s. for all } n \in \mathbb{N}.$$

By the definition of $\mathcal{K}$, for fixed $\tau_n \geq 0$, we have

$$\begin{aligned}
[\mathcal{K}(a_{\tau_n})](\omega) &= \int_{-\infty}^{0} \Phi(-s) \inf\{h(\varphi(t, \theta_{-t}\bullet)x) : t \geq \tau_n\}(\theta_s\omega) ds + \int_{-\infty}^{0} \Phi(-s)\sigma dW(s) \\
&= \int_{-\infty}^{0} \Phi(-s) \inf\{h(\varphi(t, \theta_{-t+s}\omega)x) : t \geq \tau_n\} ds + \int_{-\infty}^{0} \Phi(-s)\sigma dW(s) \\
&= \lim_{\substack{\tilde{t}\to\infty \\ \tilde{t}\geq\tau_n}} \left\{ \Phi(\tilde{t})x + \int_{\tau_n-\tilde{t}}^{0} \Phi(-s) \inf\{h(\varphi(t, \theta_{-t+s}\omega)x) : t \geq \tau_n\} ds \right. \\
&\quad \left. + \int_{-\tilde{t}}^{0} \Phi(-s)\sigma dW(s) \right\} \\
&= \lim_{\substack{\tau\to\infty \\ \tau\geq\tau_n}} \inf \left\{ \Phi(\tilde{t})x + \int_{\tau_n-\tilde{t}}^{0} \Phi(-s) \inf\{h(\varphi(t, \theta_{-t+s}\omega)x) : t \geq \tau_n\} ds \right. \\
&\quad \left. + \int_{-\tilde{t}}^{0} \Phi(-s)\sigma dW(s) : \tilde{t} \geq \tau \right\} \\
&\leq \lim_{\substack{\tau\to\infty \\ \tau\geq\tau_n}} \inf \left\{ \Phi(\tilde{t})x + \int_{\tau_n-\tilde{t}}^{0} \Phi(-s) \{h(\varphi(\tilde{t}+s, \theta_{-\tilde{t}}\omega)x)\} ds \right. \\
&\quad \left. + \int_{-\tilde{t}}^{0} \Phi(-s)\sigma dW(s) : \tilde{t} \geq \tau \right\} \\
&\leq \lim_{\tau\to\infty} \inf \left\{ \Phi(\tilde{t})x + \int_{-\tilde{t}}^{0} \Phi(-s) \{h(\varphi(\tilde{t}+s, \theta_{-\tilde{t}}\omega)x)\} ds \right. \\
&\quad \left. + \int_{-\tilde{t}}^{0} \Phi(-s)\sigma dW(s) : \tilde{t} \geq \tau \right\} \\
&= [\theta - \underline{\lim}\, \varphi](\omega),
\end{aligned}$$

where the fourth equality has used Lemma 3.1, while the second-to-last inequality has applied the positivity of $\Phi(t)$ [32] and h. The proof is complete. ◻

Lemma 3.4. *Assume that conditions* (H$_1$) *and* (A) *hold. Let $\varphi(t, \omega)x$ be a solution of* (2.1) *with initial value $x \in \mathbb{R}^d$. Then we have the following:*

(i) *If h is order-preserving, then*

$$h(\theta - \underline{\lim}\, \varphi) \leq \theta - \underline{\lim}\, h(\varphi) \leq \theta - \overline{\lim}\, h(\varphi) \leq h(\theta - \overline{\lim}\, \varphi), \quad \mathbb{P}\text{-}a.s. \tag{3.2}$$

(ii) *If h is anti-order-preserving, then*

$$h(\theta - \overline{\lim}\, \varphi) \leq \theta - \underline{\lim}\, h(\varphi) \leq \theta - \overline{\lim}\, h(\varphi) \leq h(\theta - \underline{\lim}\, \varphi), \quad \mathbb{P}\text{-}a.s. \tag{3.3}$$

Proof. Indeed, the proof of the first inequality in (3.2) is adequate and the rest of the results of this lemma can be obtained analogously. Observe that h is order-preserving for fixed $\tau \geq 0$ and $x \in \mathbb{R}^d$; then for all $t \geq \tau$, we have

$$h(\inf\{\varphi(t, \theta_{-t}\omega)x : t \geq \tau\}) \leq h(\varphi(t, \theta_{-t}\omega)x)$$

and

$$h(\inf\{\varphi(t, \theta_{-t}\omega)x : t \geq \tau\}) \leq \inf\{h(\varphi(t, \theta_{-t}\omega)x) : t \geq \tau\}. \tag{3.4}$$

Let $\tau \to \infty$ in (3.4). Then

$$\begin{aligned} h(\theta - \underline{\lim}\, \varphi)(\omega) &= \lim_{\tau\to\infty} h(\inf\{\varphi(t,\theta_{-t}\omega)x : t \geq \tau\}) \\ &\leq \lim_{\tau\to\infty} \inf\{h(\varphi(t,\theta_{-t}\omega)x) : t \geq \tau\} = \theta - \underline{\lim}\, h(\varphi)(\omega). \end{aligned}$$

The proof is complete. ∎

Lemma 3.5. *Assume that conditions* (H$_1$) *and* (A) *hold. Then, for stochastic system* (2.1)*, let* $\varphi(t,\omega)x$ *be a solution of* (2.1) *with initial value* $x \in \mathbb{R}^d$*; then we have*

$$\mathcal{K}(a_\tau^h) \leq \theta - \underline{\lim}\, \varphi \leq \theta - \overline{\lim}\, \varphi \leq \mathcal{K}(b_\tau^h), \quad \mathbb{P}\text{-}a.s.,\ \tau \geq 0, \tag{3.5}$$

where $a_\tau^h(\omega)$ *and* $b_\tau^h(\omega)$ *are as defined in Proposition* 3.2*. Furthermore, define* $\mathcal{K}^h := h \circ \mathcal{K}$ *as a gain operator. Then we have the following:*

