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Homogenization limit of a parabolic equation with nonlinear boundary conditions ☆

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ABSTRACT

Consider the parabolic equation

$$u_t = a(u_x)u_{xx} + f(u_x), \quad -1 < x < 1,\ t > 0, \tag{E}$$

with nonlinear boundary conditions:

$$\begin{aligned} u_x(-1,t) &= g\big(u(-1,t)/\varepsilon\big), \\ u_x(1,t) &= -g\big(u(1,t)/\varepsilon\big), \end{aligned} \tag{NBC}$$

where $\varepsilon > 0$ is a parameter, g is a function which takes values near its supremum “frequently”. Each almost periodic function is a special example of g. We consider a time-global solution u^ε of (E)–(NBC) and show that its homogenization limit as $\varepsilon \to 0$ is the solution η of (E) with linear boundary conditions:

$$\eta_x(-1,t) = \sup g, \qquad \eta_x(1,t) = -\sup g, \tag{LBC}$$

provided η moves upward monotonically. When g is almost periodic, Lou (preprint) [21] obtained the (unique) almost periodic traveling wave U^ε of (E)–(NBC). This paper proves that the homogenization limit of U^ε is a classical traveling wave of (E)–(LBC).

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1. Introduction

Consider a quasilinear parabolic equation

$$u_t = a(u_x)u_{xx} + f(u_x) \tag{E}$$

for $-1 < x < 1$, $t > 0$, with nonlinear boundary conditions:

$$u_x(-1,t) = g\big(u(-1,t)/\varepsilon\big), \qquad u_x(1,t) = -g\big(u(1,t)/\varepsilon\big) \tag{NBC}$$

for $t > 0$, where $\varepsilon > 0$ is a parameter, a, f and g are smooth functions, say $a, g \in C^2$ and $f \in C^1$ with $a(\cdot) > 0$ and

$$\|g\|_{L^\infty} = G < \infty, \qquad \|g\|_{C^2} := \sum_{j=0}^{2} \big\|g^{(j)}\big\|_{L^\infty} < \infty.$$

Eq. (E) includes the following examples as special cases. Example 1:

$$u_t = \frac{u_{xx}}{1+u_x^2} + A\sqrt{1+u_x^2}, \quad -1 < x < 1,\ t > 0, \tag{1}$$

where $A > 0$ is a constant. Example 2:

$$u_t = \big[\tilde{a}(u_x)\big]_x, \quad -1 < x < 1,\ t > 0. \tag{2}$$

Our motivation for studying (E), (1) and (2) comes from the curve shortening problem (or, flow by mean curvature) and from the theory of phase transitions. See, for example, [1,3,4,6,9,11,13–16,19,20, 25,27] etc. These papers studied the following (mean) curvature flow equation

$$V = \hat{a}(\mathbf{n})\kappa + \hat{f}(\mathbf{n}) \quad \text{on } \Gamma_t \subset \mathbb{R}^2, \tag{3}$$

and its various special cases. Here, for a simple plane curve Γ_t, we use $V, \mathbf{n}$ and κ to denote the normal velocity, the unit normal vector and the curvature of Γ_t, respectively. If Γ_t is the graph of a function $y = u(x,t)$, then

$$V = u_t/\sqrt{1+u_x^2}, \qquad \mathbf{n} = (-u_x, 1)/\sqrt{1+u_x^2}, \qquad \kappa = u_{xx}/\big(1+u_x^2\big)^{3/2},$$

and so (3) is expressed in the form of Eq. (E). The special case of (3) where $\hat{a}(\mathbf{n}) \equiv 1$, $\hat{f}(\mathbf{n}) \equiv A$ is expressed by (1). The special case of (3) where $\hat{a}(\mathbf{n}) = a'(u_x)(1+u_x^2)^{3/2}$, $\hat{f}(\mathbf{n}) \equiv 0$ is expressed by (2).

We remark that a and f in (E) are not assumed to be even like those in (1). So the solutions of (E)–(NBC) are not necessarily to be even in x. This makes our analysis complicated (see Sections 5.2 and 5.3).

