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Traveling waves of a curvature flow in almost periodic media

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ABSTRACT

In a plane media with almost periodic vertical striations, we study a curvature flow and construct two kinds of traveling waves, one having a straight line like profile and the other having a V shaped profile. For each of the first-kind traveling waves, its profile is the graph of a function whose derivative is almost periodic. For each of the second-kind traveling waves, its profile is like a pulsating cone, whose two tails approach asymptotically the profiles of the first-kind traveling waves. Also we consider a homogenization problem and provide an explicit formula for the homogenized traveling speed.

1. Introduction

We study traveling waves of a curvature flow in heterogeneous media on the plane:

$$V = a(x)\kappa + b(x), \quad (x, y) \in \Gamma_t \subset \mathbb{R}^2,\ t > 0, \tag{1}$$

where Γ_t is a simple curve on the xy-plane, V denotes the normal velocity, κ denotes the curvature and

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$$\begin{cases} a(x) \text{ and } b(x) \text{ are continuous, \textit{almost periodic} functions (Bohr's definition)}, \\ a_m := \inf a(x) > 0, \quad a_M := \sup a(x) < \infty, \quad b_m := \inf b(x) > 0, \quad b_M := \sup b(x) < \infty. \end{cases}$$

The (mean) curvature flow equation (1), having its own geometrical aspect [10], is quite often derived as the singular $\epsilon \to 0$ limit of the zero level sets of solutions of reaction–diffusion equations of type

$$u_t = \nabla \cdot \big(A(\mathbf{x})\nabla u\big) + \epsilon^{-2}\big(1-u^2\big)\big(u + \epsilon B(\mathbf{x})\big), \quad \mathbf{x} \in \mathbb{R}^N,\ t > 0;$$

see, for example, [1,6,11,16]. In the last two decades, many researchers studied traveling waves of reaction–diffusion equations in periodic media (e.g. [2–4,12,20,21]), or in almost periodic media (e.g. [15]). Existence, uniqueness and stability results, as well as variational formulas for the average speed were given. On the other hand, the shape of the traveling fronts, the explicit expression of the wave speed have not been studied so much, even in periodic media. In this paper, we study traveling waves in almost periodic media for the curvature flow equation (1). We give two kinds of traveling waves, one having a line like shape Γ_t and the other a V like shape Γ_t. We will also derive an explicit formula for a homogenized traveling speed.

We consider the case where Γ_t is the graph of a function $y = h(x,t)$. With an appropriate sign convention,

$$V = \frac{h_t}{\sqrt{1+h_x^2}}, \qquad \kappa = \frac{h_{xx}}{(1+h_x^2)^{\frac{3}{2}}},$$

so the curvature flow equation can be written as

$$h_t = a(x)\frac{h_{xx}}{1+h_x^2} + b(x)\sqrt{1+h_x^2}, \quad x \in \mathbb{R},\ t > 0.$$

Here by a *traveling wave* it means a solution of the form $h(x,t) = v(x) + ct$ for some v and c. The graph of v is called the *profile* of the traveling wave and c is called the traveling *speed* (in the vertical direction). For traveling waves, the curvature flow equation becomes

$$c = a(x)\frac{v''}{1+v'^2} + b(x)\sqrt{1+v'^2}, \quad x \in \mathbb{R}.$$

After a further transformation $\varphi(x) := \arctan v'(x)$, the equation is equivalent to

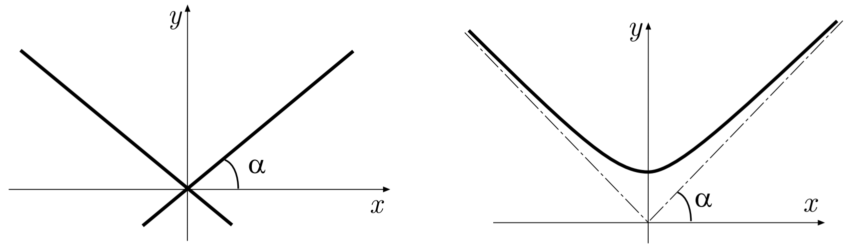
$$\varphi'(x) = \frac{c}{a(x)} - \frac{b(x)}{a(x)\cos\varphi(x)}, \quad \varphi(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\ \forall x \in \mathbb{R}. \tag{2}$$

In the sequel we study traveling waves of (1) by means of studying solutions of the ode (2). The solutions of our interest are prototyped by the homogeneous case where $a = a_0$ and $b = b_0$ are positive constants. In this case we can solve Eq. (2) via separation of variables to see the following:

(1) *Traveling waves with straight-line profiles*: For any $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, there is a solution such that $\Gamma_t = \{(x,y) \mid y = x\tan\alpha + ct\}$ where $c = b_0/\cos\alpha$. This corresponds to a traveling wave whose profile is a straight line with the prescribed inclination angle α; see Fig. 1(a).

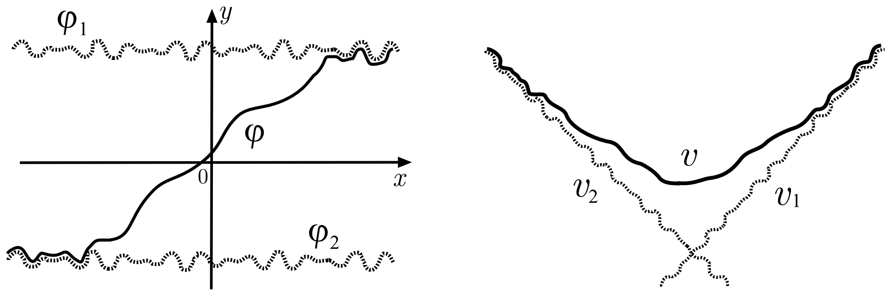
(2) *Traveling waves with convex conical profiles* (also called V-form or V shaped curves); see Fig. 1(b). For any inclination angle $\alpha \in (0, \frac{\pi}{2})$, there is a solution given by $\Gamma_t = \{(x,y) \mid y = v(x) + ct\}$ where

$$c = \frac{b_0}{\cos\alpha}, \qquad \lim_{x\to\pm\infty}\big\{v(x) - |x|\tan\alpha\big\} = 0.$$



(a) straight-line profile (b) convex conical profile

Fig. 1.



(a) $\varphi(x)$ approaches $\varphi_i(x)$ as $x \to \pm\infty$ (b) V shaped profile

Fig. 2.

Besides these two kinds of traveling waves, there are many other kinds of traveling waves, see [7,17,18] and references therein. In this paper, we shall only concern about line and V shaped profiles.

For some of our results, we need to restrict α to the range

$$\alpha \in \left(\arccos\frac{b_m}{b_M}, \frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, -\arccos\frac{b_m}{b_M}\right) \quad (\Leftrightarrow \quad b_m > b_M \cos\alpha). \tag{3}$$

We also use $\mathcal{M}[h]$ to denote the arithmetic mean of an almost periodic function $h(x)$.

Theorem A. *For any α in the range given by* (3), *there exists a unique pair $(c, \varphi) \in \mathbb{R} \times C^1(\mathbb{R})$ such that* (i) (c, φ) *satisfies* (2), (ii) φ *is almost periodic, and* (iii) $\mathcal{M}[\tan\varphi] = \tan\alpha$.

In addition, as a function of α, the unique solution (c, φ) satisfies the following estimates

$$\phi_2 := \arccos\frac{b_M \cos\alpha}{b_m} \leqslant \operatorname{sgn}(\alpha)\varphi(x) \leqslant \phi_1 := \arccos\frac{b_m \cos\alpha}{b_M} \quad \textit{for } x \in \mathbb{R}, \tag{4}$$

$$0 \leqslant \operatorname{sgn}(\alpha)\frac{dc}{d\alpha} \leqslant \frac{3a_M b_M^3}{a_m b_m^2 \cos^4\alpha}, \tag{5}$$

where sgn *is the signature function:* $\operatorname{sgn}(\alpha) = 1$ *if* $\alpha > 0$ *and* $= -1$ *if* $\alpha < 0$.

