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Periodic Rotating Waves of a Geodesic Curvature Flow on the Sphere

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Given a zone on the unit sphere S^2 with periodic undulating boundaries, we consider the motion of a curve in this zone which is driven by its geodesic curvature. First, we give a necessary and sufficient condition for the existence of periodic rotating waves. Then we study how the average rotating speed of the periodic rotating wave depends on the geometry of the boundaries. We find that when the period of the boundaries tends to 0, the homogenization limit of the rotating speed depends only on the maximum slope of the domain boundaries.

Keywords Geodesic curvature flow; Homogenization limit; Periodic rotating waves.

Mathematics Subject Classification 35K55; 53C44; 58J32.

1. Introduction

Let $S^2 := \{(\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta) \in \mathbb{R}^3 \mid \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \varphi \in \mathbb{R}\}$ be the unit sphere in $\mathbb{R}^3$ and $\Omega_{mn} \subset S^2$ be a zone with undulating boundaries, which will be specified below. We consider the motion of simple curve Γ_t immersed in Ω_{mn}. The law of the motion is

$$V = \kappa_g, \tag{1.1}$$

where V denotes the velocity of the curve at point $P \in \Gamma_t$ along the normal direction on the tangent plane $T_P S^2$, κ_g denotes the geodesic curvature of Γ_t at P. Domain Ω_{mn} is defined as follows: Let $b_1(s), b_2(s)$ be 2π-periodic smooth functions satisfying

$$b_i(0) = 0, \quad b_i(s) \geq 0, \quad \max_s b_i'(s) = \tan\alpha_i, \quad \min_s b_i'(s) = -\tan\beta_i \ \ (i = 1, 2).$$

for some $\alpha_i, \beta_i \in (0, \frac{\pi}{2})$. Given $\theta_0 \in (0, \frac{\pi}{2})$, for any $m, n \in \mathbb{N}$ define

$$b_{1m}(s) := \frac{\cos\theta_0}{m} \cdot b_1(ms), \quad b_{2n}(s) := \frac{\cos\theta_0}{n} \cdot b_2(ns).$$

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If we use spherical coordinate (θ, φ) to denote the point $(\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta) \in S^2$, then zone Ω_{mn} is defined as

$$\Omega_{mn} := \{(\theta, \varphi) \in S^2 \mid -\theta_0 - b_{1m}(\varphi) < \theta < \theta_0 + b_{2n}(\varphi), \varphi \in \mathbb{R}\}.$$

Denote the boundaries $\theta = -\theta_0 - b_{1m}(\varphi)$ and $\theta = \theta_0 + b_{2n}(\varphi)$ by $\partial_1\Omega_{mn}$ and $\partial_2\Omega_{mn}$, respectively (see Figure 1).

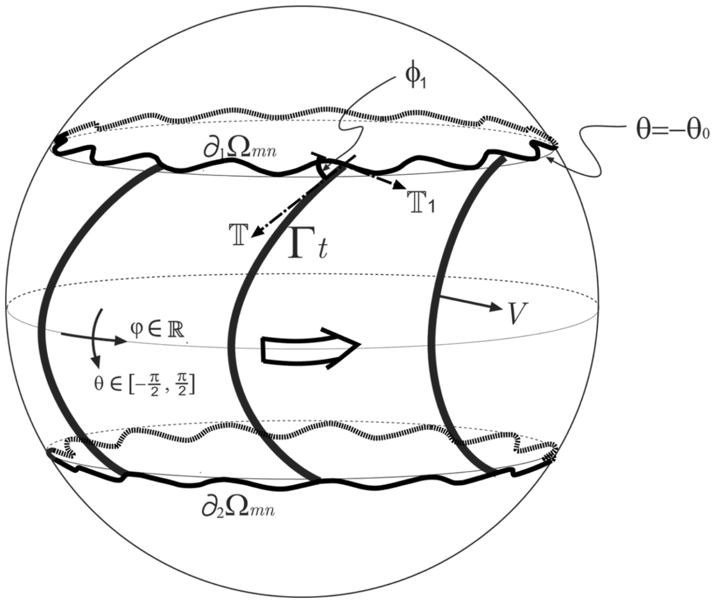

Figure 1. Zone with undulating boundaries on the unit sphere.

By a solution of (1.1) we mean a time-dependent simple curve $\Gamma_t \subset \Omega_{mn}$ which satisfies (1.1) and contacts the boundaries $\partial_i\Omega_{mn}$ with angle $\phi_i \in (0, \frac{\pi}{2})$ $(i = 1, 2)$, respectively (see details below). In this paper we are interested in curves which rotate in φ-direction periodically, as well as their average rotating speeds.

Various kinds of curvature flows in manifolds has been studied recently. To name only a few, Gage and Hamilton (1986); Chou and Zhu (2001) and references therein studied mean curvature flows on the plane; Alikakos and Freire (2003); Wang (2001), etc. studied mean curvature flows in manifolds; Andrews (2000), etc. studied Gauss curvature flows. Besides these, there are also some studies about geodesic flows under Ricci curvature, geodesic curvature flows, etc.

Most of these works concern the existence and asymptotic behavior of the flows. As far as we know, very little is known about (periodic) traveling/rotating surfaces in manifolds, though traveling/rotating wave solutions of reaction diffusion equations in Euclidean spaces have been studied a lot (cf. Xin, 2000 and references therein).

On the other hand, it is known that some curvature flows arise in physics, chemistry, biology and material science. For example, mean curvature flows appears in Belousov-Zhabotinskii reactions, Gauss curvature flows appears in the study of rolling stones. Sometimes, the reactions or phenomena may take place in a domain with reticulated structures (cf. Cioranescu and Saint Jean Paulin, 1999), and so we should consider problem in a domain with undulating boundaries like Ω_{mn}.

In this paper we study a geodesic curvature flow in $\Omega_{mn} \subset S^2$, prove the existence and uniqueness of *periodic rotating wave*, also we estimate the average rotating speed for homogenization limit problem. We believe that periodic traveling or rotating waves for other curvature flows in some other manifolds can be studied in a similar way.

To avoid sign confusion, the normal vector ν to Γ_t on TS^2 will always be chosen to be the increasing direction of φ, the sign of the normal velocity V and the geodesic curvature κ_g will be understood in accordance with this choice (see details below). We will consider the case that each curve is the graph of a function $\varphi = u(\theta, t)$, that is, the curve is $\Gamma_t = \{(\theta, u(\theta, t))\} | -\theta_0 - b_{1m}(u) \le \theta \le \theta_0 + b_{2n}(u)\} \subset \Omega_{mn}$. The unit tangent vector (pointing to the positive direction of θ) of Γ_t is

$$\mathbb{T} = \frac{1}{\sqrt{1 + u_\theta^2 \cos^2\theta}} \begin{pmatrix} -\sin\theta\cos u - \cos\theta\sin u \cdot u_\theta \\ -\sin\theta\sin u + \cos\theta\cos u \cdot u_\theta \\ \cos\theta \end{pmatrix}.$$

On the other hand, for a curve $\{(\theta(s), \varphi(s))\}_{s=s_1}^{s=s_2} \subset \Omega_{mn}$ with parameter s, its geodesic curvature is

$$\kappa_g = \cos\theta \cdot \det\begin{pmatrix} \frac{d\theta}{ds} & \frac{d^2\theta}{ds^2} + \sin\theta\cos\theta\left(\frac{d\varphi}{ds}\right)^2 \\ \frac{d\varphi}{ds} & \frac{d^2\varphi}{ds^2} - 2\tan\theta\ \frac{d\theta}{ds}\frac{d\varphi}{ds} \end{pmatrix}.$$

So for curve $\varphi = u(\theta, t)$, its geodesic curvature is

$$\kappa_g = \cos\theta \cdot \frac{u_{\theta\theta} - 2\tan\theta \cdot u_\theta - \sin\theta\cos\theta \cdot {u_\theta}^3}{(1 + u_\theta^2\cos^2\theta)^{3/2}}$$

since when we use arc length s as parameter we have

$$\frac{d\theta}{ds} = \frac{1}{\sqrt{1 + {u_\theta}^2\cos^2\theta}}, \quad \frac{d\varphi}{ds} = \frac{u_\theta}{\sqrt{1 + {u_\theta}^2\cos^2\theta}},$$
$$\frac{d^2\theta}{ds^2} = \frac{u_\theta^2\sin\theta\cos\theta - u_\theta u_{\theta\theta}\cos^2\theta}{(1 + u_\theta^2\cos^2\theta)^2}, \quad \frac{d^2\varphi}{ds^2} = \frac{u_{\theta\theta} + u_\theta^3\sin\theta\cos\theta}{(1 + u_\theta^2\cos^2\theta)^2}.$$