To study the above equations in a domain with boundary, one needs some boundary conditions. Among others the conormal boundary condition is a widely used one (cf. [10,19]). In one dimension, it means that the graph of the solution contacts the domain boundary with a certain angle. For example, [1,2,5,12,13,17], etc. required that the contact angle is a constant, this corresponds to the case $g \equiv const.$ in (NBC). When $g(y) \not\equiv const.$, condition (NBC) means that the contact angle depends on the position of the contacting point.

As far as we know, nonlinear conormal boundary conditions like (NBC) have not been studied so much for the motion by mean curvature, though, as a kind of oblique derivative boundary conditions, the initial–boundary value problems have been studied a lot [19,26,29]. Especially, the nature of the propagation of surfaces satisfying (NBC) is far from being well understood compared with the case $g \equiv const$. As one can see below, the effects of nonlinear boundary conditions are completely different from and much more complicated than linear cases. Despite the notable difference between nonlinear and linear boundary conditions, linear boundary conditions as the following play important roles in studying the homogenization problems of (E)–(NBC).

$$u_x(-1,t) = \tan\alpha^\delta, \qquad u_x(1,t) = -\tan\alpha^\delta \quad \text{for } t>0, \tag{LBC$_\delta$}$$

where $\alpha^\delta = \alpha - \delta$ for small $\delta \geqslant 0$, and α is defined by

$$\tan\alpha := \sup_{y\in\mathbb{R}} g(y). \tag{4}$$

In what follows, the solution of (E)–(NBC) (resp. (E)–(LBC)$_\delta$) with some initial data is denoted by u^ε (resp. η^δ).

In Section 2 we state our main results: Theorems 2.3 and 2.6. Theorem 2.3 says that the homogenization limit of u^ε is η^0 if $\eta^0_t > 0$ and if g can take values near its supremum "frequently" (more precisely, g has property (H) below). Theorem 2.6 says that, when g is almost periodic, the homogenization limit of the almost periodic traveling wave U^ε of (E)–(NBC) is the classical traveling wave $\varphi_0(x) + c_0 t$ of (E)–(LBC)$_0$. In Section 2.2 we also present the definition of almost periodic traveling waves, recall its existence and uniqueness. These results have been given in [21].

We emphasize that our homogenization problems, including the periodicity hypothesis of g in Theorem 2.3 and the homogenization limits in both theorems, are completely different from other typical homogenization problems.

Firstly, in a typical homogenization problem (e.g. [4,7,20]), one requires that the problem or its coefficients, like g in (NBC), are spatially periodic or almost periodic. But here, in Theorem 2.3, we will only require that g has the following property (H):

(H) For any $\delta > 0$, the set $\{y \in \mathbb{R} \mid g(y) > \sup_{y\in\mathbb{R}} g(y) - \delta\}$ is relatively dense in $\mathbb{R}$.

Here, a set $S \subset \mathbb{R}$ is called *relatively dense set* in $\mathbb{R}$ if there exists $M > 0$ such that any interval of the form $[r, r+M]$ contains a point in S. Roughly speaking, our assumption requires that g takes values near its supremum "frequently", but we do not care what g is when $\sup g - g \geqslant \delta$. Hence, property (H) is very weak, and is not a real "periodicity". On the contrary, each periodic, or almost periodic function (see Definition 2.4) has property (H). The following example has property (H) but is not an almost periodic function.

$$\tilde g(y) := \begin{cases} \sin y, & \text{if } 2n\pi \leqslant y \leqslant (2n+1)\pi,\ n\in\mathbb{Z}; \\ 2^{-|n|}, & \text{if } 2n\pi + \frac{5\pi}{4} \leqslant y \leqslant 2n\pi + \frac{7\pi}{4},\ n\in\mathbb{Z}; \\ \text{smooth}, & \text{otherwise.} \end{cases}$$

Secondly, in Theorems 2.3 and 2.6, the homogenization limits of solutions of (E)–(NBC) are solutions of (E)–(LBC)$_0$. In other words, the homogenized problems depend only on $\tan\alpha = \sup g$ rather than on various means of g. Such a result is quite surprising and is completely different from other typical homogenization problems. In the latter cases, the homogenized problems usually depend on the harmonic or arithmetic means of the spatially inhomogeneous coefficients.