Theorem B. *Assume that $\alpha_1 > 0$ and $\alpha_2 < 0$ satisfy* (3). *Also assume that (c, φ_1) and (c, φ_2) are solutions of* (2), *that φ_1 and φ_2 are almost periodic and $\mathcal{M}[\tan\varphi_i] = \tan\alpha_i$, $i = 1, 2$. Then, for any small initial value $\varphi(0)$, problem* (2) (*with the same c*) *admits a unique solution φ and the solution satisfies the following* (*see Fig.* 2):

(i) $\varphi_2(x) < \varphi(x) < \varphi_1(x)$ *for all x*;
(ii) *There exist positive constants L and ν such that*

$$\varphi_1(x) - L\exp(-\nu x) \leqslant \varphi(x) \leqslant \varphi_2(x) + L\exp(\nu x) \quad \forall x \in \mathbb{R}.$$

(iii) *There exist a unique x_0 and a unique $S_0 > 0$ such that the functions defined by*

$$v_1(x) := \int_{x_0}^{x} \tan\varphi_1(x)\,dx, \qquad v_2(x) := \int_{x_0}^{x} \tan\varphi_2(x)\,dx, \qquad v(x) := \int_{x_0}^{x} \tan\varphi(x)\,dx + S_0$$

satisfy, for some $\hat{L} > 0$ and the same ν as above,

$$\max\{v_1(x), v_2(x)\} < v(x) < \max\{v_1(x), v_2(x)\} + \hat{L}\exp(-\nu|x|) \quad \forall x \in \mathbb{R}.$$

In order to give an explicit estimate for the traveling speed c in terms of α, we consider a homogenization problem:

$$\frac{d}{dx}\varphi^\varepsilon(x) = \frac{c^\varepsilon}{a^\varepsilon(x)} - \frac{b^\varepsilon(x)}{a^\varepsilon(x)\cos\varphi^\varepsilon(x)}, \quad \varphi^\varepsilon(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\ \forall x \in \mathbb{R}, \tag{2}_\varepsilon$$

where

$$a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right), \qquad b^\varepsilon(x) = b\left(\frac{x}{\varepsilon}\right).$$

Denote

$$A = \left(\mathcal{M}\left[\frac{1}{a(x)}\right]\right)^{-1}, \qquad B = \mathcal{M}\left[\frac{b(x)}{a(x)}\right]A.$$

Theorem C. *For any α satisfying* (3), *let $(c^\varepsilon, \varphi^\varepsilon)$ be the solution of* $(2)_\varepsilon$ *as in Theorem* A. *Then*

$$\lim_{\varepsilon \searrow 0}\|\varphi^\varepsilon - \alpha\|_{L^\infty(\mathbb{R})} = 0, \qquad \lim_{\varepsilon \searrow 0} c^\varepsilon = \frac{B}{\cos\alpha}. \tag{6}$$

If in addition assume that for some $L_1, L_2 > 0$,

$$\left|\int_0^x \left(\frac{1}{a(x)} - \frac{1}{A}\right)dx\right| \leqslant L_1, \qquad \left|\int_0^x \left(\frac{b(x)}{a(x)} - \frac{B}{A}\right)dx\right| \leqslant L_2 \quad \textit{for } x \in \mathbb{R}, \tag{7}$$

then there exists a positive constant Q depending only on $L_1, L_2, a_m, a_M, b_m, b_M$ and α such that

$$\|\varphi^\varepsilon(\cdot) - \alpha\|_{L^\infty(\mathbb{R})} + \left|c^\varepsilon - \frac{B}{\cos\alpha}\right| \leqslant Q\,\varepsilon \quad \forall \varepsilon > 0. \tag{8}$$

Remark 1.1. Under the new variable

$$\tilde{x} = \int_0^x \frac{ds}{a(s)}$$

the ode system (2) can be written as

$$\frac{d\varphi}{d\tilde{x}} = c - \frac{\tilde{b}}{\cos\varphi}.$$

This system is simpler than (2). However, it is not guaranteed that $\tilde{b}$ is almost periodic. Hence, we shall work on (2).

In Section 2, we study traveling waves with pulsating profiles whose derivatives are almost periodic and prove Theorem A. In Section 3, we construct traveling waves with V shaped profiles and prove Theorem B. Finally, in Section 3, we consider the homogenized speed and prove Theorem C.

2. Almost periodic traveling waves

In this section, we prove Theorem A. The main idea is as follows.

We first study the ode in (2) on a bounded interval $[-n, n]$ with boundary conditions $\varphi(n) = \varphi(-n) = p$ where p is a parameter ranged in $(-\pi/2, \pi/2)$. The two boundary conditions for the first order ode can be imposed by selecting a unique $c = c(p)$.

Next we apply some of the ideas in [8,9] showing that the average of $\tan\varphi$ (or any $f(\varphi)$ where f is increasing) on the interval $[-n, n]$ is a monotonic function of p. Thus, for each fixed α, there is a unique p, such that this average is exactly $\tan\alpha$. We denote this solution (with α fixed) by (c_n, φ_n).

Sending n to infinity, we obtain a solution of (2) with certain specific c that depends on α. The key issue here is to identify the limit. By imposing the almost periodicity of a, b and restricting α to the range specified in (3), we are able to show that the limit of φ_n, denoted by φ, is also almost periodic and has the required arithmetic mean: $\mathcal{M}[\tan\varphi] = \tan\alpha$.

2.1. Eq. (2) on bounded intervals

Here we consider Eq. (2) on bounded interval. For each fixed $n > 0$ and $\alpha \in (-\pi/2, \pi/2)$, we want to find $(c, p, \varphi) \in \mathbb{R} \times (-\pi/2, \pi/2) \times C([-n, n])$ such that

$$\begin{cases} \varphi'(x) = G\big(x, \varphi(x), c\big) := \dfrac{c}{a(x)} - \dfrac{b(x)}{a(x)\cos\varphi(x)} \quad \forall x \in (-n, n), \\ \varphi(-n) = \varphi(n) = p, \qquad \dfrac{1}{2n}\displaystyle\int_{-n}^{n} \tan\varphi(x)\,dx = \tan\alpha. \end{cases} \tag{9}$$

A similar problem has been studied in [8,9]. Here we follow basically the idea presented there.

Lemma 2.1. *For any $p \in (-\pi/2, \pi/2)$, there exists a unique $c = c(p) \in \mathbb{R}$ such that the boundary value problem*

$$\begin{cases} \varphi' = G(x, \varphi, c) \quad \forall x \in (-n, n), \\ \varphi(-n) = \varphi(n) = p, \end{cases} \tag{10}$$

has a unique solution. Moreover, the solution satisfies

$$b_m \leqslant c\cos\varphi(\cdot) \leqslant b_M, \qquad \max\cos\varphi(\cdot) \leqslant \frac{b_M}{b_m}\min\cos\varphi(\cdot). \tag{11}$$

Proof. For $p \in (-\pi/2, \pi/2)$ and $c \in \mathbb{R}$, we denote by $\Phi(\cdot, p, c)$ the solution and by $(-n, z^*)$ the maximal existence interval to the initial value problem

$$\begin{cases} \Phi' = G(x, \Phi, c), & \Phi(x, p, c) \in (-\pi/2, \pi/2) \quad \forall x \in (-n, z^*), \\ \Phi(x, p, c)|_{x=-n} = p. \end{cases}$$

(a) On their existence interval, we denote $\Phi_c := \frac{\partial \Phi}{\partial c}$ and $\Phi_p := \frac{\partial \Phi}{\partial p}$. Then

$$\Phi_p' = G_\Phi \cdot \Phi_p, \qquad \Phi_p|_{x=-n} = 1,$$
$$\Phi_c' = G_\Phi \cdot \Phi_c + G_c, \qquad \Phi_c|_{x=-n} = 0.$$

Hence, $\Phi_p = \exp(\int_{-n}^x G_\Phi \, dx) > 0$. Also, set $\mu := \Phi_c / \Phi_p$. Then

$$\mu' = \frac{G_c}{\Phi_p}, \qquad \mu|_{x=-n} = \frac{\Phi_c}{\Phi_p}\bigg|_{x=-n} = 0.$$

Since $G_c = 1/a(x) > 0$ we have $\mu' > 0$, $\mu > 0$, and also $\Phi_c > 0$.

(b) Since $G(x, \frac{\pi}{2} - 0, c) = -\infty$, we see that $\Phi < \pi/2$ on its existence interval. Also, for $c_1 := b_M / \cos p$ we have $G(\cdot, p, c_1) \geqslant 0$, so, by comparison, when $c > c_1$ we have $p < \Phi(\cdot, c, p) < \pi/2$ on $(-n, \infty)$.