(i) *If* h *is order-preserving, then for fixed* $\tau \geq 0$,

$$(\mathcal{K}^h)^k(a_\tau^h) \leq \theta - \underline{\lim}\, h(\varphi) \leq \theta - \overline{\lim}\, h(\varphi) \leq (\mathcal{K}^h)^k(b_\tau^h), \quad \mathbb{P}\text{-}a.s.,\ k \in \mathbb{N}. \tag{3.6}$$

(ii) *If* h *is anti-order-preserving, then for fixed* $\tau \geq 0$,

$$(\mathcal{K}^h)^{2k}(a_\tau^h) \leq \theta - \underline{\lim}\, h(\varphi) \leq \theta - \overline{\lim}\, h(\varphi) \leq (\mathcal{K}^h)^{2k}(b_\tau^h), \quad \mathbb{P}\text{-}a.s.,\ k \in \mathbb{N}. \tag{3.7}$$

Proof. By the definitions of a_τ^h and b_τ^h, it is evident that

$$a_\tau^h \leq \theta - \underline{\lim}\, h(\varphi) \leq \theta - \overline{\lim}\, h(\varphi) \leq b_\tau^h, \quad \tau \geq 0.$$

Observe that Φ is monotone based on assumption (A); then $\mathcal{K}$ is also monotone with respect to u, and consequently

$$\mathcal{K}(a_\tau^h) \leq \mathcal{K}(\theta - \underline{\lim}\, h(\varphi)) \leq \mathcal{K}(\theta - \overline{\lim}\, h(\varphi)) \leq \mathcal{K}(b_\tau^h), \quad \mathbb{P}\text{-a.s.},\ \tau \geq 0,$$

which together with Lemma 3.3 implies

$$\mathcal{K}(a_\tau^h) \leq \theta - \underline{\lim}\, \varphi \leq \theta - \overline{\lim}\, \varphi \leq \mathcal{K}(b_\tau^h), \quad \mathbb{P}\text{-a.s.},\ \tau \geq 0.$$

That is, (3.5) holds.

In what follows, we claim that (3.6) and (3.7) hold. We consider two cases as follows:

Case (i). If h is order-preserving, then it deduces that h preserves the inequalities in (3.5):

$$\mathcal{K}^h(a_\tau^h) \leq h(\theta - \underline{\lim}\, \varphi) \leq h(\theta - \overline{\lim}\, \varphi) \leq \mathcal{K}^h(b_\tau^h), \quad \mathbb{P}\text{-a.s.},\ \tau \geq 0.$$

As a consequence of Lemma 3.4, it follows that

$$\mathcal{K}^h(a_\tau^h) \leq \theta - \underline{\lim}\, h(\varphi) \leq \theta - \overline{\lim}\, h(\varphi) \leq \mathcal{K}^h(b_\tau^h), \quad \mathbb{P}\text{-a.s.},\ \tau \geq 0.$$

This proves that (3.6) is true for $k = 1$.

Next we assume that, for some $k \in \mathbb{N}$, we have obtained

$$(\mathcal{K}^h)^k(a_\tau^h) \leq \theta - \underline{\lim}\, h(\varphi) \leq \theta - \overline{\lim}\, h(\varphi) \leq (\mathcal{K}^h)^k(b_\tau^h), \quad \mathbb{P}\text{-a.s.},\ \tau \geq 0.$$

From the monotonicity of $\mathcal{K}$ and h, and Lemmas 3.3 and 3.4, we can have

$$\begin{aligned} \mathcal{K}[(\mathcal{K}^h)^k(a_\tau^h)] &\leq \mathcal{K}(\theta - \underline{\lim}\, h(\varphi)) \leq \theta - \underline{\lim}\, \varphi \\ &\leq \theta - \overline{\lim}\, \varphi \leq \mathcal{K}(\theta - \overline{\lim}\, h(\varphi)) \leq \mathcal{K}[(\mathcal{K}^h)^k(b_\tau^h)], \quad \mathbb{P}\text{-a.s.},\ \tau \geq 0. \end{aligned}$$

Acted both sides by h in the above inequalities, we get that

$$\begin{aligned}(\mathcal{K}^h)^{k+1}(a^h_\tau) &\le h(\theta-\underline{\lim}\,\varphi) \le \theta-\underline{\lim}\,h(\varphi)\\ &\le \theta-\overline{\lim}\,h(\varphi) \le h(\theta-\overline{\lim}\,\varphi) \le (\mathcal{K}^h)^{k+1}(b^h_\tau), \quad \mathbb{P}\text{-a.s.},\ \tau\ge 0.\end{aligned}$$

Therefore, we conclude that (3.6) holds by mathematical induction.