In Section 3 we study the homogenization limit of u^ε and prove Theorem 2.3. In Section 3.1 we first present the time-global existence and some a priori estimates for the solution u^ε of (E)–(NBC) with initial data u^ε_0. These results have been obtained in our paper [21]. In Sections 3.2–3.4, we study the homogenization limit by giving a precise estimate for u^ε when $0 < \varepsilon \ll 1$. The main difficulty is that, as ε becomes smaller and smaller, the graph of u^ε fluctuates with higher and higher temporal frequency due to the nonlinearity of the boundary conditions. So it is not easy to construct a pair of lower and upper solutions, which have similar speeds and can give a precise estimate for u^ε. As one can see below, we will use η^0 as a lower solution and disturb η^δ (for small $\delta > 0$) with order $\sqrt{\varepsilon}$ to construct an upper solution. Such a pair of lower and upper solutions is good enough to give the desired estimate.

In Section 4 we study the classical traveling wave $\varphi_\delta(x) + c_\delta t$ of (E)–(LBC)$_\delta$ and the asymptotic limit of η^δ. Some of the conclusions will be used in the next section.

In Section 5 we study the homogenization limit of U^ε and prove Theorem 2.6. In Section 5.1, we construct lower and upper solutions similar as studying u^ε to give the homogenization limit of the average speed of U^ε. In Section 5.2 we use the comparison principle and a Lyapunov function to study the homogenization limit of the profile of U^ε.

2. Main theorems

2.1. Homogenization limit of u^ε

When we consider problem (E)–(NBC) with an initial data u_0^ε, we always assume that u_0^ε satisfies the compatibility conditions. More precisely, we choose $u_0^\varepsilon \in C^1_{\mathrm{com}}$, where

$$C^1_{\mathrm{com}} := \{ v \in C^1([-1,1]) \mid v'(-1) = g(v(-1)/\varepsilon),\ v'(1) = -g(v(1)/\varepsilon) \}.$$

Definition 2.1. A function $u^\varepsilon(x,t)$ defined in $[-1,1] \times [0,T)$ is said to be a classical solution of (E)–(NBC) in the time interval $[0,T)$ if u^ε, u^ε_x are continuous in $[-1,1]\times[0,T)$, u^ε_{xx}, u^ε_t are continuous in $(-1,1)\times(0,T)$ and if u^ε satisfies (E)–(NBC) in $(-1,1)\times(0,T)$. It is called a time-global classical solution if $T = +\infty$.

Remark 2.2. In what follows, when we say u^ε is a classical solution in the closed time-interval $[0,T]$, we mean that u^ε is a classical solution in $[0,T)$ and that u^ε, u^ε_x u^ε_{xx}, u^ε_t are continuous up to $t = T$.

Similarly, when we consider problem (E)–(LBC)$_\delta$ with an initial data η_0^δ we always choose η_0^δ from $C^1_{\mathrm{com},\delta}$, where

$$C^1_{\mathrm{com},\delta} := \{ \eta \in C^1([-1,1]) \mid \eta'(-1) = \tan\alpha^\delta,\ \eta'(1) = -\tan\alpha^\delta \}.$$

Classical solutions of (E)–(LBC)$_\delta$ are defined similarly as in Definition 2.1 and Remark 2.2.

In [21] we obtained the existence of time-global solution u^ε of (E)–(NBC) for any initial data $u_0^\varepsilon \in C^1_{\mathrm{com}} \cap C^{1+\lambda}([-1,1])$ $(\lambda > 0)$. This implies the existence of time-global solution η^δ of (E)–(LBC)$_\delta$ for any initial data $\eta_0^\delta \in C^1_{\mathrm{com},\delta} \cap C^{1+\lambda}([-1,1])$. Now we can state our first main result.