(c) It is not difficult to see that when $c \ll -1$, $z^* < n$. Hence, there exists $c_2 \in \mathbb{R}$ such that $z^* = n$ and $\Phi(n - 0, p, c_2) = -\pi/2$. When $c \in (c_2, c_1]$, $z^* > n$ and $\Phi(n, p, c)$ is a differentiable strictly increasing function of c. Hence, there exists a unique $c(p) \in (c_2, c_1]$ such that $\Phi(n, p, c(p)) = p$. It is easy to see from $\Phi_c > 0$ that the pair $c = c(p)$ and $\varphi(x) \equiv \Phi(x, p, c(p))$ is the unique solution of (10).

Next we verify (11). Since $\varphi(x)$ takes the same values at $x = \pm n$, for any $\vartheta \in [\min \varphi(\cdot), \max \varphi(\cdot)]$, there exist $x_1, x_2 \in [-n, n)$ such that $\varphi(x_1) = \varphi(x_2) = \vartheta$, $\varphi'(x_1) \geqslant 0$, $\varphi'(x_2) \leqslant 0$, so that

$$b_m \leqslant b(x_1) \leqslant c \cos \vartheta \leqslant b(x_2) \leqslant b_M.$$

Since $\vartheta \in [\min \varphi(x), \max \varphi(x)]$ can be chosen arbitrarily, we obtain the first two inequalities in (11). Note that these two inequalities imply $\max \cos \varphi(\cdot) \leqslant b_M / c$ and $b_m \leqslant c \min \cos \varphi(\cdot)$. Combing these we obtain the last inequality in (11). $\square$

Lemma 2.2. *Let $\alpha \in (-\pi/2, \pi/2)$ be given. For each $n > 0$, problem (9), for unknown $(c, p, \varphi) \in \mathbb{R} \times (-\pi/2, \pi/2) \times C^1([-n, n])$, admits a unique solution. Denote the solution by (c_n, p_n, φ_n). We have*

$$\frac{b_m}{\cos \alpha} < \frac{b_m}{\min \cos \varphi_n} \leqslant c_n \leqslant \frac{b_M}{\max \cos \varphi_n} \leqslant \frac{b_M}{\cos \alpha}, \tag{12}$$

$$\left|\varphi_n(x)\right| \leqslant \phi_1 := \arccos \frac{b_m \cos \alpha}{b_M} \quad \forall x \in [-n, n]. \tag{13}$$

Proof. The solution of (10) is $\varphi(x) \equiv \Phi(x, p, c(p))$. Set $I(p) = \int_{-n}^n \tan \Phi(x, p, c(p)) \, dx$. We shall show that there exists a unique $p \in (-\pi/2, \pi/2)$ such that $I(p) = 2n \tan \alpha$.

(a) From the first inequality in (11), we see that as $p \to \pi/2$, $c(p) \to \infty$. This, together with the second inequality in (11), implies that as $p \to \pi/2$, $\Phi(x, p, c(p)) \to \pi/2$ uniformly in $x \in [-n, n]$. Hence, $I(p) \to \infty$ as $p \to \pi/2$. Similarly, as $p \to -\pi/2$, we have $\Phi(\cdot, p, c(p)) \to -\pi/2$ and $I(p) \to -\infty$.

Thus, to complete the proof, we need only show that $I(p)$ is monotonic in p.

(b) Denote $\overline{\Phi}(p, c) := \Phi(n, p, c)$. Differentiating $\overline{\Phi}(p, c(p)) = p$ we have $\overline{\Phi}_p + \overline{\Phi}_c \cdot \frac{dc(p)}{dp} = 1$, so $\frac{dc(p)}{dp} = \frac{1 - \overline{\Phi}_p}{\overline{\Phi}_c}$. Using notation μ as in the proof of Lemma 2.1 and with $c = c(p)$, we have

$$\frac{dI(p)}{dp} = \int_{-n}^{n} \sec^2 \Phi \left(\Phi_p + \Phi_c \cdot \frac{dc(p)}{dp} \right) dx = \int_{-n}^{n} \sec^2 \Phi \, \frac{\Phi_p \overline{\Phi}_c + \Phi_c - \Phi_c \overline{\Phi}_p}{\overline{\Phi}_c} \, dx$$
$$= \int_{-n}^{n} \sec^2 \Phi \, \frac{\mu(n)\Phi_p \overline{\Phi}_p + \mu(x)\Phi_p - \mu(x)\Phi_p \overline{\Phi}_p}{\mu(n)\overline{\Phi}_p} \, dx > 0.$$

Hence, there exists a unique solution p to $I(p) = 2n\tan\alpha$. This also implies that there is a unique solution to (9). We denote the solution by (c_n, p_n, φ_n).

(c) Since $\tan\varphi_n(x)$ has average $\tan\alpha$ and $\varphi_n(x) \not\equiv \alpha$, we have $\min\varphi_n(x) < \alpha < \max\varphi_n(x)$. When $\alpha \neq 0$,

$$0 < \min\cos\varphi_n < \cos\alpha < \max\cos\varphi_n.$$

Estimate (12) then follows from (11). By (12),

$$\frac{b_m\cos\alpha}{b_M} < \cos\varphi_n < \frac{b_M\cos\alpha}{b_m} \quad \text{for all } x \in [-n, n]. \tag{14}$$

Estimate (13) then follows from the first inequality. When $\alpha = 0$, the proof is similar. This completes the proof of the lemma. □

2.2. Eq. (2) on $\mathbb{R}$

Lemma 2.3. *Let $\alpha \in (-\pi/2, \pi/2)$ be fixed. For each $n \in \mathbb{N}$ let $(c_n, \varphi_n(x))$ be the unique solution of (9). Then there exist a subsequence $\{n_i\}$ of $\{n\}$, $c > 0$ and $\varphi(x) \in C^1(\mathbb{R})$ such that, as $i \to \infty$, $c_{n_i} \to c$, and*

$$\varphi_{n_i}(x) \to \varphi(x), \qquad \varphi'_{n_i}(x) \to \varphi'(x) \quad \text{in } L^\infty_{loc}(\mathbb{R}).$$

In addition, (c, φ) solves (2) on $\mathbb{R}$ and satisfies the estimates

$$\frac{b_m}{\cos\alpha} \leqslant \frac{b_m}{\inf\cos\varphi} \leqslant c \leqslant \frac{b_M}{\sup\cos\varphi} \leqslant \frac{b_M}{\cos\alpha}, \tag{15}$$

and

$$-\phi_1 \leqslant \varphi(x) \leqslant \phi_1 \quad \text{for } x \in \mathbb{R}. \tag{16}$$

Proof. For any $X > 0$, when $n \geqslant X$, φ_n is defined on $[-X, X]$. By (12), (13), and by the equation of φ_n, we see that

$$\|\varphi'_n\|_{L^\infty[-X,X]} \leqslant Q_1 \quad \text{for all } n > X,$$

where $Q_1 = Q_1(\alpha)$ is independent of n. Using the theorem of Ascoli–Arzelà and using the diagonal process, we can select a subsequence $\{n_i\}$ from $\{n\}$ such that, as $i \to \infty$,

$$c_{n_i} \to c, \qquad \|\varphi_{n_i} - \varphi\|_{L^\infty[-X,X]} \to 0 \quad \text{for any } X > 0,$$

for some $c > 0$ and $\varphi \in C(\mathbb{R})$.

From $\varphi'_n = G(x, \varphi_n, c_n)$ we also see that $\varphi'_{n_i} \to \varphi'$ locally uniformly and that $\varphi' = G(x, \varphi, c)$.

Finally, the estimates (15) and (16) follow from (12), (13). This proves the lemma. □

2.3. Almost periodicity

Lemma 2.4. *Assume that* (3) *holds. Then the limits c and φ in Lemma* 2.3 *satisfies* (4) *and* (5).

Proof. It is clear that to prove the lemma we only need to consider the solution (c_n, φ_n) of (9) and show that

$$\phi_2 < \operatorname{sgn}(\alpha)\varphi_n(x) < \phi_1, \tag{17}$$

$$0 < \operatorname{sgn}(\alpha)\frac{dc_n}{d\alpha} < \frac{3a_M b_M^3}{a_m b_m^2 \cos^4\alpha} \quad \text{for large } n. \tag{18}$$

When (3) holds, we have $\cos\varphi_n < 1$ by (14). Hence, φ_n does not change sign. This means that, when $\alpha > 0$ (resp. $\alpha < 0$), $\varphi_n(x) > 0$ (resp. $\varphi_n(x) < 0$) for all $x \in [-n, n]$. Therefore, we have (17) by (14).