Case (ii). Assume that h is anti-order-preserving, similar to Case (i); we deduce that

$$\mathcal{K}^h(b^h_\tau) \le h(\theta-\overline{\lim}\,\varphi) \le h(\theta-\underline{\lim}\,\varphi) \le \mathcal{K}^h(a^h_\tau), \quad \mathbb{P}\text{-a.s.},\ \tau\ge 0.$$

Using (3.3) in Lemma 3.4, we have

$$\mathcal{K}^h(b^h_\tau) \le \theta-\underline{\lim}\,h(\varphi) \le \theta-\overline{\lim}\,h(\varphi) \le \mathcal{K}^h(a^h_\tau), \quad \mathbb{P}\text{-a.s.},\ \tau\ge 0.$$

Combining the monotonicity of $\mathcal{K}$ and Lemma 3.3, it shows that

$$\mathcal{K}[\mathcal{K}^h(b^h_\tau)] \le \theta-\underline{\lim}\,\varphi \le \theta-\overline{\lim}\,\varphi \le \mathcal{K}[\mathcal{K}^h(a^h_\tau)], \quad \mathbb{P}\text{-a.s.},\ \tau\ge 0,$$

which together with the antimonotonicity of h and (3.3) in Lemma 3.4 implies

$$(\mathcal{K}^h)^2(a^h_\tau) \le \theta-\underline{\lim}\,h(\varphi) \le \theta-\overline{\lim}\,h(\varphi) \le (\mathcal{K}^h)^2(b^h_\tau), \quad \mathbb{P}\text{-a.s.},\ \tau\ge 0.$$

The rest of the proof of (3.7) can be obtained analogously to Case (i) by mathematical induction. The proof is complete. ∎

4. Main results.

In this section, we will state our main result on the stability of nonlinear stochastic system (2.1) and present its proof. We begin with a lemma.

LEMMA 4.1. *Assume that conditions* (H$_1$), (H$_2$), *and* (A) *hold. Let* $\mathcal{M}^b_{\mathscr{F}}(\Omega;[0,N])$ *be the space of* $\mathscr{F}$*-measurable functions* $f:\Omega\to[0,N]$, *where* $N=(N_1,\dots,N_d)$, $N_i=\sup_{x\in\mathbb{R}^d}|h_i(x)|$, $i=1,\dots,d$. *We introduce a metric on* $\mathcal{M}^b_{\mathscr{F}}(\Omega;[0,N])$ *as follows:*

$$\rho(f_1,f_2) := |f_1-f_2|_\infty = \sup_{\omega\in\Omega}|f_1(\omega)-f_2(\omega)| \quad \text{for all } f_1,f_2\in\mathcal{M}^b_{\mathscr{F}}(\Omega;[0,N]);$$

then $(\mathcal{M}^b_{\mathscr{F}},\rho)$ *is a complete metric space and the gain operator* $\mathcal{K}^h := h\circ\mathcal{K} : \mathcal{M}^b_{\mathscr{F}}\to\mathcal{M}^b_{\mathscr{F}}$ *is a contraction mapping, where the definition of input-to-state characteristic operator* $\mathcal{K}$ *can be chosen as an* $\mathbb{R}^d$*-valued version for all* $\omega\in\Omega$.

Proof. It is clear that $(\mathcal{M}^b_{\mathscr{F}},\rho)$ is a metric space. Now we will show that the metric space $\mathcal{M}^b_{\mathscr{F}}$ is complete with respect to ρ. To prove this, we can choose a Cauchy sequence $\{f_n, n\in\mathbb{N}\}$ in $(\mathcal{M}^b_{\mathscr{F}},\rho)$; we denote a function f as follows:

$$f(\omega) := \lim_{n\to\infty} f_n(w)\in[0,N] \quad \text{for all } \omega\in\Omega,$$

which holds based on the fact that $\{f_n(\omega), n\in\mathbb{N}\}$ is a Cauchy sequence in $\mathbb{R}^d$ for fixed $\omega\in\Omega$. It is noticed that the limit of a family of $\mathscr{F}$-measurable functions is an $\mathscr{F}$-measurable function [6]. This shows that $f\in\mathcal{M}^b_{\mathscr{F}}(\Omega;[0,N])$.

In what follows, we will prove that $|f-f_n|_\infty\to 0$ as $n\to\infty$. Observing that $\{f_n(\omega), n\in\mathbb{N}\}$ is a Cauchy sequence, we know that for all $\varepsilon>0$, there exists $N_0=N_0(\varepsilon)\in\mathbb{N}$ such that for $n,m\ge N_0$,

$$\sup_{\omega\in\Omega}|f_m(\omega)-f_n(\omega)| < \varepsilon.$$

Let $m \to \infty$; then

$$\sup_{\omega\in\Omega} |f(\omega) - f_n(\omega)| \le \varepsilon \quad \text{for all } n \ge N_0,$$

which implies that $|f - f_n|_\infty \to 0$ as $n \to \infty$. Thus $(\mathcal{M}^b_{\mathscr{F}}, \rho)$ is a complete metric space.

Next we claim that $\mathcal{K}^h : \mathcal{M}^b_{\mathscr{F}} \to \mathcal{M}^b_{\mathscr{F}}$ is a contraction mapping. First, we should show that $\mathcal{K}^h : \mathcal{M}^b_{\mathscr{F}} \to \mathcal{M}^b_{\mathscr{F}}$ is well defined. From (H_1), it follows that $h : \mathbb{R}^d \to [0, N]$. By the definition of $\mathcal{K}$, the measurability of θ, and the Fubini theorem, it is evident that $\mathcal{K}(u)$ is an $\mathscr{F}$-measurable function, which yields $\mathcal{K}^h : \mathcal{M}^b_{\mathscr{F}} \to \mathcal{M}^b_{\mathscr{F}}$.