The unit normal vector to Γ_t on TS^2 is

$$\nu = \frac{1}{\sqrt{1 + u_\theta^2\cos^2\theta}} \begin{pmatrix} \sin\theta\cos\theta\cos u \cdot u_\theta - \sin u \\ \sin\theta\cos\theta\sin u \cdot u_\theta + \cos u \\ -\cos^2\theta \cdot u_\theta \end{pmatrix},$$

and so

$$V = \begin{pmatrix} -\cos\theta\sin u \cdot u_t \\ \cos\theta\cos u \cdot u_t \\ 0 \end{pmatrix} \cdot \nu = \frac{\cos\theta \cdot u_t}{\sqrt{1 + u_\theta^2\cos^2\theta}}.$$

Thus (1.1) is equivalent to

$$u_t = \frac{u_{\theta\theta} - 2\tan\theta \cdot u_\theta - \sin\theta\cos\theta \cdot {u_\theta}^3}{1 + u_\theta^2 \cos^2\theta} \quad \text{for } \eta_1(t) < \theta < \eta_2(t),\ t > 0, \tag{1.2}$$

where $\eta_i(t)$ (with $\eta_1(t) < 0 < \eta_2(t)$) denote the θ-coordinates of the end points of Γ_t lying on $\partial_i\Omega_{mn}$, i.e., $\eta_1(t) = -\theta_0 - b_{1m}(u(\eta_1(t), t))$, $\eta_2(t) = \theta_0 + b_{2n}(u(\eta_2(t), t))$.

Denote the unit tangent vector of $\partial_i\Omega_{mn}$ by $\mathbb{T}_i$ $(i = 1, 2)$ (both toward the increasing direction of φ, see Figure 1), then

$$\mathbb{T}_2 = \frac{1}{\sqrt{{b'_{2n}}^2 + \cos^2\theta}} \begin{pmatrix} -b'_{2n}\sin\theta\cos\varphi - \cos\theta\sin\varphi \\ -b'_{2n}\sin\theta\sin\varphi + \cos\theta\cos\varphi \\ -b'_{2n}\cos\theta \end{pmatrix} \quad (\theta = \theta_0 + b_{2n}(\varphi)),$$

$\mathbb{T}_1$ is calculated similarly. Hereafter when we say a curve Γ_t contacts $\partial_i\Omega_{mn}$ with angle $\phi_i \in (0, \frac{\pi}{2})$, we mean $\cos\phi_1 = -\mathbb{T}\cdot\mathbb{T}_1$ on $\partial_1\Omega_{mn}$, $\cos\phi_2 = \mathbb{T}\cdot\mathbb{T}_2$ on $\partial_2\Omega_{mn}$. These equalities are nothing but our *boundary conditions*, which can be expressed as

$$u_\theta(\theta, t) = -\frac{\cos\phi_1\cos\theta - b'_{1m}(u)\sin\phi_1}{\cos\theta(\sin\phi_1\cos\theta + b'_{1m}(u)\cos\phi_1)} =: \mathscr{F}_1(u) \quad \text{for } \theta = \eta_1(t),$$

$$u_\theta(\theta, t) = \frac{\cos\phi_2\cos\theta - b'_{2n}(u)\sin\phi_2}{\cos\theta(\sin\phi_2\cos\theta + b'_{2n}(u)\cos\phi_2)} =: \mathscr{F}_2(u) \quad \text{for } \theta = \eta_2(t). \tag{1.3}$$

Let $\Omega_0 = \{(\theta, \varphi) \in S^2 | -\theta_0 < \theta < \theta_0\}$ be a trivial zone which is the formal limit of Ω_{mn} as $m, n \to \infty$. For Ω_0 we consider problem (1.2) with boundary condition

$$u_\theta(-\theta_0, t) = -\frac{\cot\phi_1}{\cos\theta_0} \quad u_\theta(\theta_0, t) = \frac{\cot\phi_2}{\cos\theta_0}. \tag{1.4}$$

As we will see in Lemma 2.1 below, there exists a unique $c_0 = c_0(\phi_1, \phi_2) > 0$ such that problem (1.2), (1.4) has a unique *rotating wave* $U_0(\theta) + c_0 t$, which has a fixed profile and a constant rotating speed c_0.

On the other hand, in Ω_{mn}, as Γ_t propagates, its shape and speed fluctuate along with the undulation of the boundaries of Ω_{mn}. In such a situation, we adopt a generalized definition of rotating waves. Denote the greatest common divisor of m and n by (m, n) ($(m, n) = 1$ if the least common multiple of m and n is mn), then both boundaries of Ω_{mn} have same period $\frac{2\pi}{(m,n)}$. A solution $U_{mn}(\theta, t)$ of (1.2)–(1.3) is called a *periodic rotating wave* if it satisfies

$$U_{mn}(\theta, t + T_{mn}) = U_{mn}(\theta, t) + \frac{2\pi}{(m, n)} \quad \text{for some } T_{mn} > 0.$$

The *average rotating speed* of $U_{mn}(\theta, t)$ is defined by $c_{mn} = \frac{2\pi}{(m,n)T_{mn}}$. In what follows we concentrate on periodic rotating waves with average speed of order 1 as m, $n \to \infty$.

Before stating our results, we give some assumptions.

$$\phi_1 + \alpha_1 + \phi_2 + \alpha_2 < \pi. \tag{H1}$$

$$\alpha_2 + \beta_i < \phi_i, \quad \beta_2 + \beta_i < \phi_i, \quad \alpha_1 + \beta_i < \phi_i, \quad \beta_1 + \beta_i < \phi_i \ (i = 1, 2). \tag{H2}$$

Roughly speaking these conditions require that α_i and β_i are not large, that is, the undulation of the boundaries is gradual. One example is that $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $\phi_1 = \phi_2 = \phi$, in this case, our assumptions reduce to $\phi + \alpha < \frac{\pi}{2}$, $\alpha + \beta < \phi$, $2\beta < \phi$. (H1) guarantees the existence of lower solutions rotating in a positive speed (see Lemma 2.2 below). Conditions $\alpha_i + \beta_i < \frac{\pi}{2}$ in (H2) exclude the possible singularity that the curve touches the boundaries besides at two endpoints, otherwise the curve may split into multiple components; $\beta_i < \phi_i$ in (H2) ensure that $|u_\theta|$ is bounded on the boundaries; Also (H2) guarantees that we can transform $u(\theta, t)$ into a new variable $v(z, t)(z \in (0, 1))$ rigorously, and use v to discuss the global existence, just as we did in Matano et al. (2006). However we do not think (H2) is essential in the homogenization argument. In fact, we believe that Theorem 2 below remains true even if the curve develops singularities near the boundaries.

Our existence result is

Theorem 1. *Assume* (H2) *holds, m and n are sufficiently large. Then* (1.2)–(1.3) *has a periodic rotating wave if and only if* (H1) *holds. Moreover, the periodic rotating wave is unique up to time-shift when it exists.*

In fact, the unique periodic rotating wave is asymptotically stable (see, for example, Matano et al., 2006).

Another aim in this paper is to study how the periodic rotating wave and its average speed depend on the geometry of the boundaries. This problem is important in the study of traveling/rotating waves and very little is known so far. We estimate the average rotating speed c_{mn} and determine its homogenization limit as $m, n \to \infty$.