Now we prove (18). Note that p_n and $c_n = c(p_n)$ depend on α. Differentiating the following equalities in α

$$\int_{-n}^{n} \tan\Phi(x, p_n, c_n)\, dx = 2n\tan\alpha, \qquad \overline{\Phi}(p_n, c_n) = p_n,$$

we have

$$\int_{-n}^{n} \sec^2\Phi\left(\Phi_p\frac{dp_n}{d\alpha} + \Phi_c\frac{dc_n}{d\alpha}\right)dx = 2n\sec^2\alpha, \qquad \overline{\Phi}_p\frac{dp_n}{d\alpha} + \overline{\Phi}_c\frac{dc_n}{d\alpha} = \frac{dp_n}{d\alpha}.$$

Substituting $dp_n/d\alpha$ in the second equation into the first equation and utilizing $\mu = \Phi_c/\Phi_p$ as in the proof of Lemma 2.1 we obtain

$$\frac{dc_n}{d\alpha} = \frac{2n(1 - \overline{\Phi}_p)\sec^2\alpha}{\int_{-n}^{n} \sec^2\Phi[(\mu(n) - \mu(x))\overline{\Phi}_p\Phi_p + \mu(x)\Phi_p]\, dx}. \tag{19}$$

We now focus on the case $\alpha > 0$; the proof for the case $\alpha < 0$ is similar. Since

$$G_\Phi = \frac{-b(x)\sin\Phi}{a(x)\cos^2\Phi},$$

we have $-\lambda_1 < G_\Phi < -\lambda_2$ by (17), where

$$\lambda_1 := \frac{b_M\sin\phi_1}{a_m\cos^2\phi_1}, \qquad \lambda_2 := \frac{b_m\sin\phi_2}{a_M\cos^2\phi_2}. \tag{20}$$

Hence

$$\exp\bigl(-\lambda_1(x+n)\bigr) \leqslant \Phi_p(x, p_n, c_n) = \exp\left(\int_{-n}^{x} G_\Phi\right) \leqslant \exp\bigl(-\lambda_2(x+n)\bigr).$$

Especially, $\exp(-2n\lambda_1) \leqslant \overline{\Phi}_p = \Phi_p(n, p_n, c_n) \leqslant \exp(-2n\lambda_2)$. Hence

$$\frac{1}{2} < 1 - \overline{\Phi}_p < \frac{3}{2} \quad \text{for large } n. \tag{21}$$

Therefore, $\frac{dc_n}{d\alpha} > 0$ since in (19) the denominator is positive.

Next, we provide an upper bound for $\frac{dc_n}{d\alpha}$. By $\mu' = 1/(a\Phi_p)$ and $\mu(-n) = 0$ we have

$$\mu(x) = \int_{-n}^{x} \frac{ds}{a(s)\Phi_p(s, p_n, c_n)} \geqslant \frac{1}{a_M} \int_{-n}^{x} \exp\left(-\int_{-n}^{s} G_\Phi\right) ds.$$

So,

$$\mu(x)\Phi_p(x, p_n, c_n) \geqslant \frac{1}{a_M} \int_{-n}^{x} \exp\left(\int_{s}^{x} G_\Phi\right) ds \geqslant \frac{1}{a_M} \int_{-n}^{x} \exp\big(\lambda_1(s - x)\big)\, ds = \frac{1 - \exp(-\lambda_1(x + n))}{a_M\lambda_1}.$$

Therefore,

$$\int_{-n}^{n} \mu(x)\Phi_p(x, p_n, c_n)\, dx \geqslant \frac{1}{a_M\lambda_1}\left(2n - \frac{1}{\lambda_1}\right) > \frac{n}{a_M\lambda_1} \quad \text{provided } n > \frac{1}{\lambda_1}.$$

Consequently, by (19) and (21) we have $\frac{dc_n}{d\alpha} < 3a_M\lambda_1 \sec^2\alpha$ for large n. This proves the inequalities in (18) in the case $\alpha > 0$. Thus we can prove the lemma. $\square$

Recall that a and b are assumed to be almost periodic. By Bohr's definition and Bochner's criterion (see e.g. [5,14]), we have the following claim

Claim 2.5. *For every $\delta > 0$, there exists $M_\delta > 0$ such that*

$$\mathscr{A}_\delta \cap [l, l + M_\delta] \neq \emptyset \quad \textit{for any } l \in \mathbb{R}, \tag{22}$$

where

$$\mathscr{A}_\delta := \left\{\tau \;\Big|\; \sup_{x\in\mathbb{R}}\big|a(x) - a(x + \tau)\big| < \delta,\ \sup_{x\in\mathbb{R}}\big|b(x) - b(x + \tau)\big| < \delta\right\}. \tag{23}$$

Now we are ready to prove the following

Lemma 2.6. *Assume that a, b are almost periodic and α satisfies* (3)*. Then the limit φ in Lemma* 2.3 *is almost periodic.*

Proof. To prove the almost periodicity of the limit φ, it is sufficient to show that, for any $\delta > 0$ and $\tau \in \mathscr{A}_\delta$,

$$\sup_{x\in\mathbb{R}}\big|\varphi(x) - \varphi(x + \tau)\big| \leqslant Q\,\delta, \tag{24}$$

for some positive constant Q that is independent of δ and τ. For this purpose, we denote

$$\widetilde{G}(a,b,\phi,c) := \frac{c}{a} - \frac{b}{a\cos\phi}.$$

Then $\phi(x) = \varphi(x+\tau)$ is the solution of $\phi' = \widetilde{G}(a(x+\tau), b(x+\tau), \phi, c)$. When we write $\psi_1(x) := \varphi(x) - \phi(x)$ we have

$$\psi_1' = g_1(x)\psi_1 + r_1(x),$$

where

$$g_1(x) := \widetilde{G}_\phi\big(a_1, b_1, \xi_1(x), c\big),$$
$$r_1(x) := \widetilde{G}_a(a_1, b_1, \xi_1, c)\big(a(x) - a(x+\tau)\big) + \widetilde{G}_b(a_1, b_1, \xi_1, c)\big(b(x) - b(x+\tau)\big)$$

where a_1, b_1, ξ_1 lie between $a(x)$ and $a(x+\tau)$, $b(x)$ and $b(x+\tau)$, $\varphi(x)$ and $\phi(x)$, respectively.

We need only consider the case $\alpha > 0$ since the case $\alpha < 0$ is analogous. By the assumption (3) and (17), we have $\varphi, \xi_1 \in [\phi_2, \phi_1]$, and so

$$-\lambda_1 \leqslant g_1(x) \leqslant -\lambda_2, \qquad \big|r_1(x)\big| \leqslant Q_2\delta,$$

where $Q_2 = Q_2(\alpha)$. For any $x \in \mathbb{R}$ and $z < x$, solving for ψ_1 on $[z, x]$ we have

$$\psi_1(x) = \psi_1(z)\left(\exp\int_z^x g_1(s)\,ds\right) + \int_z^x r_1(t)\exp\left(\int_t^x g_1(s)\,ds\right)dt.$$

Sending $z \to -\infty$ we have

$$\big|\psi_1(x)\big| \leqslant Q_2\delta\int_{-\infty}^x \exp\left(\int_t^x g_1(s)\,ds\right)dt \leqslant Q_2\delta\int_{-\infty}^x \exp\big(-\lambda_2(x-t)\big)\,dt = \frac{Q_2\delta}{\lambda_2}.$$

This implies that (24) holds for $Q = Q_2/\lambda_2$. This proves the lemma. $\square$

2.4. *The arithmetic mean*

Now we show that for the limit φ in Lemma 2.3, the arithmetic mean of $\tan\varphi$ is $\tan\alpha$.

Denote $h = h(\delta) := \frac{1}{\lambda_2}\ln\frac{1}{\delta}$ for any $\delta > 0$. Then there exists $\delta_0 > 0$ small such that

$$\max\left\{M_\delta, \frac{1}{\delta}\right\} > h \quad \text{for } \delta \in (0, \delta_0). \tag{25}$$

In what follows, we always consider $\delta \in (0, \delta_0)$. For any $\tau \in \mathscr{A}_\delta$, denote

$$x_- := \max\{-n, -n-\tau\}, \qquad x_+ := \min\{n, n-\tau\}.$$

Lemma 2.7. *Let (c_n, φ_n) be the unique solution of* (9) *for given α satisfying* (3). *If $\tau \in \mathscr{A}_\delta$ is chosen such that $h + x_- < x_+$, then*

$$\big|\tan\varphi_n(x) - \tan\varphi_n(x+\tau)\big| \leqslant \widetilde{Q}\,\delta \quad \textit{for } x \in [h + x_-, x_+], \tag{26}$$

for some $\widetilde{Q} = \widetilde{Q}(Q, \alpha)$.