Finally, we prove that $\mathcal{K}^h$ is a contraction mapping. By (H_1) and (H_2), we can have

$$\sup_{x\in\mathbb{R}^d} \|Dh(x)\| \le L,$$

where $Dh(x)$ is the Jacobian of h. Let f_1 and f_2 be two elements in $(\mathcal{M}^b_{\mathscr{F}}, \rho)$. By the fact that $|\Phi x| \le d\|\Phi\| \cdot |x|$ for all $x \in \mathbb{R}^d$ and $\Phi \in \mathbb{R}^{d\times d}$, we get

$$\begin{aligned}
|\mathcal{K}^h(f_1) - \mathcal{K}^h(f_2)|_\infty &= |h[\mathcal{K}(f_1)] - h[\mathcal{K}(f_2)]|_\infty \\
&= |Dh[\mathcal{K}(f_2) + \mu(\mathcal{K}(f_1) - \mathcal{K}(f_2))] \cdot [\mathcal{K}(f_1) - \mathcal{K}(f_2)]|_\infty \\
&\le d \sup_{x\in\mathbb{R}^d} \|Dh(x)\| \cdot |\mathcal{K}(f_1) - \mathcal{K}(f_2)|_\infty \\
&\le Ld \left| \int_{-\infty}^0 \Phi(-s) f_1(\theta_s\omega) ds - \int_{-\infty}^0 \Phi(-s) f_2(\theta_s\omega) ds \right|_\infty \\
&\le Ld^2 \int_{-\infty}^0 \|\Phi(-s)\| \cdot |f_1 - f_2|_\infty ds \\
&\le Ld^2 \int_{-\infty}^0 \mathrm{e}^{-\lambda s} ds |f_1 - f_2|_\infty \\
&= -\frac{Ld^2}{\lambda} |f_1 - f_2|_\infty,
\end{aligned}$$

where $0 < \mu < 1$, $-\frac{Ld^2}{\lambda} < 1$, and the second-to-last inequality holds due to condition (A). The proof is complete. ∎

Theorem 4.2 (small-gain theorem). *Assume that conditions* (H_1), (H_2), *and* (A) *hold. Then the gain operator* $\mathcal{K}^h := h \circ \mathcal{K} : \mathcal{M}^b_{\mathscr{F}} \to \mathcal{M}^b_{\mathscr{F}}$ *possesses a unique nonnegative fixed point* $u \in \mathcal{M}^b_{\mathscr{F}}(\Omega; [0, N])$ *such that for all* $x \in \mathbb{R}^d$,

$$\lim_{t\to\infty} \varphi(t, \theta_{-t}\omega)x = [\mathcal{K}(u)](\omega), \quad \mathbb{P}\text{-a.s.} \tag{4.1}$$

Moreover, $\varphi(t, \omega)[\mathcal{K}(u)](\omega) = [\mathcal{K}(u)](\theta_t\omega)$, $\mathbb{P}$*-a.s.,* $t > 0$*; i.e., the image* $[\mathcal{K}(u)](\cdot)$ *at the fixed point u for the input-to-state characteristic operator is a random equilibrium.*

Proof. In view of Lemma 3.5, regardless of the monotonicity or antimonotonicity for h, for fixed $\tau \ge 0$, we have

$$(\mathcal{K}^h)^{2k}(a^h_\tau) \le \theta - \underline{\lim}\, h(\varphi) \le \theta - \overline{\lim}\, h(\varphi) \le (\mathcal{K}^h)^{2k}(b^h_\tau), \quad \mathbb{P}\text{-a.s.}, \ k \in \mathbb{N}, \tag{4.2}$$

where a^h_τ and b^h_τ are as defined in Proposition 3.2. By Proposition 3.2, a^h_τ and b^h_τ are bounded $\mathscr{F}$-measurable functions in $(\mathcal{M}^b_{\mathscr{F}}, \rho)$. Since $\mathcal{K}^h$ is a contraction mapping on the complete metric space $\mathcal{M}^b_{\mathscr{F}}$, by the Banach fixed point theorem [36], there exists a unique nonnegative random variable $u : \Omega \to [0, N]$ for $\mathcal{K}^h$ such that

$$[\mathcal{K}^h(u)](\omega) = u(\omega) \quad \text{for all } \omega \in \Omega$$

and

$$\lim_{k\to\infty}[(\mathcal{K}^h)^{2k}(a_\tau^h)](\omega) = u(\omega) = \lim_{k\to\infty}[(\mathcal{K}^h)^{2k}(b_\tau^h)](\omega) \quad \text{for all } \omega \in \Omega. \tag{4.3}$$

Moreover, u is evidently independent of $x \in \mathbb{R}^d$. Combining (4.2) and (4.3), we can have

$$[\theta - \underline{\lim}\, h(\varphi)](\omega) = [\theta - \overline{\lim}\, h(\varphi)](\omega) = u(\omega), \quad \mathbb{P}\text{-a.s.},$$

which together with Lemma 3.3 implies that

$$[\theta - \underline{\lim}\, \varphi](\omega) = [\theta - \overline{\lim}\, \varphi](\omega) = [\mathcal{K}(u)](\omega), \quad \mathbb{P}\text{-a.s.}$$

In order to prove (4.1), it remains to show that

$$[\theta - \underline{\lim}\, \varphi](\omega) = [\theta - \overline{\lim}\, \varphi](\omega) = \lim_{t\to\infty} \varphi(t, \theta_{-t}\omega)x, \quad \mathbb{P}\text{-a.s.},\ x \in \mathbb{R}^d. \tag{4.4}$$

By the definitions of infimum and supremum, it is clear that

$$\inf\{\varphi(t, \theta_{-t}\omega)x : t \geq \tau\} \leq \varphi(\tau, \theta_{-\tau}\omega)x \leq \sup\{\varphi(t, \theta_{-t}\omega)x : t \geq \tau\}, \quad \mathbb{P}\text{-a.s.},\ x \in \mathbb{R}^d.$$

Let $\tau \to \infty$ in the above inequality; then (4.4) holds, and so (4.1) holds.