Theorem 2. *Assume* (H1) *and* (H2) *hold.*

(i) *If m and n are sufficiently large, then there exists $C > 0$ independent of m, n such that*

$$c^* - \max\left\{\frac{C}{m}, \frac{C}{n}\right\} < c_{mn} < c^* + \max\left\{\frac{C}{\sqrt{m}}, \frac{C}{\sqrt{n}}\right\} < c_0, \tag{1.5}$$

where $c^* = c^*(\alpha_1, \alpha_2, \phi_1, \phi_2) > 0$ is given by the unique solution $(c^*, \Phi^*(\theta; c^*))$ of

$$\begin{cases} c = \dfrac{\Phi_{\theta\theta} - 2\tan\theta \cdot \Phi_\theta - \sin\theta\cos\theta \cdot \Phi_\theta^3}{1 + {\Phi_\theta}^2 \cos^2\theta}, & -\theta_0 < \theta < \theta_0, \\ \Phi_\theta(-\theta_0) = -\dfrac{\cot(\phi_1 + \alpha_1)}{\cos\theta_0}, \quad \Phi_\theta(\theta_0) = \dfrac{\cot(\phi_2 + \alpha_2)}{\cos\theta_0}, & \Phi(0) = 0, \end{cases} \tag{1.6}$$

c_0 is given by the unique rotating wave $U_0(\theta) + c_0 t$ of (1.2), (1.4) *in Ω_0.*

(ii) *As $m, n \to \infty$, periodic rotating wave $U_{mn}(\theta, t)$ converges to $\Phi^*(\theta; c^*) + c^* t + C$ in $C^{2,1}_{\mathrm{loc}}((-\theta_0, \theta_0) \times \mathbb{R})$, where C is a constant.*

$c_{mn} < c_0$ in (1.5) implies that boundary undulation always lowers the speed of the rotating wave, $c^* < c_0$ implies that the effect of spatial inhomogeneity of c_{mn} is left to the homogenization limit. Moreover, it is a surprising result that the homogenized speed c^* depends *only* on the maximum slope of the domain boundaries (corresponding to angles α_1, α_2). The effect coming from other information of the boundaries should appear in the error $\max\{\frac{C}{\sqrt{m}}, \frac{C}{\sqrt{n}}\}$.

In Matano et al. (2006), we studied a mean curvature flow in an undulating band domain, obtained periodic traveling waves and estimated its average speed. Problem in that paper is different from the present one in several points. First, since the boundaries of a zone on S^2 have period 2π anyway, above theorems are true even if $(m, n) = 1$. In such a case, the period of periodic rotating wave is 2π, not necessarily to be small as that in Matano et al. (2006). Second, the problems and backgrounds are different. A mean curvature flow in an *unbounded* band domain is reduced from a traveling front or a traveling pulse, but problem in this paper is about geodesic curvature flows in *bounded* zone on the sphere, which is a geometrical problem.

In Section 2, we prove Theorem 1. First, we give a *global solution* of an initial-boundary value problem; Next, we use the global solution to construct an *entire solution* by using *renormalization method*; Then we prove the uniqueness (upto a time-shift) of entire solution, this immediately implies the existence and uniqueness of *periodic rotating wave*. Necessity of (H1) is proved in Subsection 2.5.

In Section 3 we prove Theorem 2: estimate the average rotating speed by constructing a precise upper solution.

2. Existence and Uniqueness of Periodic Rotating Wave

2.1. *Global Solutions of Initial-Boundary Value Problem*

In this part, we give a global solution of (1.2)–(1.3) with some initial data. As preliminaries, we first study rotating waves in trivial zones (zones with flat boundaries). Denote

$$\theta_1 = \theta_0 + \max_s b_{1m}(s), \quad \theta_2 = \theta_0 + \max_s b_{2n}(s).$$

For $B_1, B_2 \in \mathbb{R}$, consider the following problem

$$\begin{cases} c = \dfrac{\Phi_{\theta\theta} - 2\tan\theta \cdot \Phi_\theta - \sin\theta\cos\theta \cdot {\Phi_\theta}^3}{1 + \Phi_\theta^2 \cos^2\theta}, & -\theta_1 < \theta < \theta_2, \\ \Phi_\theta(-\theta_1) = -B_1, \quad \Phi_\theta(\theta_2) = B_2, \quad \Phi(0) = 0. \end{cases} \tag{2.1}$$

If c and $\Phi(\theta)$ satisfy (2.1), then we call the pair $(c, \Phi(\theta))$ to be a solution of (2.1). This solution determines a rotating wave $\Phi(\theta) + ct$ of (1.2) in zone $\{(\theta, \varphi) | -\theta_1 < \theta < \theta_2\}$. Assume the graph of $\Phi(\theta)$ contacts $\theta = -\theta_1$ with angle γ_1, contacts $\theta = \theta_2$ with angle γ_2, then $B_1 = \frac{\cot\gamma_1}{\cos\theta_1}$, $B_2 = \frac{\cot\gamma_2}{\cos\theta_2}$.

Lemma 2.1. (2.1) *has a unique solution* $(c, \Phi(\theta))$ *with* $c > 0$ *provided* (i) $B_1 > 0, B_2 > 0$; *or* (ii) $B_1 + B_2 > 0$ *and* m, n *are large. Moreover,* $c = c(B_1, B_2)$ *is increasing in both* B_1 *and* B_2.

Proof. Set $\Psi(\theta) = \Phi_\theta(\theta)$, and consider the following initial value problem

$$\begin{cases} \Psi' = c(1 + \Psi^2 \cos^2 \theta) + 2\Psi \tan\theta + \Psi^3 \sin\theta\cos\theta, & \theta \geq -\theta_1, \\ \Psi(-\theta_1) = -B_1. \end{cases} \tag{2.2}$$

For each c, denote the solution of (2.2) by $\Psi(\theta; c)$. It is clear that $\Psi(\theta; c)$ is strictly increasing in c, and depends on c continuously.

First, when $c > 0$ is very large, we have $\Psi(\theta_2; c) > B_2$. Next, when $c = 0$, the solution of (2.2) is

$$\Psi(\theta; 0) = \frac{d}{\cos\theta\sqrt{\cos^2\theta - d^2}} \ (\theta \in [-\theta_1, \theta_2]) \quad \text{with } d = \frac{-B_1 \cos^2\theta_1}{\sqrt{1 + B_1^2 \cos^2\theta_1}}.$$

In case (i), $d < 0$ and $\Psi(\theta_2; 0) < 0 < B_2$. In case (ii), since m and n are large, we have

$$d = \frac{-B_1 \cos^2\theta_0}{\sqrt{1 + B_1^2 \cos^2\theta_0}} + O\left(\frac{1}{m}\right), \quad \Psi(\theta_2; 0) = -B_1 + O\left(\frac{1}{m}\right) + O\left(\frac{1}{n}\right) < B_2.$$

Therefore in each case there exists a unique $c > 0$ such that $\Psi(\theta_2; c) = B_2$, which gives a solution of (2.1): $\Phi(\theta) = \int_0^\theta \Psi(\varsigma; c) d\varsigma$.

Also, one can see from the proof that $c = c(B_1, B_2)$ is increasing in both B_1 and B_2. $\square$

Now we use this lemma to construct lower and upper solutions of (1.2)–(1.3). We will use positive constants like $\zeta_1, \zeta_2, \mu, \mu_1, \ldots, C, C_1, \ldots$, which may be different from line to line and may depend on some of $b_{1m}, b_{2n}, \theta_0, m, n$, but are independent of t.

Lemma 2.2. *Assume* (H1) *holds and m, n are large, then* (1.2)–(1.3) *has lower solution $\Phi^l(\theta) + c^l t$ and upper solution $\Phi^u(\theta) + c^u t$. Moreover, $c^u > c_0 > c^l > 0$, where c_0 is given by the solution of* (1.2) *and* (1.4).

Proof. Denote $\varepsilon = \max\{\frac{1}{m}, \frac{1}{n}\}$, and

$$B_1^u = \frac{\cot(\phi_1 - \beta_1)}{\cos\theta_0}, \quad B_2^u = \frac{\cot(\phi_2 - \beta_2)}{\cos\theta_0}, \quad B_1^l = \frac{\cot(\phi_1 + \alpha_1)}{\cos\theta_0}, \quad B_2^l = \frac{\cot(\phi_2 + \alpha_2)}{\cos\theta_0}.$$

Consider (2.1) with $B_1 = B_1^l - \zeta_1\varepsilon$, $B_2 = B_2^l - \zeta_2\varepsilon$. (H1) implies that (i) or (ii) of Lemma 2.1 holds. Hence (2.1) has a unique solution $(c^l, \Phi^l(\theta))$, which determines a rotating wave $\Phi^l(\theta) + c^l t$ in zone $\{(\theta, \varphi) | -\theta_0 \leq \theta \leq \theta_0, \ \varphi \in \mathbb{R}\}$.