Proof. Define $\psi_n(x) := \varphi_n(x) - \varphi_n(x+\tau)$. A similar proof as proving (24) shows that

$$\left|\varphi_n(x) - \varphi_n(x+\tau)\right| \leqslant Q\,\delta \quad \text{for } x \in [h + x_-, x_+].$$

Then by (17),

$$\left|\tan\varphi_n(x) - \tan\varphi_n(x+\tau)\right| = \sec^2\xi(x)\cdot\left|\varphi_n(x) - \varphi_n(x+\tau)\right| \leqslant \widetilde{Q}\,\delta,$$

where $\xi(x)$ lies between $\varphi_n(x)$ and $\varphi_n(x+\tau)$ and $\widetilde{Q} = Q\sec^2\phi_1$. $\square$

Lemma 2.8. *Let (c_n, φ_n) be the unique solution of* (9) *for given α satisfying* (3)*. For any $T_\delta \in \mathscr{A}_\delta$ satisfying $T_\delta \geqslant \max\{M_\delta, \frac{1}{\delta}\}$ we have*

$$\left|\int_0^{T_\delta}\left[\tan\varphi_n(x) - \tan\alpha\right]dx\right| < 2\widetilde{Q}\,\delta T_\delta + 4\tan\phi_1 \quad \textit{for } n > (T_\delta)^2. \tag{27}$$

Proof. For $T_\delta \geqslant \max\{M_\delta, \frac{1}{\delta}\}$, we have $T_\delta > h$ when $\delta \in (0, \delta_0)$.

For any given $n > (T_\delta)^2$, we choose $k \in \mathbb{N}$ such that $kT_\delta \leqslant n - h < (k+1)T_\delta$. Since a and b are almost periodic, there exists

$$\tau_i \in \mathscr{A}_\delta \cap [iT_\delta, iT_\delta + M_\delta] \subset \left[iT_\delta, (i+1)T_\delta\right],$$

for each $i \in \{-k, -k+1, \ldots, -1, 0, 1, 2, \ldots, k-1\}$.

It is easy to see that when $i \in \{-k, -k+1, \ldots, -1\}$, $\tau_i < 0$, $x_- = -n - \tau_i$, $x_+ = n$. Hence

$$[h + x_-, x_+] = [h - n - \tau_i, n] \supset \left[-(k+i)T_\delta, kT_\delta\right]. \tag{28}$$

When $i \in \{0, 1, \ldots, k-1\}$, $\tau_i \geqslant 0$, $x_- = -n$, $x_+ = n - \tau_i$. Hence

$$[h + x_-, x_+] = [-n + h, n - \tau_i] \supset \left[-kT_\delta, (k-i-1)T_\delta\right]. \tag{29}$$

Therefore, by (26), (28) and (29) we have

$$\begin{array}{lll}
(E_{-k}) & \left|\tan\varphi_n(x) - \tan\varphi_n(x+\tau_{-k})\right| \leqslant \widetilde{Q}\,\delta, & x \in [0, kT_\delta], \\
 & \vdots & \\
(E_{-1}) & \left|\tan\varphi_n(x) - \tan\varphi_n(x+\tau_{-1})\right| \leqslant \widetilde{Q}\,\delta, & x \in \left[(-k+1)T_\delta, kT_\delta\right], \\
(E_0) & \left|\tan\varphi_n(x) - \tan\varphi_n(x+\tau_0)\right| \leqslant \widetilde{Q}\,\delta, & x \in \left[-kT_\delta, (k-1)T_\delta\right], \\
(E_1) & \left|\tan\varphi_n(x) - \tan\varphi_n(x+\tau_1)\right| \leqslant \widetilde{Q}\,\delta, & x \in \left[-kT_\delta, (k-2)T_\delta\right], \\
 & \vdots & \\
(E_{k-1}) & \left|\tan\varphi_n(x) - \tan\varphi_n(x+\tau_{k-1})\right| \leqslant \widetilde{Q}\,\delta, & x \in [-kT_\delta, 0].
\end{array}$$

Also we have

$$(E)\quad \bigl|\tan\varphi_n(x)-\tan\varphi_n(x+T_\delta)\bigr|\leqslant \widetilde{Q}\,\delta,\quad x\in\bigl[-kT_\delta,(k-1)T_\delta\bigr].$$

Now

$$\begin{aligned}\int_{-n}^{n}\tan\varphi_n(x)\,dx &= \left(\int_{-n}^{-kT_\delta}+\int_{-kT_\delta}^{(-k+1)T_\delta}+\cdots+\int_{-T_\delta}^{0}+\int_{0}^{T_\delta}+\int_{T_\delta}^{2T_\delta}+\cdots+\int_{(k-1)T_\delta}^{kT_\delta}+\int_{kT_\delta}^{n}\right)\tan\varphi_n(x)\,dx\\ &=: I_-+I_{-k}+\cdots+I_{-1}+I_0+I_1+\cdots+I_{k-1}+I_+.\end{aligned}$$

Since $0<n-kT_\delta<2T_\delta$ and $\varphi_n\in[\phi_2,\phi_1]$ we have $|I_\pm|\leqslant 2T_\delta\tan\phi_1$.

In what follows, for any $P\in\mathbb{R}$ and $P>0$, we use $\langle P\rangle$ to denote functions which take values in $[-P,P]$. By (E_{k-1}) we have

$$\begin{aligned}I_{-k} &= \int_{-kT_\delta}^{-kT_\delta+T_\delta}\bigl[\tan\varphi_n(x)-\tan\varphi_n(x+\tau_{k-1})\bigr]\,dx+\int_{-kT_\delta}^{-kT_\delta+T_\delta}\tan\varphi_n(x+\tau_{k-1})\,dx\\ &= \langle\widetilde{Q}\,\delta T_\delta\rangle+\int_{\tau_{k-1}-kT_\delta}^{\tau_{k-1}-kT_\delta+T_\delta}\tan\varphi_n(s)\,ds\\ &= \langle\widetilde{Q}\,\delta T_\delta\rangle+\int_{\tau_{k-1}-kT_\delta}^{0}\tan\varphi_n(s)\,ds+\int_{T_\delta}^{\tau_{k-1}-kT_\delta+T_\delta}\tan\varphi_n(s)\,ds+\int_{0}^{T_\delta}\tan\varphi_n(s)\,ds\\ &= \langle\widetilde{Q}\,\delta T_\delta\rangle+\int_{\tau_{k-1}-kT_\delta}^{0}\bigl[\tan\varphi_n(s)-\tan\varphi_n(s+T_\delta)\bigr]\,ds+\int_{0}^{T_\delta}\tan\varphi_n(s)\,ds.\end{aligned}$$

Since $-T_\delta<\tau_{k-1}-kT_\delta<0$, by (E) we have

$$I_{-k}=\langle 2\widetilde{Q}\,\delta T_\delta\rangle+\int_{0}^{T_\delta}\tan\varphi_n(s)\,ds.$$

In a similar way, for $i=-k+1,\dots,-1,0,1,\dots,k-1$ by (E_{-i-1}) we have

$$I_i=\langle 2\widetilde{Q}\,\delta T_\delta\rangle+\int_{0}^{T_\delta}\tan\varphi_n(s)\,ds.$$

In the case $\alpha>0$, $\varphi_n\in[\phi_2,\phi_1]$, so

$$\begin{aligned}\int_{-n}^{n}\tan\varphi_n(s)\,ds &\leqslant 2k\left(\int_{0}^{T_\delta}\tan\varphi_n(s)\,ds+2\widetilde{Q}\,\delta T_\delta\right)+4\tan\phi_1 T_\delta\\ &\leqslant \frac{2n}{T_\delta}\left(\int_{0}^{T_\delta}\tan\varphi_n(s)\,ds+2\widetilde{Q}\,\delta T_\delta\right)+4\tan\phi_1 T_\delta.\end{aligned}$$

Since $\int_{-n}^{n} \tan\varphi_n(s)\,ds = 2n\tan\alpha$ and $n > T_\delta^2$ we have

$$0 \leqslant \int_0^{T_\delta} \left[\tan\varphi_n(s) - \tan\alpha\right] ds + 2\widetilde{Q}\,\delta T_\delta + 2\tan\phi_1.$$

Noting $\frac{2n}{T_\delta} \geqslant 2k \geqslant \frac{2n}{T_\delta} - 4$ and using a similar discussion as above we have

$$\int_0^{T_\delta} \left[\tan\varphi_n(s) - \tan\alpha\right] ds \leqslant 2\widetilde{Q}\,\delta T_\delta + 4\tan\phi_1.$$

Combining the last two estimates we then obtain (27) in the case $\alpha > 0$. The case $\alpha < 0$ can be similarly considered. □

Lemma 2.9. *Assume* (3) *and let φ be as in Lemma* 2.3. *Then the arithmetic mean of* $\tan\varphi$ *is* $\tan\alpha$.