Furthermore, by (4.1) and the property of continuity and cocycle for (θ, φ), one can show that for fixed $t > 0$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
\varphi(t,\omega)[\mathcal{K}(u)](\omega) &= \varphi(t,\omega) \lim_{s\to\infty} \varphi(s, \theta_{-s}\omega)x \\
&= \lim_{s\to\infty} \varphi(t,\omega) \circ \varphi(s, \theta_{-s}\omega)x \\
&= \lim_{s\to\infty} \varphi(t+s, \theta_{-s}\omega)x \\
&= \lim_{s\to\infty} \varphi(t+s, \theta_{-(t+s)} \circ \theta_t\omega)x \\
&= [\mathcal{K}(u)](\theta_t\omega) \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

The proof is complete. ∎

Remark 2. Since $(t, x) \mapsto \varphi(t, \theta_{-t}\omega)x$ is a continuous mapping from $\mathbb{R}_+ \times \mathbb{R}^d$ into $\mathbb{R}^d$ [5, Remark 1.5.1, p. 34], we can have an undistinguished version of the stochastic process $\{\varphi(t, \theta_{-t}\omega)x,\ t \in \mathbb{R}_+\}$ for fixed $x \in \mathbb{R}^d$, which leads us to believe that all conclusions presented in this paper would hold for all $\omega \in \Omega$.

Remark 3. It is worth pointing out that we cannot obtain our main results directly from Theorem 4.4 in [14]. To be more precise, the perfection of the crude cocycle $\varphi(t, \omega, x, u)$ is not easy and the small-gain condition (Definition 4.2 in [14]) meets some difficulties for stochastic systems. In fact, the key role of the small-gain condition is to guarantee the existence on the unique, globally attracting fixed point of $\mathcal{K}^h$. In this paper, motivated by this thought, we directly consider the existence and uniqueness of globally attracting fixed points by using the Banach fixed point theorem. It is worth noting that the idea established in this paper has a potential to apply to other stochastic systems with feedback interconnections, for example, to those driven by multiplicative white noise. However, specific measurability problems, which need to be overcome, appear while realizing this idea; see [17].

5. Examples. In this section, we present several examples to illustrate the use of our small-gain theorem. Note that our result can be applied to stochastic cooperative, competitive, and predator-prey systems, or even to others. As far as the authors know, there has been no criterion to guarantee them to be globally stable so far. Although we can construct many higher-dimensional stochastic systems, we only give three-dimensional systems in the following.

Example 5.1. Consider *stochastic cooperative system*

$$dx_i = [(Ax)_i + h_i(x_i)]dt + \sigma_i dW_t^i, \qquad i = 1, 2, 3, \tag{5.1}$$

where

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \tag{5.2}$$

with three eigenvalues $\lambda_1 = -1$, $\lambda_{2,3} = \frac{-3\pm\sqrt{5}}{2}$ and

$$h_i(x_i) := \frac{1}{6 + g_i(x_i)}, \qquad i = 1, 2, 3, \tag{5.3}$$

where $g_i(x_i) = \frac{\pi}{2} - \arctan x_i$ is decreasing with respect to x_i, $i = 1, 2, 3$. It is clear that (5.1) is a cooperative system. By direct calculation, we obtain

$$\Phi(t) = \begin{bmatrix} \frac{5+\sqrt{5}}{10}\mathrm{e}^{\frac{-3+\sqrt{5}}{2}t} + \frac{5-\sqrt{5}}{10}\mathrm{e}^{\frac{-3-\sqrt{5}}{2}t} & \frac{\sqrt{5}}{5}\mathrm{e}^{\frac{-3+\sqrt{5}}{2}t} - \frac{\sqrt{5}}{5}\mathrm{e}^{\frac{-3-\sqrt{5}}{2}t} & 0 \\ \frac{\sqrt{5}}{5}\mathrm{e}^{\frac{-3+\sqrt{5}}{2}t} - \frac{\sqrt{5}}{5}\mathrm{e}^{\frac{-3-\sqrt{5}}{2}t} & \frac{5-\sqrt{5}}{10}\mathrm{e}^{\frac{-3+\sqrt{5}}{2}t} + \frac{5+\sqrt{5}}{10}\mathrm{e}^{\frac{-3-\sqrt{5}}{2}t} & 0 \\ -\mathrm{e}^{-t} + \frac{5+\sqrt{5}}{10}\mathrm{e}^{\frac{-3+\sqrt{5}}{2}t} + \frac{5-\sqrt{5}}{10}\mathrm{e}^{\frac{-3-\sqrt{5}}{2}t} & \frac{\sqrt{5}}{5}\mathrm{e}^{\frac{-3+\sqrt{5}}{2}t} - \frac{\sqrt{5}}{5}\mathrm{e}^{\frac{-3-\sqrt{5}}{2}t} & \mathrm{e}^{-t} \end{bmatrix}.$$

It is not difficult to estimate that for any $t \geq 0$,

$$\|\Phi(t)\| := \max\{|\Phi_{ij}(t)| : i, j = 1, 2, 3\} \leq \mathrm{e}^{\lambda_2 t},$$

that is, (2.4) holds. Moreover, it is easy to see that $\max_{1\leq i\leq 3} \mathrm{Re}\lambda_i = \lambda_2 < 0$, $L \leq \frac{1}{36}$. So

$$-\frac{9L}{\lambda_2} \leq \frac{1}{2(3-\sqrt{5})} < 1.$$

By the small-gain theorem (Theorem 4.2), stochastic cooperative system (5.1) possesses a unique globally asymptotically stable random equilibrium; that is, (5.1) has a unique stationary solution, which is globally attractive in pull-back trajectories.

The same conclusion holds if we replace $g_i(x_i)$ by $g_i(x_1 + x_2 + x_3)$.