Denote by $\eta_1^l(t)$ (resp. $\eta_2^l(t)$) the θ-coordinate of the point where the graph of $\Phi^l(\theta) + c^l t$ meets $\partial_1\Omega_{mn}$ (resp. $\partial_2\Omega_{mn}$). Then $\eta_1^l(t) + \theta_1 = O(\frac{1}{m})$ and we have

$$\Phi_\theta^l(\eta_1^l(t)) \geq \Phi_\theta^l(-\theta_1) - \frac{C}{m} = -B_1^l + \zeta_1\varepsilon - \frac{C}{m} \quad \text{for some } C > 0.$$

On the other hand, since $\frac{\partial\mathcal{F}_1(u)}{\partial(b'_{1m})} > 0$ and $b'_{1m}(u) \leq \cos\theta_0 \tan\alpha_1$ we have

$$\mathcal{F}_1(\Phi^l(\eta_1^l)) = \frac{b'_{1m}(u)\sin\phi_1 - \cos\phi_1\cos\eta_1^l}{\cos\eta_1^l(\sin\phi_1\cos\eta_1^l + b'_{1m}(u)\cos\phi_1)}$$

$$
\begin{aligned}
&\le \frac{\cos\theta_0 \tan\alpha_1 \sin\phi_1 - \cos\phi_1 \cos\eta_1^l}{\cos\eta_1^l(\sin\phi_1 \ \cos\eta_1^l + \cos\theta_0 \tan\alpha_1 \cos\phi_1)} \\
&\le \frac{\cos\theta_0 \tan\alpha_1 \sin\phi_1 - \cos\phi_1 \cos\theta_0}{\cos\theta_0(\sin\phi_1 \ \cos\theta_0 + \cos\theta_0 \tan\alpha_1 \cos\phi_1)} + \frac{C}{m} \\
&= -B_1^l + \frac{C}{m} \le \Phi_\theta^l(\eta_1^l(t))
\end{aligned}
$$

provided ζ_1 is large. Similarly, $\Phi_\theta^l(\eta_2^l) \le \mathcal{F}_2(\Phi^l(\eta_2^l))$ provided ζ_2 is large.

Therefore $\Phi^l(\theta) + c^l t$ (for θ with $(\theta, \Phi^l(\theta) + c^l t) \in \Omega_{mn}$) is a lower solution of (1.2)–(1.3).

In a similar way as above, (2.1) with $B_1 = B_1^u + \zeta_1\varepsilon$, $B_2 = B_2^u + \zeta_2\varepsilon$ has solution $(c^u, \Phi^u(\theta))$, which determines a rotating wave $\Phi^u(\theta) + c^u t$. For large ζ_1, ζ_2 with order $O(1)$, $\Phi^u(\theta) + c^u t$ is an upper solution of (1.2)–(1.3).

Finally, when m and n are large, Lemma 2.1 implies that $c^u > c_0 > c^l > 0$. $\square$

Using above lemmas and using a similar argument as proving Lemmas 3.18 and 3.19 in Matano et al. (2006) we have

Lemma 2.3. *Assume* (H1) *and* (H2) *hold, m and n are sufficiently large. Then there exists appropriate smooth function $u_0(x)$ such that,* (1.2)–(1.3) *with initial data $u_0(x)$ has a unique, time-global solution $u(\theta, t)$. Moreover,*

(i) $u \in C^{2+\mu, 1+\frac{\mu}{2}}(\overline{Q})$ *for some* $\mu \in (0, 1)$, *where* $Q := \{(\theta, t) \mid \eta_1(t) < \theta < \eta_2(t), t > 0\}$;

(ii) *There exists* $C > 0$ *such that*

$$
\|u_\theta(\theta, t)\|_{C(\overline{Q_T})} \le C \quad \textit{and} \quad \|u(\theta, t)\|_{C^{2+\mu, 1+\frac{\mu}{2}}(\overline{Q_T})} \le c^u T + C,
$$

where $Q_T := \{(\theta, t) \mid \eta_1(t) < \theta < \eta_2(t) \text{ and } 0 < t \le T\}$ *for any* $T > 0$;

(iii) *There exists* $C > 0$ *such that for any* $t \ge 0$, $t_0 \ge 0$,

$$
c^l t - C \le u(\theta, t + t_0) - u(\theta, t_0) \le c^u t + C;
$$

(iv) $u_t(\theta, t) \ge 0$ *for* $(\theta, t) \in \overline{Q}$.

2.2. *Existence of Entire Solutions*

A solution of (1.2)–(1.3) defined on $t \in (-\infty, \infty)$ is called an *entire solution*. We use *renormalization method* to show the existence of entire solutions.

Lemma 2.4. *Assume* (H1) *and* (H2) *hold, m and n are sufficiently large. Then* (1.2)–(1.3) *has entire solution $U(\theta, t)$ such that U satisfies* (i) *and* (ii) *of Lemma* 2.3, *$U_t(\theta, t) > 0$ and*

$$
c^l t - C \le U(\theta, t + t_0) - U(\theta, t_0) \le c^u t + C \quad \textit{for } t_0 \in \mathbb{R} \textit{ and } t \ge 0. \tag{2.3}
$$

Proof. Let u be the global solution of (1.2)–(1.3) obtained in Lemma 2.3. Take $t_k \to \infty$ in the following way: $\max_\theta u(\theta, t_k) = \frac{2k\pi}{(m,n)}$ for $k = k_0, k_0 + 1, \dots,$ where k_0

is a large integer. Set

$$u_k(\theta, t) := u(\theta, t + t_k) - k \cdot \frac{2\pi}{(m, n)},$$

then u_k also satisfies (1.2)–(1.3) for $-t_k \le t < \infty$, $\max_\theta u_k(\theta, 0) = 0$ and $\frac{\partial u_k}{\partial t} \ge 0$.

By (ii) and (iii) of Lemma 2.3, there exists $C > 0$ such that

$$c^l t - C \le u_k(\theta, t) \le c^u t + C \quad \text{for } -t_k \le t < \infty. \tag{2.4}$$

For any given $T > 0$ and for any k with $t_k > T$, consider the problem about u_k on $[-T, T]$. One can see that u_k satisfies (i) and (ii) of Lemma 2.3 for some μ and some $C = C_1$, where $\mu \in (0, 1)$ and $C_1 > 0$ are independent of T and k. So there exist $\mu_1 \in (0, \mu)$, $U(\theta, t) \in C^{2+\mu_1, 1+\frac{\mu_1}{2}}(\overline{Q_T^U})$ and a subsequence $k_j \to \infty$ $(j \to \infty)$ such that

$$u_{k_j}(\theta, t) \to U(\theta, t) \quad \text{in } C^{2+\mu_1, 1+\frac{\mu_1}{2}}(\overline{Q_T^U}),$$

where $Q_T^U := \{(\theta, t) | t \in [-T, T], \theta \quad \text{with } (\theta, U(\theta, t)) \in \Omega_{mn}\}$.

Taking $T \to \infty$ and using Cantor's diagonal argument, one find that there exist a subsequence of $\{k_j\}$, still write it as $\{k_j\}$, and $U(\theta, t) \in C^{2+\mu_1, 1+\frac{\mu_1}{2}}(\overline{Q_\infty^U})$ (with $Q_\infty^U = \lim_{T\to\infty} Q_T^U$) such that, for any $T > 0$,

$$u_{k_j}(\theta, t) \to U(\theta, t) \quad \text{in } C^{2+\mu_1, 1+\frac{\mu_1}{2}}(\overline{Q_T^U}).$$

Hence U is an entire solution of (1.2)–(1.3), which satisfies (i) and (ii) of Lemma 2.3 for $\mu = \mu_1$, $C = C_1$. Also, it is easy to see that U satisfies (2.3) for some $C > 0$, $\max_\theta U(\theta, 0) = 0$.

Finally, $U_t(\theta, t) > 0$ follows from strong maximum principle and $U_t(\theta, t) \ge 0$.