Proof. Since φ is almost periodic, so is $\tan\varphi$. Hence the arithmetic mean $\mathcal{M}[\tan\varphi]$ exists (e.g. [14]), and

$$\mathcal{M}[\tan\varphi] := \lim_{T\to\infty} \frac{1}{T} \int_l^{l+T} \tan\varphi(x)\,dx \quad \text{uniformly in } l \in \mathbb{R}. \tag{30}$$

Fix $\delta \in (0, \delta_0)$ and fix T_δ. The estimate (27) holds for any $n = n_i > (T_\delta)^2$. Taking limit as $i \to \infty$ we have

$$\left| \frac{1}{T_\delta} \int_0^{T_\delta} \tan\varphi(s)\,ds - \tan\alpha \right| \leqslant 2\widetilde{Q}\,\delta + \frac{4\tan\phi_1}{T_\delta}.$$

Then, taking limits in both sides as $\delta \to 0$ (note that $T_\delta \to \infty$ since we have chosen $T_\delta > \frac{1}{\delta}$) and using (30) we have

$$\mathcal{M}[\tan\varphi] = \tan\alpha. \qquad \square$$

Remark 2.10. One sees that if $M_\delta \leqslant \frac{k}{\delta}$ as $\delta \to 0$ for some $k > 0$, then we can choose $T_\delta \in [\frac{k}{\delta}, \frac{2k}{\delta}]$ in the proof of Lemma 2.8 and obtain

$$\left| \int_0^{T_\delta} \left[\tan\varphi(x) - \tan\alpha\right] dx \right| \leqslant 4k\widetilde{Q} + 4\tan\phi_1$$

by (27). This implies that $\int_0^x \tan\varphi(x)\,dx - x\tan\alpha$ is bounded (and so is almost periodic by The Bohl–Bohr–Amerio theorem (see e.g. [14])). However, $\delta M_\delta = O(1)$ is not a condition that can be easily verified. We conjecture that to guarantee that $\int_0^x \tan\varphi(x)\,dx - x\tan\alpha$ is almost periodic, (3) and (7) are sufficient conditions.

2.5. Uniqueness of positive or negative solutions of (2)

Lemma 2.11. *Let* (c_1, φ_1) *be a solution of* (2)*. Assume that either* $\inf_{\mathbb{R}} \varphi_1 \geqslant 0$ *or* $\sup_{\mathbb{R}} \varphi_1 \leqslant 0$*. Then*

$$\frac{b_m}{c_1} \leqslant \cos\varphi_1(x) \leqslant \frac{b_M}{c_1} \quad \forall x \in \mathbb{R}.$$

Proof. If $\varphi_1'(x) = 0$ we have $c_1 \cos\varphi_1(x) = b(x) \in [b_m, b_M]$. Now let $\{x_j\}$ be a sequence such that as $j \to \infty$, $\varphi_1(x_j) \to \sup\varphi_1$ and $\varphi_1'(x_j) \to 0$. Then we find that $c_1 \cos(\sup\varphi_1) \in [b_m, b_M]$. Similarly, let $\{x_j\}$ be a sequence such that as $j \to \infty$, $\varphi_1(x_j) \to \inf\varphi_1$ and $\varphi_1'(x_j) \to 0$. Then we find that $c_1 \cos(\inf\varphi_1) \in [b_m, b_M]$. Hence, in the case $\inf_{\mathbb{R}} \varphi_1 \geqslant 0$ or $\sup_{\mathbb{R}} \varphi_1 \leqslant 0$, we have the assertion of the lemma. □

Lemma 2.12. *Let* $(c, \varphi) = (c_1, \varphi_1)$ *and* $(c, \varphi) = (c_2, \varphi_2)$ *be solutions of* (2)*. Assume that* $c_1 \geqslant c_2$ *and*

$$\min\Big\{\inf_{\mathbb{R}}\varphi_1, \inf_{\mathbb{R}}\varphi_2\Big\} > 0 \quad \Big(\textit{or} \quad \max\Big\{\sup_{\mathbb{R}}\varphi_1, \sup_{\mathbb{R}}\varphi_2\Big\} < 0\Big).$$

Then there exist positive constant λ *and* Λ *such that*

$$\lambda(c_1 - c_2) \leqslant \varphi_1(x) - \varphi_2(x) \leqslant \Lambda(c_1 - c_2) \quad \forall x \in \mathbb{R}$$
$$\big(\textit{or} \quad \lambda(c_1 - c_2) \leqslant \varphi_2(x) - \varphi_1(x) \leqslant \Lambda(c_1 - c_2) \quad \forall x \in \mathbb{R}\big).$$

Proof. We consider only the case $m := \min\{\inf\varphi_1, \inf\varphi_2\} > 0$. The other case can be proven in a similar manner.

Set $\psi_2(x) := \varphi_1(x) - \varphi_2(x)$. Then $\psi_2' = r_2(x) - g_2(x)\psi_2$, where, for some ξ_2 lying between φ_1 and φ_2,

$$g_2(x) := \frac{b(x)}{a(x)} \frac{\sin\xi_2(x)}{\cos\varphi_1 \cos\varphi_2}, \qquad r_2(x) := \frac{c_1 - c_2}{a(x)}.$$

By assumption and the previous lemma, we have

$$\inf_{\mathbb{R}} g_2 \geqslant g_0 := \frac{b_m \sin m}{a_M}, \qquad \sup_{\mathbb{R}} g_2 \leqslant g^0 := \frac{b_M}{a_m} \frac{c_1}{b_m} \frac{c_2}{b_m}.$$

Solving $\psi_2' = r_2 - g_2\psi_2$ we have, for any $z < x$,

$$\psi_2(x) = \psi_2(z) \exp\Bigg(-\int_z^x g_2(s)\,ds\Bigg) + \int_z^x r_2(s) \exp\Bigg(-\int_s^x g_2(t)\Bigg)\,ds.$$

Since g_2 is uniformly positive, we can sending $z \to -\infty$ to obtain

$$\psi_2(x) = \int_{-\infty}^x \frac{c_1 - c_2}{a(s)} \exp\Bigg(-\int_s^x g_2(t)\,dt\Bigg)\,ds.$$

Using the lower and upper bounds of g_2 we obtain

$$\frac{c_1 - c_2}{a_M g^0} \leqslant \psi_2(x) \leqslant \frac{c_1 - c_2}{a_m g_0}.$$

This completes the proof. □

Corollary 2.13. *Assume that α satisfies* (3)*.*

(1) *There exists a unique solution (c, φ) of* (2) *such that φ is almost periodic and $\mathcal{M}[\tan\varphi] = \tan\alpha$.*
(2) *The whole sequence $\{(c_n, \varphi_n)\}_{n\in\mathbf{N}}$ obtained in Lemma 2.2 converges to (c, φ) as $n \to \infty$.*

Proof. We need only consider the case $\alpha \in (\arccos\frac{b_m}{b_M}, \frac{\pi}{2})$ since the other case $\alpha \in (-\pi/2, \arccos\frac{b_m}{b_M}]$ can be proven similarly.

Suppose that (c, φ) solves (2), φ is almost periodic, and $\mathcal{M}[\tan\varphi] = \tan\alpha$. Then from $\mathcal{M}[\tan\varphi] = \tan\alpha$ and the ode $a\varphi'\cos\varphi = c\cos\varphi - b$ we can derive that $c\cos\alpha \in [b_m, b_M]$. We claim that $\varphi(x) \neq 0$ for all $x \in \mathbb{R}$. Indeed, if this is not true, then there will be a point $x_0 \in \mathbb{R}$ such that $\varphi(x_0) = 0 \geqslant \varphi'(x_0)$, which implies by (2) that $c \leqslant b(x_0) \leqslant b_M$. This, however, implies that $b_m \leqslant c\cos\alpha \leqslant b_M\cos\alpha$, contradicting (3). Thus, we must have either $\inf\varphi \geqslant 0$ or $\sup_{\mathbb{R}}\varphi \leqslant 0$. Since $\mathcal{M}[\tan\varphi] = \tan\alpha > 0$, we must have $\inf_{\mathbb{R}}\varphi \geqslant 0$. Indeed, $\alpha > \arccos\frac{b_m}{b_M}$ is equivalent to $b_m > b_M\cos\alpha$, and so $c\cos\alpha \geqslant b_m > b_M\cos\alpha$. This implies that $c > b_M$ and $\sup\cos\varphi \leqslant b_M/c < 1$ by Lemma 2.11, so $\inf\varphi > 0$.