Example 5.2. Consider *stochastic competitive system*

$$dx_i = [a_i x_i + h_i(x_{i-1})]dt + \sigma_i dW_t^i, \qquad i = 1, 2, 3, \tag{5.4}$$

where $a_1 = -1, a_2 = -2, a_3 = -3$ $(x_0 = x_3)$, and

$$h_i(x_{i-1}) = \frac{1}{5 + \mathrm{th}\ x_{i-1}} := \frac{1}{4 + g_i(x_{i-1})}, \qquad i = 1, 2, 3. \tag{5.5}$$

Since $g_i(x_{i-1}) = 1 + \mathrm{th}\ x_{i-1} = 1 + \frac{e^{x_{i-1}} - e^{-x_{i-1}}}{e^{x_{i-1}} + e^{-x_{i-1}}}$ is increasing with respect to x_{i-1}, $i = 1, 2, 3$, (5.4) is a stochastic competitive biochemical circuit. Furthermore, by (5.4) and (5.5), letting $\lambda = -1$ and $L \leq \frac{1}{16}$, it follows that

$$\|\Phi(t)\| := \max\{|\Phi_{ij}(t)| : i, j = 1, 2, 3\} = \mathrm{e}^{-t}, \ t \geq 0,$$

and

$$-\frac{9L}{\lambda} \leq \frac{9}{16} < 1.$$

Applying the small-gain theorem (Theorem 4.2), stochastic competitive system (5.4) admits a unique globally asymptotically stable random equilibrium, which produces an ergodic stationary solution for (5.4).

Note that the same result is true when $g_i(x_{i-1})$ is replaced by $g_i(x_1 + x_2 + x_3)$.

Example 5.3. Consider *stochastic predator-prey system*

$$dx_i = [(Ax)_i + h_i(x_{i-1})]dt + \sigma_i dW_t^i, \qquad i = 1, 2, 3, \tag{5.6}$$

where $x_0 = x_3$, $x_4 = x_1$, and

$$A = \begin{bmatrix} -1 & \sqrt[3]{2} & 0 \\ 0 & -2 & \sqrt[3]{2} \\ \sqrt[3]{2} & 0 & -4 \end{bmatrix} \tag{5.7}$$

with three eigenvalues $\lambda_1 = -3$, $\lambda_{2,3} = -2 \pm \sqrt{2}$ and

$$h_i(x_{i-1}) = \frac{1}{4 + \frac{\pi}{2} + \arctan x_{i-1}} := \frac{1}{4 + g_i(x_{i-1})}, \qquad i = 1, 2, 3. \tag{5.8}$$

Let $f(x) = Ax + h(x)$, $x \in \mathbb{R}^3$. Then let $\frac{\partial f_i}{\partial x_{i+1}}(x) = \sqrt[3]{2} > 0$ and $\frac{\partial f_{i+1}}{\partial x_i}(x) = -\frac{1}{(4+\frac{\pi}{2}+\arctan x_i)^2} \cdot \frac{1}{1+x_i^2} < 0$ for $i = 1, 2, 3$, which implies that (5.6) is a stochastic predator-prey system. Direct calculation of Φ shows that

$$\Phi(t) = \left[\begin{matrix} \mathrm{e}^{-3t} + \frac{\sqrt{2}}{2}\mathrm{e}^{(-2+\sqrt{2})t} - \frac{\sqrt{2}}{2}\mathrm{e}^{-(2+\sqrt{2})t} \\ -\sqrt[3]{4}\mathrm{e}^{-3t} + \frac{\sqrt[3]{4}-\sqrt[6]{2}}{2}\mathrm{e}^{(-2+\sqrt{2})t} + \frac{\sqrt[3]{4}+\sqrt[6]{2}}{2}\mathrm{e}^{-(2+\sqrt{2})t} \\ \sqrt[3]{2}\mathrm{e}^{-3t} + \frac{\sqrt[6]{2^5}-\sqrt[3]{2}}{2}\mathrm{e}^{(-2+\sqrt{2})t} - \frac{\sqrt[6]{2^5}+\sqrt[3]{2}}{2}\mathrm{e}^{-(2+\sqrt{2})t} \end{matrix}\right.$$

$$\begin{matrix} -\sqrt[3]{2}\mathrm{e}^{-3t} + \frac{\sqrt[3]{2}}{2}\mathrm{e}^{(-2+\sqrt{2})t} + \frac{\sqrt[3]{2}}{2}\mathrm{e}^{-(2+\sqrt{2})t} \\ 2\mathrm{e}^{-3t} + \frac{\sqrt{2}-1}{2}\mathrm{e}^{(-2+\sqrt{2})t} - \frac{\sqrt{2}+1}{2}\mathrm{e}^{-(2+\sqrt{2})t} \\ -\sqrt[3]{4}\mathrm{e}^{-3t} + \frac{\sqrt[3]{4}-\sqrt[6]{2}}{2}\mathrm{e}^{(-2+\sqrt{2})t} + \frac{\sqrt[3]{4}+\sqrt[6]{2}}{2}\mathrm{e}^{-(2+\sqrt{2})t} \end{matrix}$$

$$\left.\begin{matrix} -\sqrt[3]{4}\mathrm{e}^{-3t} + \frac{\sqrt[3]{4}-\sqrt[6]{2}}{2}\mathrm{e}^{(-2+\sqrt{2})t} + \frac{\sqrt[3]{4}+\sqrt[6]{2}}{2}\mathrm{e}^{-(2+\sqrt{2})t} \\ 2\sqrt[3]{2}\mathrm{e}^{-3t} + (\frac{3\sqrt[6]{2^5}}{4} - \sqrt[3]{2})\mathrm{e}^{(-2+\sqrt{2})t} - (\frac{3\sqrt[6]{2^5}}{4} + \sqrt[3]{2})\mathrm{e}^{-(2+\sqrt{2})t} \\ -2\mathrm{e}^{-3t} + (\frac{3}{2} - \sqrt{2})\mathrm{e}^{(-2+\sqrt{2})t} + (\frac{3}{2} + \sqrt{2})\mathrm{e}^{-(2+\sqrt{2})t} \end{matrix}\right],$$

and it is not difficult to prove that for any $t \geq 0$,

$$\|\Phi(t)\| := \max\{|\Phi_{ij}(t)| : i, j = 1, 2, 3\} \leq \mathrm{e}^{(-2+\sqrt{2})t} = \mathrm{e}^{\lambda_2 t}.$$