□

2.3. *Uniqueness of Entire Solution*

Assume $U(\theta, t)$ and $W(\theta, t)$ are two $C^{2+\mu, 1+\frac{\mu}{2}}$ entire solutions of (1.2)–(1.3), then by Lemmas 2.3 and 2.4 as well as their proofs, one can see that they both satisfy the properties in Lemma 2.4. We shall prove that U is a time-shift of W. Define

$$\Lambda_{U,W}(t) := \inf\{\Lambda > 0 \mid \exists\, a \in \mathbb{R} \quad \text{such that } U(\theta, t + a) \le W(\theta, t) \le U(\theta, t + a + \Lambda)\}.$$

In Matano et al. (2006) we proved

Lemma 2.5. (i) *$\Lambda_{U,W}(t)$ is monotone decreasing, and there exists Λ_M such that $0 \le \Lambda_{U,W}(t) \le \Lambda_M$ for $t \in \mathbb{R}$.*

(ii) *If $\Lambda_{U,W}(t_0) = 0$ for some t_0, then there exists $a \in \mathbb{R}$ such that $U(\theta, t + a) \equiv W(\theta, t)$ for $t \ge t_0$. If $\Lambda_{U,W}(t_0) > 0$ for some t_0, then $\Lambda_{U,W}(t) > 0$ and is strictly decreasing for $t < t_0$.*

Lemma 2.6. *$W(\theta, t)$ is a time-shift of $U(\theta, t)$.*

Proof. We only need to show that $\Lambda_{U,W}(t) = 0$ for all $t \in \mathbb{R}$. Suppose $\Lambda_{U,W}(t_0) > 0$ for some t_0. By the monotonicity and boundedness of $\Lambda_{U,W}(t)$, we have $\lim_{t\to-\infty} \Lambda_{U,W}(t) =: \bar{\Lambda}$ for some $\bar{\Lambda}$ satisfying $0 < \Lambda_{U,W}(t_0) < \bar{\Lambda} \le \Lambda_M$.

Set

$$l_k = \left[\frac{\max_\theta U(\theta, -k) \cdot (m, n)}{2\pi}\right]$$

and define

$$U_k(\theta, t) := U(\theta, t-k) - l_k \frac{2\pi}{(m, n)}, \quad W_k(\theta, t) := W(\theta, t-k) - l_k \frac{2\pi}{(m, n)}.$$

Then both U_k and W_k satisfy (2.4), and similar discussion as in the proof of Lemma 2.4 shows that there exist a subsequence $\{k_j\}$ with $k_j \to \infty$ $(j \to \infty)$, $\mu_2 \in (0, 1)$ and U_∞, W_∞ such that $U_\infty, W_\infty \in C^{2+\mu_2, 1+\frac{\mu_2}{2}}$ are entire solutions of (1.2)–(1.3), and, as $j \to \infty$

$$U_{k_j}(\theta, t) \to U_\infty(\theta, t) \quad \text{for } t \in \mathbb{R} \text{ and } \theta \quad \text{with } (\theta, U_\infty(\theta, t)) \in \Omega_{mn},$$
$$W_{k_j}(\theta, t) \to W_\infty(\theta, t) \quad \text{for } t \in \mathbb{R} \text{ and } \theta \quad \text{with } (\theta, W_\infty(\theta, t)) \in \Omega_{mn}.$$

It is easily seen that $\Lambda_{U_\infty, W_\infty}(t) = \lim_{j\to\infty} \Lambda_{U_{k_j}, W_{k_j}}(t)$.

On the other hand, since $\Lambda_{U_{k_j}, W_{k_j}}(t) = \Lambda_{U,W}(t - k_j)$, we have $\Lambda_{U_\infty, W_\infty}(t) = \lim_{j\to\infty} \Lambda_{U,W}(t-k_j) = \bar{\Lambda}$, that is, $\Lambda_{U_\infty, W_\infty}(t) \equiv \bar{\Lambda}(t \in \mathbb{R})$. Using (ii) of Lemma 2.5 to functions U_∞ and W_∞ we see that this is true only if $\bar{\Lambda} = 0$, contradicts to $\bar{\Lambda} > \Lambda_{U,W}(t_0) > 0$.

Therefore, $\Lambda_{U,W}(t) = 0$ $(\forall t \in \mathbb{R})$, and so there exists a_0 such that $U(\theta, t + a_0) \equiv W(\theta, t)$ for $t \in \mathbb{R}$. □

2.4. *Existence and Uniqueness of Periodic Rotating Wave*

Proof of Theorem 1. ((H1) $\Rightarrow$ *Existence and Uniqueness of Periodic Rotating Wave*). In the previous subsection we obtain an entire solution $U(\theta, t)$ of (1.2)–(1.3). Clearly, $U(\theta, t) + \frac{2\pi}{(m,n)}$ is also an entire solution, Lemma 2.6 implies that $U(\theta, t) + \frac{2\pi}{(m,n)}$ is a time-shift of $U(\theta, t)$, i.e., there exists $T_{mn} > 0$ such that

$$U(\theta, t) + \frac{2\pi}{(m, n)} = U(\theta, t + T_{mn}) \quad \text{for } t \in \mathbb{R} \text{ and } \theta \text{ with } (\theta, U(\theta, t)) \in \Omega_{mn}.$$

In other words, $U(\theta, t)$ is a periodic rotating wave.

The uniqueness follows from the uniqueness of entire solution (Lemma 2.6). □

2.5. *Necessity of (H1)*

Proof of Theorem 1 *(Existence of Periodic Rotating Wave* $\Rightarrow$ (H1)*).* We show that (H1) is a necessary condition for the existence of a periodic rotating wave with average rotating speed $O(1)$. Without loss of generality, assume $n = m^p$ for some $p \ge 1$.

(i) In case $\alpha_1 + \phi_1 + \alpha_2 + \phi_2 = \pi$, we show that the average speed c_{mn} is less than $O(m^{-1/4})$, which is not the case we are interested in this paper. We prove this by showing that the periodic rotating wave use time longer than $O(m^{-3/4})$ to pass one period of $\partial_1\Omega_{mn}$, whose φ-length is $\frac{2\pi}{m}$. (see Figure 2).

From the proof of Lemma 2.1 it is not difficult to see that (1.6) has unique solution $(0, \Phi(\theta; 0))$ in case $\alpha_1 + \phi_1 + \alpha_2 + \phi_2 = \pi$. The solution of (2.1) depends on θ_i and B_i continuously, so when we choose $B_1 = B_1^l + \frac{\zeta_1}{m}$, $B_2 = B_2^l + \frac{\zeta_2}{m}$ ($\zeta_i = O(1)$), (2.1) has a unique solution $(\epsilon, \Phi(\theta; \epsilon))$ with $\epsilon = O(\frac{1}{m})$.

Without loss of generality, suppose that the graph of $\Phi(\theta; \epsilon)$ contacts $\partial_1\Omega_{mn}$ at point $P_1 = (-\theta_0 - b_{1m}(s_1), s_1)$ for s_1 satisfying $b'_{1m}(s_1) = \cos\theta_0 \tan\alpha_1$, suppose the graph of $\Phi(\theta; \epsilon)$ contacts $\partial_2\Omega_{mn}$ at $P'_2 = (\theta_0 + b_{1n}(s'_2), s'_2)$ for some s'_2. Choose $s_2 \in (s'_2, s'_2 + \frac{2\pi}{n})$ such that $b'_{1n}(s_2) = \cos\theta_0 \tan\alpha_2$. Denote $P_2 = (\theta_0 + b_{1n}(s_2), s_2)$.

For convenience, we use some other notations. Denote the period of $\partial_1\Omega_{mn}$ containing P_1 by I, the segment of I with $\varphi \in [s_1 - m^{-3/2}, s_1 + m^{-3/2}]$ by I^*. Denote the period of $\partial_2\Omega_{mn}$ containing P_2 by II_1, the segment of II_1 with $\varphi \in [s_2 - n^{-q}, s_2 + n^{-q}]$ by II_1^*, where

$$q = \begin{cases} \dfrac{7}{6}, & \text{if } p \le \frac{3}{2}; \\ 1 + \dfrac{1}{4p}, & \text{if } p > \frac{3}{2}. \end{cases}$$

Denote $II_k = II_1 + \frac{2k\pi}{n}$, $II_k^* = II_1^* + \frac{2k\pi}{n}$ for $k = 1, 2, \ldots, [\frac{n}{m}]$. Denote the beginning point and the end point of I^* (resp. II_k^*) by P_1^b and P_1^e (resp. P_2^{k1} and P_2^{k2}), respectively. Moreover, we write $a \sim b$ in the sense that $a = O(b)$ as $m, n \to \infty$.