Now suppose $(\tilde{c}, \tilde{\varphi})$ is another solution of (2) having the property that $\tilde{\varphi}$ is almost periodic and $\mathcal{M}[\tan\tilde{\varphi}] = \tan\alpha$. Then $\inf\tilde{\varphi} > 0$. It then follows from Lemma 2.12 and $\mathcal{M}[\tan\varphi] = \mathcal{M}[\tan\tilde{\varphi}]$ that $c = \tilde{c}$ and $\varphi \equiv \tilde{\varphi}$. This proves the first assertion. The second assertion follows from the first assertion. $\square$

Proof of Theorem A. The assertion follows from Lemmas 2.3, 2.6. 2.9 and Corollary 2.13. $\square$

3. Traveling waves with V shaped profiles

In this section we study traveling waves with V shaped profiles; namely, we prove Theorem B. We use the following assumption:

Both (c, φ_1) and (c, φ_2) solve (2), both φ_1 and φ_2 are almost periodic, $\alpha_1 := \arctan\mathcal{M}[\tan\varphi_1] \in (\arctan\frac{b_m}{b_M}, \frac{\pi}{2})$, and $\alpha_2 := \arctan\mathcal{M}[\tan\varphi_2] \in (-\frac{\pi}{2}, -\arccos\frac{b_m}{b_M})$.

Such a choice for α_1 and α_2 is possible since c depends on α continuously by Theorem A. By Theorem A again,

$$\phi_4 \leqslant \varphi_1(x), -\varphi_2(x) \leqslant \phi_3, \qquad c \geqslant \frac{b_m}{\cos\alpha_i} > b_M, \tag{31}$$

where

$$\phi_3 := \max\left\{\arccos\frac{b_m\cos\alpha_1}{b_M}, \arccos\frac{b_m\cos\alpha_2}{b_M}\right\},$$
$$\phi_4 := \min\left\{\arccos\frac{b_M\cos\alpha_1}{b_m}, \arccos\frac{b_M\cos\alpha_2}{b_m}\right\}.$$

Proof of Theorem B. By (31), one can choose $\phi_5 = \phi_5(\alpha_1, \alpha_2) \in (0, \phi_4)$ small and $\delta_1 = \delta_1(\alpha_1, \alpha_2) > 0$ small such that

$$\frac{b_m}{\cos\alpha_i} \geqslant (1 + \delta_1)\frac{b_M}{\cos\phi_5}. \tag{32}$$

(i) Let φ be a (local in x) solution of (2) with initial value $\varphi(0) \in (\varphi_2(0), \varphi_1(0))$. Then all φ_1, φ, and φ_2 satisfy the same ode (2), so by comparison, φ exists on $\mathbb{R}$ and $\varphi_2(x) < \varphi(x) < \varphi_1(x)$ for all $x \in \mathbb{R}$.

(ii) Assume that $\varphi(0) \in (-\phi_5, \phi_5)$. Set $J := [-\phi_5, \phi_5]$,

$$x_1 = \sup\{x' > 0 \mid \varphi(x) \in J \text{ on } [0, x']\}, \qquad x_2 = \inf\{x' < 0 \mid \varphi(x) \in J \text{ on } [x', 0]\}.$$

On $[0, x_1]$, by (31) and (32) we have

$$\varphi' \geqslant \frac{1}{a(x)}\left(\frac{b_m}{\cos\alpha_1} - \frac{b(x)}{\cos\varphi}\right) \geqslant \frac{1}{a(x)}\left(\frac{b_m}{\cos\alpha_1} - \frac{b_M}{\cos\phi_5}\right) \geqslant \delta_2 = \delta_2(\alpha_1) := \frac{\delta_1 b_M}{a_M \cos\phi_5}.$$

So $\varphi(x) \geqslant \delta_2 x + \varphi(0)$ on $[0, x_1]$. This implies that $x_1 \leqslant (\phi_5 - \varphi(0))/\delta_2$. Moreover, $\varphi' > 0$ for $\varphi \in [-\phi_5, \phi_5]$ implies that $\varphi(x) > \phi_5$ for $x > x_1$.

Now denote $\psi_3(x) := \varphi_1(x) - \varphi(x)$, then on $[x_1, \infty)$, $0 < \psi_3 < \phi_3$ and $\psi_3' = g_3(x)\psi_3$, where for some $\xi_3(x) \in (\varphi(x), \varphi_1(x)) \subset [\phi_5, \phi_3]$,

$$g_3(x) := -\frac{b(x)\sin\xi_3(x)}{a(x)\cos\varphi\cos\varphi_1} \leqslant -\lambda_3 := -\frac{b_m \sin\phi_5}{a_M \cos^2\phi_5} \quad \text{on } [x_1, \infty).$$

Therefore on $[x_1, \infty)$, $\psi_3' \leqslant -\lambda_3\psi_3$ and so $0 < \psi_3(x) \leqslant \psi_3(x_1)\exp(\lambda_3 x_1 - \lambda_3 x)$ for $x \in [x_1, \infty)$.

In a similar way, denote $\psi_4(x) := \varphi(x) - \varphi_2(x)$, then $0 < \psi_4(x) \leqslant \psi_4(x_2)\exp(\lambda_3 x - \lambda_3 x_2)$ for $x \in (-\infty, x_2]$.

(iii) Define

$$V_1(x) = \int_x^\infty \left[\tan\varphi_1(x) - \tan\varphi(x)\right] dx, \qquad V_2(x) = \int_{-\infty}^x \left[\tan\varphi(x) - \tan\varphi_2(x)\right] dx \quad \text{for } x \in \mathbb{R}.$$

It is easy to see that V_1 is strictly decreasing and $V_1(-\infty) = \infty$, $V_1(\infty) = 0$; V_2 is strictly increasing and $V_2(-\infty) = 0$, $V_2(\infty) = \infty$. Hence there exists a unique x_0 such that $V_2(x_0) = V_1(x_0) =: S_0$. Define $v_1(x)$, $v_2(x)$ and $v(x)$ as in Theorem B. Then for $x > x_0$,

$$v(x) - v_1(x) = \int_{x_0}^x \left[\tan\varphi(x) - \tan\varphi_1(x)\right] dx + V_1(x_0) = V_1(x) > 0,$$

and when $x > \max\{x_0, x_1\}$, for some $\xi_4(x) \in [\phi_5, \phi_3]$, we have

$$v(x) - v_1(x) = V_1(x) = \int_x^\infty \sec^2\xi_4(x) \cdot \psi_3(x)\, dx \leqslant \lambda_3^{-1}\left(\sec^2\phi_3\right)\psi_3(x_1)\exp(\lambda_3 x_1 - \lambda_3 x).$$

In a similar way we have

$$0 < v(x) - v_2(x) \leqslant \lambda_3^{-1}\left(\sec^2\phi_3\right)\psi_4(x_2)\exp(\lambda_3 x - \lambda_3 x_2) \quad \text{for } x < \min\{x_0, x_2\}.$$

This proves Theorem B. $\square$

4. Homogenized speed

This section is devoted to the proof of Theorem C.

Let α be a fixed constant satisfying (3). For definiteness, we assume that α is positive (the case α negative is analogous). By Theorem A, for each $\varepsilon > 0$, $(2)_\varepsilon$ has a unique solution $(c^\varepsilon, \varphi^\varepsilon)$ in which φ^ε is almost periodic and $\mathcal{M}[\tan\varphi^\varepsilon] = \tan\alpha$. In addition, for every $x \in \mathbb{R}$ and $\varepsilon > 0$,

$$b_m \leqslant c^\varepsilon \cos\varphi^\varepsilon(x) \leqslant b_M, \quad c^\varepsilon \in \left[\frac{b_m}{\cos\alpha}, \frac{b_M}{\cos\alpha}\right],$$
$$\varphi^\varepsilon(x) \in \left[\arccos\frac{b_M\cos\alpha}{b_m}, \arccos\frac{b_m\cos\alpha}{b_M}\right].$$

To prove the assertion of Theorem C, we need only show that φ^ε is almost a constant.