Specifically, since $\Phi_{ij}(t) \geq 0, i, j = 1, 2, 3$, for any $t \geq 0$, due to the cooperativity of A, it is enough to show that $\Psi_{ij}(t) := \frac{\Phi_{ij}(t)}{\mathrm{e}^{(-2+\sqrt{2})t}} \leq 1, i, j = 1, 2, 3$, for any $t \geq 0$. More precisely, for $\Psi_{31}(t)$, we have

$$\Psi_{31}(t) = \sqrt[3]{2}\mathrm{e}^{-(1+\sqrt{2})t} + \frac{\sqrt[6]{2^5} - \sqrt[3]{2}}{2} - \frac{\sqrt[6]{2^5} + \sqrt[3]{2}}{2}\mathrm{e}^{-2\sqrt{2}t}, \; t \geq 0,$$

and

$$\frac{\mathrm{d}\Psi_{31}(t)}{\mathrm{dt}} = -\sqrt[3]{2}(1 + \sqrt{2})\mathrm{e}^{-(1+\sqrt{2})t} + \sqrt{2}(\sqrt[6]{2^5} + \sqrt[3]{2})\mathrm{e}^{-2\sqrt{2}t}, \; t \geq 0,$$

which together with $\Psi'_{31}(0) > 0$ implies that there exists a unique local maximum point $t_0 > 0$ for $\Psi_{31}(t), t \geq 0$. In fact, it is clear that $t_0 > 0$ is also a global maximum point for $\Psi_{31}(t), t \geq 0$. By direct calculation, it follows that $t_0 > \frac{4}{5}$, and so

$$\Psi_{31}(t) \leq \Psi_{31}(t_0) \leq \sqrt[3]{2}\mathrm{e}^{-(1+\sqrt{2})\times\frac{4}{5}} + \frac{\sqrt[6]{2^5} - \sqrt[3]{2}}{2} < 1,\ t \geq 0.$$

Analogously, we can obtain that there exists a unique local maximum point $t_0 > 0$ for $\Psi_{23}(t)$, $t \geq 0$, which implies that $\Psi_{23}(t) \leq 1$, $t \geq 0$. Moreover, for $\Psi_{11}(t)$, we can easily see that

$$\Psi_{11}(t) = \mathrm{e}^{-(1+\sqrt{2})t} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathrm{e}^{-2\sqrt{2}t},\ t \geq 0,$$

and

$$\frac{\mathrm{d}\Psi_{11}(t)}{\mathrm{dt}} = -(1+\sqrt{2})\mathrm{e}^{-(1+\sqrt{2})t} + 2\mathrm{e}^{-2\sqrt{2}t} < 0,\ t \geq 0,$$

which implies that $\Psi_{11}(t) \leq 1$, $t \geq 0$. By the same method, we can obtain that the rest of the elements of $\Psi(t) = (\Psi_{ij}(t))_{d\times d}$, $t \geq 0$, except for $\Psi_{33}(t)$, are monotone with respect to t and consequently smaller than one. Finally, we consider the element $\Psi_{33}(t)$; it is clear that

$$\Psi_{33}(t) = -2\mathrm{e}^{-(1+\sqrt{2})t} + \left(\frac{3}{2} - \sqrt{2}\right) + \left(\frac{3}{2} + \sqrt{2}\right)\mathrm{e}^{-2\sqrt{2}t},\ t \geq 0,$$

and

$$\frac{\mathrm{d}\Psi_{33}(t)}{\mathrm{dt}} = 2(1+\sqrt{2})\mathrm{e}^{-(1+\sqrt{2})t} - 2\sqrt{2}\left(\frac{3}{2} + \sqrt{2}\right)\mathrm{e}^{-2\sqrt{2}t},\ t \geq 0,$$

which together with $\Psi'_{33}(0) < 0$ implies that there exists a unique local minimum point $t_0 > 0$ for $\Psi_{33}(t)$, $t \geq 0$. It is noticed that $\Psi_{33}(0) = 1$ and $\lim_{t\to\infty}\Psi_{33}(t) = \frac{3}{2} - \sqrt{2} < 1$. Then we conclude that $\Psi_{33}(t) \leq 1$, $t \geq 0$. Furthermore, we can choose $L \leq \frac{1}{16}$, $\lambda = -2 + \sqrt{2}$, and so

$$-\frac{9L}{\lambda} \leq \frac{9}{16(2-\sqrt{2})} < 1.$$

Using the small-gain theorem (Theorem 4.2), there exists a unique globally attractive random equilibrium for stochastic predator-prey system (5.6).