Let us see the shape of I^* on $\partial_1\Omega_{mn}$. At P_1, $b'_{1m}(s_1) = \max_s b'_{1m}(s) = \cos\theta_0 \tan\alpha_1$, $b''_{1m}(s_1) = 0$ and $b'''_{1m}(s_1) \sim m^2$, therefore for $\Delta s \in [-m^{-3/2}, m^{-3/2}]$ we have

$$b'_{1m}(s_1 + \Delta s) - b'_{1m}(s_1) = \frac{b'''_{1m}(s_1)}{2}(\Delta s)^2 + b^{(4)}_{1m}(s^*)(\Delta s)^3 \sim \frac{1}{m}.$$

Similar discussion is true on II_k^*. Therefore, we can choose ζ_1, ζ_2 in B_1, B_2 large such that, when we move the graph of $\Phi(\theta; \epsilon)$ so that it contacts $\partial_1\Omega_{mn}$ on I^*, the angles between the graph of $\Phi(\theta; \epsilon)$ and I^* are smaller than ϕ_1; when the graph of $\Phi(\theta; \epsilon)$ contacts $\partial_2\Omega_{mn}$ on II_k^*, the angles between the graph of $\Phi(\theta; \epsilon)$ and II_k^* are smaller than ϕ_2.

Construct functions $\xi_1(\theta)$ on $[-\theta_1, \theta_2]$ satisfying

$$\xi_1(\theta) \equiv s_1 \quad \text{for } \theta \in [-\theta_1, -\theta_0], \quad \xi_1(\theta) \equiv s_2 \quad \text{for } \theta \in [\theta_0, \theta_2],$$
$$\xi_1(\theta) \sim \frac{1}{m}, \quad \xi_{1\theta}(\theta) \sim \frac{1}{m}, \quad \xi_{1\theta\theta}(\theta) \sim \frac{1}{m},$$

and the graph of $\Phi_1(\theta) := \Phi(\theta; \epsilon) + \xi_1(\theta)$ contacts I^* at P_1^b, contacts II_1^* at P_2^{11}.

Set $c_s = \epsilon + \frac{\zeta}{m} \sim \frac{1}{m}$ for large $\zeta > 0$, then $\Phi_1(\theta) + c_s t$ is a *temporary* upper solution, that is, there exists $t_1 > 0$ small such that on $t \in [0, t_1]$, the graph of $\Phi_1(\theta) + c_s t$ contacts $\partial_2\Omega_{mn}$ on II_1^*. Since $c_s \sim \frac{1}{m}$, we have $t_1 \sim \frac{1}{n^q}/\frac{1}{m} = \frac{m}{n^q}$. (Of course, $\Phi_1(\theta) + c_s t$ uses time $t \le \frac{Cm}{n}$ ($C > 0$) to pass the whole II_1. However, when its graph contacts $\partial_2\Omega_{mn}$ on $II_1 \backslash II_1^*$, it may be not an upper solution).

If $p \le \frac{3}{2}$, then by the definition of q we have $t_1 \sim \frac{m}{n^q} = m^{1-pq} \ge m^{-3/4}$.

In what follows, we consider the case $p > \frac{3}{2}$. As soon as the graph of $\Phi_1(\theta) + c_s t_1$ reaches P_2^{12} on II_1^* (assume it contacts I^* at P_1^2 at time t_1), we define another ξ_2 as above ξ_1 such that $\Phi_2(\theta) := \Phi_1(\theta) + c_s t_1 + \xi_2(\theta)$ contacts $\partial_1\Omega_{mn}$ at P_1^2, contacts $\partial_2\Omega_{mn}$ at P_2^{21} (the beginning point of II_2^*). Then $\Phi_2(\theta) + c_s t$ is a *temporary* upper solution on time-interval $t \in [0, t_2]$, as long as the graph of $\Phi_2(\theta) + c_s t$ contacts $\partial_2\Omega_{mn}$ on II_2^*. As above we have $t_2 \sim \frac{m}{n^q}$.

Denote by l_0 the greatest integer such that $\bigcup_{k=1}^{l_0} II_k$ has φ-length smaller than $\frac{2}{m^{3/2}}$, which is the φ-length of I^*. Then

$$l_0 = \left[\frac{2}{m^{3/2}} \cdot \frac{n}{2\pi}\right] = \left[\frac{m^{p-\frac{3}{2}}}{\pi}\right] > 1.$$

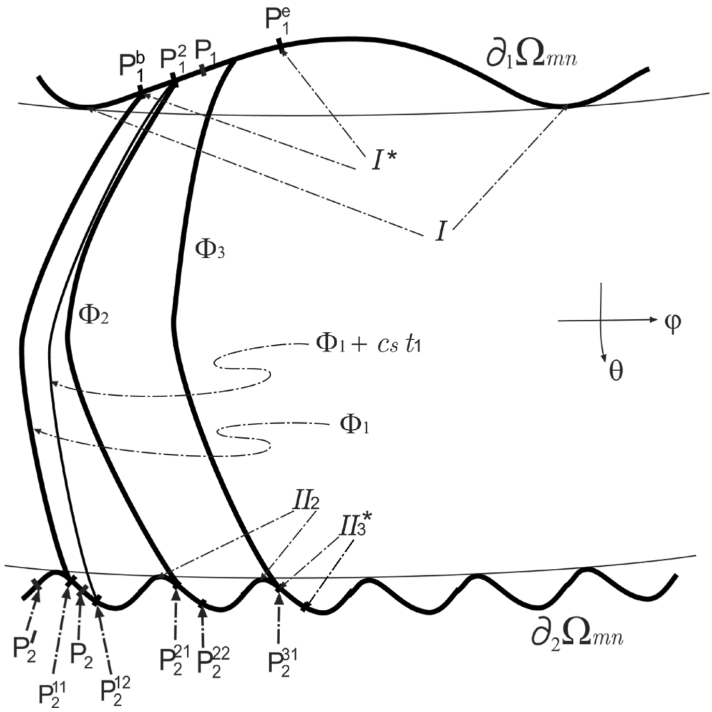

Figure 2. Temporary upper solutions.

Then above discussion is valid on the l_0 periods $\bigcup_{k=1}^{l_0} II_k \subset \partial_2\Omega_{mn}$.

On each of II_k^* $(k = 1, 2, \ldots, l_0)$, $\Phi_k(\theta) + c_s t$ is a *temporary* upper solution, then the periodic rotating wave U is blocked by these upper solutions, and so U uses time $t \geq t_1 + \cdots + t_{l_0}$ to pass $\bigcup_{k=1}^{l_0} II_k$ on $\partial_2\Omega_{mn}$, and to pass I^* on $\partial_1\Omega_{mn}$. By the definitions of l_0 and q we have

$$t_1 + \cdots + t_{l_0} \sim \frac{m}{n^q} \cdot l_0 \sim m^{p-\frac{1}{2}-pq} = m^{-3/4}.$$

Therefore, in both cases $p \leq \frac{3}{2}$ and $p > \frac{3}{2}$, U uses time longer than $O(m^{-3/4})$ to pass one period I of $\partial_1\Omega_{mn}$.

(ii) In case $\alpha_1 + \phi_1 + \alpha_2 + \phi_2 > \pi$, one can even construct a curve (like above $\Phi_k(\theta)$) such that it is a *temporary* upper solution and rotates in $-\varphi$-direction in a short period, so it blocks rotation in φ-direction.

Consequently, (H1) is a necessary condition for the existence of periodic rotating waves with average speeds $O(1)$.

3. Estimate of Average Speed – Proof of Theorem 2

First we point out that (ii) of Theorem 2 can be proved by (i) of Theorem 2 and regularity results in Section 2 as was done in Matano et al. (2006). So we only need to prove (i) of Theorem 2 in the following. We will use lower and upper solution argument to estimate the average rotating speed.