4.1. Mean oscillation of a^ε and b^ε

We define a function M that measures the mean oscillation of a and b by

$$M(\ell) = \sup_{x\in\mathbb{R}}\left|\frac{1}{\ell}\int_x^{x+\ell}\left(\frac{1}{a(s)} - \mathcal{M}\left[\frac{1}{a}\right]\right)ds\right| + \sup_{x\in\mathbb{R}}\left|\frac{1}{b_m}\frac{1}{\ell}\int_x^{x+\ell}\left(\frac{b(s)}{a(s)} - \mathcal{M}\left[\frac{b}{a}\right]\right)ds\right|. \tag{33}$$

Next we introduce

$$\alpha^\varepsilon := \arccos\frac{\mathcal{M}[b/a]}{c^\varepsilon\mathcal{M}[1/a]}, \qquad h^\varepsilon(x) = \int_0^x\left\{\frac{c^\varepsilon}{a^\varepsilon(s)} - \frac{b^\varepsilon(s)}{a^\varepsilon(s)\cos\alpha^\varepsilon}\right\}ds \quad \forall x \in \mathbb{R}.$$

The special choice of α^ε implies that

$$h^\varepsilon(x) = c^\varepsilon\int_0^x\left\{\frac{1}{a^\varepsilon(s)} - \mathcal{M}\left[\frac{1}{a}\right]\right\}dx - \frac{1}{\cos\alpha^\varepsilon}\int_0^x\left(\frac{b^\varepsilon(s)}{a^\varepsilon(s)} - \mathcal{M}\left[\frac{b}{a}\right]\right)ds.$$

Note that $1/\cos\alpha^\varepsilon = c^\varepsilon\mathcal{M}[1/a]/\mathcal{M}[b/a] \in [c^\varepsilon/b_M, c^\varepsilon/b_m]$. Hence, for every $x \in \mathbb{R}$ and $\ell > 0$, using $a^\varepsilon(x) = a(x/\varepsilon)$ and $b^\varepsilon(x) = b(x/\varepsilon)$ we obtain

$$\left|h^\varepsilon(x+\ell) - h^\varepsilon(x)\right| \leqslant c^\varepsilon\ell M\left(\frac{\ell}{\varepsilon}\right) \leqslant \frac{b_M}{\cos\alpha}\ell M\left(\frac{\ell}{\varepsilon}\right) \quad \forall \ell > 0.$$

Finally, the definition of α^ε also implies that

$$\left|c^\varepsilon - \frac{\mathcal{M}[b/a]}{\mathcal{M}[1/a]\cos\alpha}\right| = c^\varepsilon\left|1 - \frac{\cos\alpha^\varepsilon}{\cos\alpha}\right| \leqslant \frac{b_M}{\cos^2\alpha}\left\|\varphi^\varepsilon - \alpha^\varepsilon\right\|_{L^\infty(\mathbb{R})}.$$

4.2. Oscillations of the solution φ^ε

Let $\zeta^\varepsilon(x) = \varphi^\varepsilon(x) - \alpha^\varepsilon$. Then

$$\zeta^{\varepsilon\prime} = h^{\varepsilon\prime} - \frac{b^\varepsilon}{a^\varepsilon}\left\{\frac{1}{\cos\varphi^\varepsilon} - \frac{1}{\cos\alpha^\varepsilon}\right\} = h^{\varepsilon\prime} + g^\varepsilon\zeta^\varepsilon$$

where, for some $\xi^\varepsilon = \xi^\varepsilon(x)$ lying between $\varphi^\varepsilon(x)$ and α^ε,

$$g^\varepsilon = -\frac{b^\varepsilon \sin\xi^\varepsilon}{a^\varepsilon \cos\varphi^\varepsilon \cos\alpha^\varepsilon} \in [-\lambda_1, -\lambda_2]$$

where λ_1 and λ_2 are those in (20). Indeed, using $\cos\varphi^\varepsilon, \cos\alpha^\varepsilon \in [b_m \cos\alpha/b_M, b_M\cos\alpha/b_m]$, it is easily shown that

$$\lambda_1 = \frac{b_M^2\sqrt{b_M^2 - b_m^2\cos^2\alpha}}{a_m b_m^2 \cos^2\alpha} \geqslant -g^\varepsilon \geqslant \frac{b_m^2\sqrt{b_m^2 - b_M^2\cos^2\alpha}}{a_M b_M^2\cos^2\alpha} = \lambda_2.$$

Now using variation of constants we have

$$\zeta^\varepsilon(x) = \int_{-\infty}^{x} h^{\varepsilon\prime}(s)\exp\left(\int_s^x g^\varepsilon(t)\,dt\right)ds = \int_{-\infty}^{x}\left[h^\varepsilon(s) - h^\varepsilon(x)\right]g^\varepsilon(s)\exp\left(\int_s^x g^\varepsilon(t)\,dt\right)ds.$$

Hence,

$$\left|\zeta^\varepsilon(x)\right| \leqslant \int_{-\infty}^{x}\left|h^\varepsilon(s) - h^\varepsilon(x)\right|\lambda_1\exp\bigl(-\lambda_2(x-s)\bigr)\,ds \leqslant \frac{\lambda_1 b_M}{\cos\alpha}\int_0^\infty s\exp(-\lambda_2 s)M\left(\frac{s}{\varepsilon}\right)ds.$$

Thus,

$$\left\|\varphi^\varepsilon - \alpha\right\|_{L^\infty(\mathbb{R})} \leqslant 2\left\|\zeta^\varepsilon\right\|_{L^\infty(\mathbb{R})} \leqslant \frac{2\lambda_1 b_M}{\cos\alpha}\int_0^\infty s\exp(-\lambda_2 s)M\left(\frac{s}{\varepsilon}\right)ds.$$

We summarize our calculation as follows:

Theorem 4.1. *Let $(c^\varepsilon, \varphi^\varepsilon)$ be a solution of $(2)_\varepsilon$ in which φ^ε is almost periodic, $\mathcal{M}[\tan\varphi^\varepsilon] = \tan\alpha$ and α satisfies* (3)*. There exist positive constants Q and λ_2 that depend only on a_m, a_M, b_m, b_M and α such that*

$$\left\|\varphi^\varepsilon - \alpha\right\|_{L^\infty(\mathbb{R})} + \left|c^\varepsilon - \frac{\mathcal{M}[b/a]}{\mathcal{M}[1/a]\cos\alpha}\right| \leqslant Q\int_0^\infty s\exp(-\lambda_2 s)M\left(\frac{s}{\varepsilon}\right)ds$$

where $M(\cdot)$ is defined as in (33)*.*

4.3. Proof of Theorem C

The assertion (6) follows immediately from the above theorem since for almost periodic a and b,

$$\lim_{\ell\to\infty} M(\ell)=0.$$

Also, under condition (7), we have

$$0\leqslant M(\ell)\leqslant \frac{L_1 b_m+L_2}{b_m\ell}\quad \forall \ell>0.$$

This is enough to yield the estimate (8).

Remark 4.2. Another possible method to study the limit of c^ε is using the "stretched" variables $\theta := x/\varepsilon$, $\eta(\theta):=\varphi^\varepsilon(\varepsilon\theta)$ to consider the equation

$$\eta_\theta=\varepsilon\left(\frac{c^\varepsilon}{a(\theta)}-\frac{b(\theta)}{a(\theta)\cos\eta}\right).$$

One can compare $\eta(\theta)$ and the solution $\bar\eta(\theta)$ of the following homogenized problem

$$\bar\eta_\theta=\varepsilon\left(\frac{c^\varepsilon}{A}-\frac{B}{A\cos\bar\eta}\right).$$

Using *averaging method* one has $\eta(\theta)-\bar\eta(\theta)=O(\varepsilon)$ for $\theta\in[0,1/\varepsilon]$ (we refer to [13,14,19] for related results on averaging method). On the other hand, $\eta(\tau)-\eta(0)=O(\delta)$ for any $\tau\in\mathscr{A}_\delta$ since η is almost periodic. These facts imply that $\bar\eta(\tau)-\bar\eta(0)=O(\delta)+O(\varepsilon)$, and so $\bar\eta(\theta)$ cannot be strictly monotone on $[0,1/\varepsilon]$. A precise analysis along this line shows that $\bar\eta\approx\alpha$, $c^\varepsilon\approx B/\cos\bar\eta$, and $c^\varepsilon\to B/\cos\alpha$ as $\varepsilon\to 0$.

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