6. Conclusion and discussion.

In this paper, we have developed Marcondes de Freitas and Sontag's approach in random differential equations with inputs and outputs to control the problem of stochastic differential equations and proved a small-gain theorem of stochastic control under the assumptions that the output function is either order-preserving or anti-order-preserving in the usual vector order and the global Lipschitz constant of the output function is less than the absolute of the negative principal eigenvalue of the linear matrix. For the sake of convenience, let $\sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_d)$. This means that the stochastic system (2.1) has a unique globally attracting stationary solution $[\mathcal{K}(u)](\theta_t\omega)$, whose probability distribution density is the unique stationary solution of Fokker–Planck equation

(6.1)
$$p_t = \frac{1}{2}\sum_{i=1}^{d}\sigma_i^2\frac{\partial^2 p(x,t)}{\partial x_i^2} - \mathrm{div}((Ax+h(x))p),\ x \in \mathbb{R}^d,\ t > 0,\ p(x,t) \geq 0,\ \int p(x,t)dx = 1.$$

Such a stationary solution is ergodic. This reminds us to investigate a small-gain theorem in stationary solution or stationary measure manner for the stochastic system (2.1). We outline this approach as follows.

Assume that the matrix A is stable with the maximal real part $-\lambda$ of all its eigenvalues and h possesses a global Lipschitz constant L with $L < \lambda - \epsilon_0$ for a sufficiently small $\epsilon_0 > 0$. According to [30, Chapter 2, Proposition 2.10], without loss of generality, we may assume that A has the Jordan normal form with blocks along the diagonal of the form, i.e.,

$$A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_q \end{bmatrix},$$

where A_i is one of the following two types of matrices:

$$\begin{bmatrix} \lambda_i & & & \\ \varepsilon & \lambda_i & & \\ & \ddots & \ddots & \\ & & \varepsilon & \lambda_i \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} D_i & & & \\ I_\varepsilon & D_i & & \\ & \ddots & \ddots & \\ & & I_\varepsilon & D_i \end{bmatrix}, \quad i = 1, \dots, q,$$

where

$$D_i = \begin{bmatrix} a_i & -b_i \\ b_i & a_i \end{bmatrix} \quad \text{and} \quad I_\varepsilon = \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}, \ 0 < \varepsilon < \epsilon_0.$$

Define the Fokker–Planck operator by

$$LV = \frac{1}{2}\sum_{i=1}^{d} \sigma_i^2 \frac{\partial^2 V}{\partial x_i^2} + \ll \nabla V, (Ax + h(x)) \gg .$$

Here $\ll \cdot, \cdot \gg$ denotes the inner product in $\mathbb{R}^d$ and $V \in C^2(\mathbb{R}^d)$. Now let

$$V(x) := \frac{1}{2}(x_1^2 + x_2^2 + \cdots + x_d^2). \tag{6.2}$$

Then

$$\begin{aligned} LV &= \frac{1}{2}\sum_{i=1}^{d} \sigma_i^2 + \ll x, Ax \gg + \ll x, h(x) - h(0) \gg + \ll x, h(0) \gg \\ &\le \frac{1}{2}\sum_{i=1}^{d} \sigma_i^2 - (\lambda - \epsilon)|x|_2^2 + L|x|_2^2 + |h(0)|_2 \cdot |x|_2 \\ &= -(\lambda - \epsilon - L)|x|_2^2 + |h(0)|_2 \cdot |x|_2 + \frac{1}{2}\sum_{i=1}^{d} \sigma_i^2, \end{aligned}$$

where $0 < \epsilon < \epsilon_0$, $\lambda = -\max_{1 \le i \le q} \mathrm{Re}\lambda_i > L + \epsilon_0$. This implies that there is an R sufficiently large such that

$$LV \le -\frac{1}{2}(\lambda - \epsilon - L)|x|_2^2 \quad \text{for all } |x|_2 \ge R. \tag{6.3}$$

Combining (6.2), (6.3), and the Khasminskii theorem (see [22, Theorem 4.1, p. 108] and [38, p. 1163]), we know that there is a unique stationary solution for (6.1) under the condition $L < \lambda$, the stationary measure generated by this stationary solution is ergodic (see [22, Theorem 4.2, p. 110]), and the transition probability function weakly converges to the stationary measure as t tends to infinity. This stationary solution plays the same role as $[\mathcal{K}(u)](\theta_t\omega)$ in Theorem 4.2. As $\max_{1\le i\le d}|\sigma_i| \to 0$, the stationary measure sequence weakly converges to a Dirac measure at the unique equilibrium of the deterministic system without noise. This means that the stationary measure concentrates around the deterministic equilibrium when the noise intensity is small.

Example 6.1. Consider the two-dimensional stochastic system

$$\begin{aligned} dx_1 &= [-ax_1 - bx_2 + h_1(x_1, x_2)]dt + \sigma_1 dW_t^1, \\ dx_2 &= [bx_1 - ax_2 + h_2(x_1, x_2)]dt + \sigma_2 dW_t^2. \end{aligned} \tag{6.4}$$

Here a and b are positive, and $h = (h_1, h_2)^T$ has a global Lipschitz constant $L < a$. The above discussion shows that (6.4) admits a unique stationary solution, which is ergodic.

It is noticed that the matrix

$$A = \begin{bmatrix} -a & -b \\ b & -a \end{bmatrix}$$

is not cooperative. So Theorem 4.2 cannot be applied to (6.4). This may be an advantage of this method for investigating the small-gain theorem.

Comparing to the result presented in [14], the method used here can also be applied to the problem (random systems) considered in [14]. It is remarkable that the boundedness of g is not necessary for us, while the derivatives of g need to be bounded in the present work. For example, in Example 5.1, we can let

$$g_i(x_i) = \begin{cases} \frac{\pi}{2} - x_i, & x_i \le 0; \\ \frac{\pi}{2} - \arctan x_i, & x_i \ge 0. \end{cases}$$

It is easy to see that the conclusion of Example 5.1 still holds.

This result is only the first step in the study of stochastic stability, and a small-gain theorem for the stochastic control problem with multiplicative noise will be investigated in the near future; see [17].

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