Recall that in Subsection 2.1 we constructed a lower solution $\Phi^l(\theta) + c^l t$, so the average rotating speed c_{mn} satisfies $c_{mn} \geq c^l$. Denote by $(c^*, \Phi^*(\theta))$ the solution of (1.6) ((1.6) is nothing but (2.1) with θ_i, B_i being replaced by θ_0, B_i^l, respectively). Then from the proofs of Lemmas 2.1 and 2.2, it is easily seen that

$$c^* = c^l + O(\varepsilon), \quad \Phi^* = \Phi^l + C + O(\varepsilon), \quad \Phi^*_\theta = \Phi^l_\theta + O(\varepsilon).$$

Therefore,

$$c_{mn} > c^* - C\varepsilon = c^* - \max\left\{\frac{C}{m}, \frac{C}{n}\right\} \quad \text{for some } C > 0.$$

3.1. *Upper Solution*

Now we use $\Phi^l(\theta) + c^l t$ to construct an upper solution. Let $U(\theta, t)$ be the periodic rotating wave of (1.2)–(1.3). We note that $U(\theta, t)\big|_{[-\theta_0,\theta_0]}$ is nothing but the solution of

$$\begin{cases} \tilde{u}_t = \dfrac{\tilde{u}_{\theta\theta} - 2\tan\theta \cdot \tilde{u}_\theta - \sin\theta\cos\theta \cdot \tilde{u}_\theta^3}{1 + \tilde{u}_\theta^2 \cos^2\theta}, & -\theta_0 < \theta < \theta_0, \ \ t > 0, \\ \tilde{u}(-\theta_0, t) = U(-\theta_0, t), \tilde{u}(\theta_0, t) = U(\theta_0, t), & t > 0, \\ \tilde{u}(\theta, 0) = U(\theta, 0), & -\theta_0 < \theta < \theta_0. \end{cases} \tag{3.1}$$

Without loss of generality we assume $U(\theta, 0) \preceq \Phi^l(\theta)$. Hereafter, $\Psi_1(\theta) \preceq \Psi_2(\theta)$ means that $\Psi_1(\theta) \leq \Psi_2(\theta)$ for $\theta \in [-\theta_0, \theta_0]$ and $\Psi_1(\hat{\theta}) = \Psi_2(\hat{\theta})$ for some $\hat{\theta} \in [-\theta_0, \theta_0]$. Recall $\varepsilon = \max\left\{\frac{1}{m}, \frac{1}{n}\right\}$ and define

$$w(\theta, t) = E\sqrt{\varepsilon}\left(1 - e^{-a^2 t}\sin\left(a\theta + \frac{\pi}{2}\right)\right) + aEF\sqrt{\varepsilon}t \quad \text{for } |\theta| \leq \theta_0, \ t \geq 0, \tag{3.2}$$

where $E = O(1)$ is determined later, $a = \frac{\pi}{2\theta_0}$ and $F = \max_{-\theta_1 \leq \theta \leq \theta_2} |F(\theta)| + 1$ with

$$F(\theta) = \frac{2\Phi^l_\theta \Phi^l_{\theta\theta}\cos^2\theta + 2\tan\theta + (\Phi^l_\theta)^2 \sin\theta\cos\theta + (\Phi^l_\theta)^4 \sin\theta\cos^3\theta}{1 + (\Phi^l_\theta)^2\cos^2\theta}.$$

It is clear that

$$w \geq aEF\sqrt{\varepsilon}t, \quad w_t = w_{\theta\theta} + aEF\sqrt{\varepsilon} \geq aEF\sqrt{\varepsilon}, \quad \min w(\theta, 0) = 0.$$

Lemma 3.1. *$\bar{u}(\theta, t) := w(\theta, t) + \Phi^l(\theta) + c^l t$ is an upper solution of (3.1) on time-interval $t \in [0, 1]$, and hence $\bar{u}(\theta, t) \geq U(\theta, t)$ for $\theta \in [-\theta_0, \theta_0]$, $t \in [0, 1]$.*

Proof (see Figure 3). To prove the lemma, it suffices to show that

$$\bar{u}_t \geq \frac{\bar{u}_{\theta\theta} - 2\tan\theta \cdot \bar{u}_\theta - \sin\theta\cos\theta \cdot \bar{u}_\theta^3}{1 + \bar{u}_\theta^2 \cos^2\theta} \quad \text{for } -\theta_0 < \theta < \theta_0, \ t > 0, \tag{3.3}$$

and

$$U(-\theta_0, t) < \bar{u}(-\theta_0, t), \quad U(\theta_0, t) < \bar{u}(\theta_0, t) \quad \text{for } t \in [0, 1]. \tag{3.4}$$

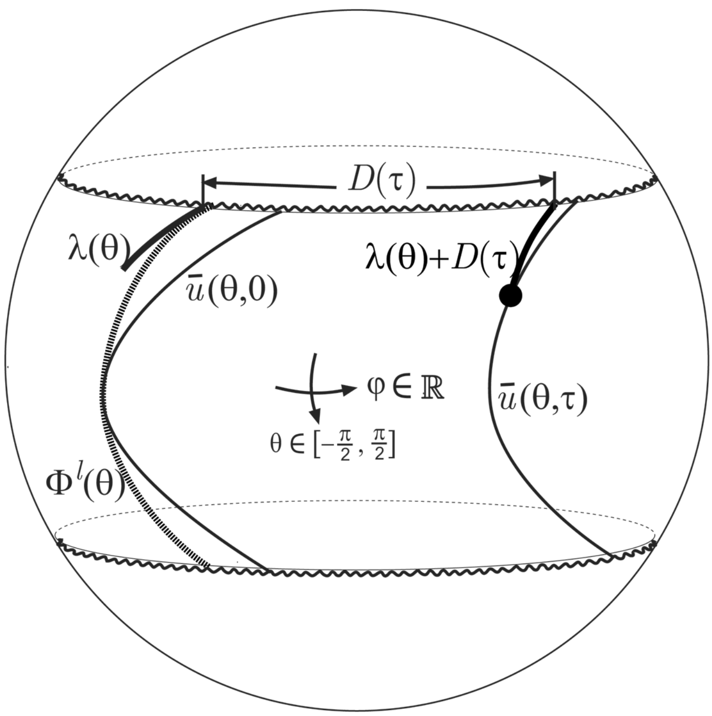

Figure 3. Upper solution.

We first prove (3.3). Since $w_{\theta\theta} \geq 0$ and $|w_\theta| \leq aE\sqrt{\varepsilon}$, a direct calculation shows that

$$\begin{aligned} &\bar{u}_t - \frac{\bar{u}_{\theta\theta} - 2\tan\theta \cdot \bar{u}_\theta - \sin\theta\cos\theta \cdot \bar{u}_\theta^3}{1 + \bar{u}_\theta^2 \cos^2\theta} \\ &\quad = w_{\theta\theta} - \frac{w_{\theta\theta}}{1 + (w_\theta + \Phi_\theta^l)^2 \cos^2\theta} + aEF\sqrt{\varepsilon} + F(\theta)w_\theta + O(\varepsilon) \geq 0. \end{aligned}$$

Next we prove (3.4). By the definitions of w and $\bar{u}$ we see that (3.4) holds on time-interval $[0, \tau]$ for some $\tau \in (0, 1)$. Thus $\bar{u}$ is an upper solution of (3.1) on $[0, \tau]$ and so

$$U(\theta, t) \le \bar{u}(\theta, t) \quad \text{for } \theta \in [-\theta_0, \theta_0], \quad t \in [0, \tau]. \tag{3.5}$$

Construct a *great circle* $\varphi = \lambda(\theta)$ on S^2 as the following. Assume $b'_{1m}(s_1) = \cos\theta_0 \tan\alpha_1$ at $s_1 \in [0, \frac{2\pi}{m})$. Denote $\theta_1^* = -\theta_0 - b_{1m}(s_1)$ and $P_1 = (\theta_1^*, s_1) \in \partial_1\Omega_{mn}$. Choose $\lambda(\theta)$ to be the great circle (geodesic curvature is 0) contacting $\partial_1\Omega_{mn}$ at P_1 with angle ϕ_1. By the proof of Lemma 2.1 we know that great circle

$$\lambda(\theta) = \int_0^\theta \frac{d}{\cos\varsigma\sqrt{\cos^2\varsigma - d^2}} d\varsigma + C$$

for suitable d and C. Just as that in the boundary condition (1.3), at P_1 we have

$$\begin{aligned}
\lambda_\theta(\theta_1^*) &= -\frac{\cos\phi_1\cos\theta_1^* - \cos\theta_0\tan\alpha_1\sin\phi_1}{\cos\theta_1^*(\sin\phi_1\cos\theta_1^* + \cos\theta_0\tan\alpha_1\cos\phi_1)} \\
&= -\frac{\cot(\phi_1 + \alpha_1)}{\cos\theta_0} + O\left(\frac{1}{m}\right) = \Phi_\theta^l(-\theta_1) + O(\varepsilon).
\end{aligned}$$

Hence, there exists $R > 1$ such that

$$|\Phi_\theta^l(\theta) - \lambda_\theta(\theta)| \le (R-1)\sqrt{\varepsilon} \quad \text{for } \theta \in [-\theta_0, -\theta_0 + \sqrt{\varepsilon}]. \tag{3.6}$$

Since $\lambda(\theta_1^*) = s_1$ we have

$$|\lambda(-\theta_0)| = |\lambda(\theta_1^*) + \lambda_\theta(\theta_1^*) \cdot b_{1m}(s_1) + O(\varepsilon^2)| \le R_1\varepsilon \quad \text{for some } R_1 > 0.$$

Suppose at time τ, $\lambda(\theta) + D(\tau)$ intersects $\bar{u}(\theta, \tau)$ at $\theta = -\theta_0 + \sqrt{\varepsilon}$, i.e. $\bar{u}(-\theta_0 + \sqrt{\varepsilon}, \tau) = \lambda(-\theta_0 + \sqrt{\varepsilon}) + D(\tau)$. Then by (3.6) we have (note that $\tau < 1$)

$$\begin{aligned}
D(\tau) &= \bar{u}(-\theta_0 + \sqrt{\varepsilon}, \tau) - \lambda(-\theta_0 + \sqrt{\varepsilon}) \\
&= w(-\theta_0 + \sqrt{\varepsilon}, \tau) + c^l\tau + \Phi^l(-\theta_0 + \sqrt{\varepsilon}) - \lambda(-\theta_0 + \sqrt{\varepsilon}) \\
&= w(-\theta_0, \tau) + w_\theta(-\theta_0, \tau)\sqrt{\varepsilon} + c^l\tau \\
&\quad + \Phi^l(-\theta_0) - \lambda(-\theta_0) + (\Phi_\theta^l(\widehat{\theta}) - \lambda_\theta(\widehat{\theta}))\sqrt{\varepsilon} + o(\varepsilon) \\
&< \bar{u}(-\theta_0, \tau) - aEe^{-a^2\tau}\varepsilon + (R-1)\varepsilon + R_1\varepsilon + o(\varepsilon) \\
&< \bar{u}(-\theta_0, \tau) + [R + R_1 - aEe^{-a^2}]\varepsilon < \bar{u}(-\theta_0, \tau) - (R_1 + 5\pi)\varepsilon
\end{aligned}$$

provided we choose E large such that $aEe^{-a^2} > R + 2R_1 + 5\pi$.

Since $\lambda(\theta)$ contacts $\partial_1\Omega_{mn}$ at P_1 with angle ϕ_1, there exists $\delta \in [0, \frac{2\pi}{m})$ such that $\lambda(\theta) + D(\tau) + \delta$ also contacts $\partial_1\Omega_{mn}$ at point $(\theta_1^*, s_1 + \frac{2k\pi}{m})$ (for some $k \in \mathbb{N}$) with angle ϕ_1, so $\lambda(\theta) + D(\tau) + \delta$ is stationary. Therefore by

$$U(-\theta_0 + \sqrt{\varepsilon}, \tau) \le \bar{u}(-\theta_0 + \sqrt{\varepsilon}, \tau) \le \lambda(-\theta_0 + \sqrt{\varepsilon}) + D(\tau) + \delta,$$

we have $U(\theta, \tau) \leq \lambda(\theta) + D(\tau) + \delta$ for $-\theta_0 \leq \theta \leq -\theta_0 + \sqrt{\varepsilon}$. Especially,

$$U(-\theta_0, \tau) \leq \lambda(-\theta_0) + D(\tau) + \delta \leq \bar{u}(-\theta_0, \tau) - 3\pi\varepsilon.$$

Since U satisfies the estimate result (ii) in Lemma 2.3, we have

$$|U_t(\theta, t)| \leq C_\varepsilon \quad \text{for all } \theta \in [-\theta_0, \theta_0] \quad \text{and} \quad t \in [0, 3],$$

where $C_\varepsilon > 1$ depends on ε but does not depend on t. Define $T_\varepsilon := \frac{2\pi\varepsilon}{C_\varepsilon}$, then $0 < T_\varepsilon < 2\pi\varepsilon$, and

$$U(-\theta_0, \tau + t) - U(-\theta_0, \tau) = \int_\tau^{\tau+t} U_t(-\theta_0, t)dt \leq C_\varepsilon \cdot t \leq 2\pi\varepsilon \quad \text{for } t \in [0, T_\varepsilon].$$

Therefore,

$$\bar{u}(-\theta_0, \tau + t) \geq \bar{u}(-\theta_0, \tau) \geq U(-\theta_0, \tau) + 3\pi\varepsilon > U(-\theta_0, \tau + t) \quad \text{for } t \in [0, T_\varepsilon].$$

In other words, the first inequality of (3.4) holds at least on $[0, \tau + T_\varepsilon]$. Similarly, the second inequality of (3.4) at $\theta = \theta_0$ also holds on $[0, \tau + T_\varepsilon]$. Consequently, (3.4) hold on $t \in [0, \tau + T_\varepsilon]$.

If $\tau + T_\varepsilon \geq 1$, then (3.4) holds on $[0, 1]$. If $\tau + T_\varepsilon < 1$, then repeat above discussion finite times we obtain (3.4) on $t \in [0, 1]$. □

3.2. *Upper Estimate of Average Speed*

From Lemma 3.1 we have

$$U(\theta, 1) \leq \bar{u}(\theta, 1) \leq \Phi^l(\theta) + (E + aEF)\sqrt{\varepsilon} + c^l \quad \text{for } \theta \in [-\theta_0, \theta_0].$$

Then there exists $d_1 \leq (E + aEF)\sqrt{\varepsilon} + c^l$ such that

$$U(\theta, 1) \preceq \Phi^l(\theta) + d_1 \quad \text{for } \theta \in [-\theta_0, \theta_0].$$

Consider the problem of $U(\theta, 1 + t)$, a similar discussion as that in previous section shows that $w(\theta, t) + \Phi^l(\theta) + d_1 + c^l t$ is an upper solution on $t \in [0, 1]$, and so

$$\begin{aligned} U(\theta, 1 + 1) \leq w(\theta, 1) + \Phi^l(\theta) + d_1 + c^l &\leq (E + aEF)\sqrt{\varepsilon} + \Phi^l(\theta) \\ &\quad + d_1 + c^l \quad \text{for } \theta \in [-\theta_0, \theta_0]. \end{aligned}$$

Hence, there exists $d_2 \leq (E + aEF)\sqrt{\varepsilon} + c^l$ such that

$$U(\theta, 2) \preceq \Phi^l(\theta) + d_1 + d_2 \quad \text{for } \theta \in [-\theta_0, \theta_0].$$

Repeating this process k times we have

$$U(\theta, k) \preceq \Phi^l(\theta) + d_1 + d_2 + \cdots + d_k \leq \Phi^l(\theta) + k(c^l + (E + aEF)\sqrt{\varepsilon}).$$

So

$$\frac{U(\theta,k)-U(\theta,0)}{k} \leq \frac{\Phi^l(\theta)-U(\theta,0)}{k} + c^l + (E+aEF)\sqrt{\varepsilon}.$$

As $k \to \infty$, the left hand side has limit c_{mn}, and so $c_{mn} \leq c^l + (E+aEF)\sqrt{\varepsilon}$. This completes the proof of (i) in Theorem 2. □

Remark 3.1. We point out that our upper solution $\bar{u}$ is a special one, since it is larger than the solution just on time-interval [0, 1], not only on one period and not for all time. However, it is good enough to give the upper bound of the rotating speed.

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