Theorem 2.3 *(Homogenization limit of u^ε). Assume that g has property* (H). *Let u^ε be the time-global solution of* (E)–(NBC) *with initial data $u_0^\varepsilon \in C^1_{\mathrm{com}} \cap C^{1+\lambda}([-1,1])$ for some $0 < \lambda < 1$, η^0 be the time-global solution of* (E)–(LBC)$_0$ *with initial data $\eta_0^0 \in C^1_{\mathrm{com},0} \cap C^2([-1,1])$. Assume also that*

(i) *$a((\eta_0^0)')(\eta_0^0)'' + f((\eta_0^0)') > 0$, and*
(ii) *$\lim_{\varepsilon\to 0} \|u_0^\varepsilon - \eta_0^0\|_C = 0$, where $\|\cdot\|_C$ denotes the maximum norm in $C([-1,1])$.*

Then u^ε converges to η^0 in $C^{2,1}_{\mathrm{loc}}((-1,1)\times(0,\infty))$ as $\varepsilon \to 0$.

Condition (i) will be used only in the proof of Lemma 3.11. However, it is not only a technical condition, and cannot be omitted completely. In fact, by (i) we have $\eta_t^0 > 0$, and so u^ε moves virtually upward since η^0 is a lower solution of (E)–(NBC). In other words, the above theorem virtually deals with an upward moving solution, and the homogenization limit is η^0 in this case. As can be expected, if $-g$ has property (H) and if (i) is replaced by some condition guaranteeing that $u_t^\varepsilon < 0$, then the homogenization limit of u^ε will be the solution of (E) with the following boundary conditions:

$$u_x(-1,t) = \inf g, \qquad u_x(1,t) = -\inf g.$$

There are some sufficient conditions which ensure the existence of η_0^0 satisfying the hypotheses in the theorem. For example, $\inf g \leqslant 0$ with $f(0) > 0$, or the conditions (8) and (9) below. The latter indeed guarantee the existence of regular almost periodic traveling waves of (E)–(NBC) (see details in [21]).

2.2. Definition and existence of almost periodic traveling waves

We recall the definition and existence of almost periodic traveling waves, which have been given in [21] (see also [22,23] for related studies).

In the classical notion, a solution of (E) is called a traveling wave if its spatial profile does not change in time and moves at a constant speed. Such a traveling wave exists only if the environment is spatially uniform in the direction of the propagation of the wave. For problem (E)–(NBC), such a classical traveling wave exists only if $g \equiv const.$, and in that case, a traveling wave is given in the form $\varphi(x) + ct$, where φ represents the profile and c the speed. In fact, [1] proved that any time-global solution of (E)–(NBC) with $f \equiv 0$, $g \equiv const.$ converges to a classical traveling wave with speed $\int_g^{-g} a(p)\,dp/2$. However, when $g(y)$ is nonconstant (for example, g is periodic, or almost periodic), classical traveling waves cannot exist. One needs an extended notion of traveling waves. The nature of such traveling waves is far from being well understood compared with classical traveling waves.

We now recall the almost periodicity of Bohr.

Definition 2.4. A bounded continuous function $h : \mathbb{R} \to \mathbb{R}$ is called *almost periodic* in the sense of Bohr if, for any $\delta > 0$, the following set is relatively dense in $\mathbb{R}$:

$$S_\delta := \bigl\{ b \in \mathbb{R} \bigm| \|\sigma_b h - h\|_{L^\infty(\mathbb{R})} < \delta \bigr\},$$

where σ_b denotes the shift operator: $\sigma_b : h(x) \mapsto h(x+b)$. In other words, for any $\delta > 0$, there exists $M_\delta > 0$ such that any interval of the form $[r, r+M_\delta]$ contains a point of S_δ. Also we denote

$$\mathcal{H}_h := \overline{\{\sigma_b h \mid b \in \mathbb{R}\}}^{L^\infty(\mathbb{R})},$$

where $\overline{X}^{L^\infty(\mathbb{R})}$ stands for the closure of a set X in the $L^\infty(\mathbb{R})$ topology.

Definition 2.5. (Cf. [21], see also [22,23].) Let g be an almost periodic function. An entire solution U (i.e. a solution defined for all $t \in \mathbb{R}$) of (E)–(NBC) is called an *almost periodic traveling wave* ("APTW" for short) if there exist a continuous map $\mathcal{W} : [-1,1] \times \mathcal{H}_g \to C([-1,1])$ and a function $b : \mathbb{R} \to \mathbb{R}$ such that

$$b'(t) > 0 \quad (t \in \mathbb{R}), \qquad b(t) \to \pm\infty \quad \text{as } t \to \pm\infty, \tag{5}$$

$$U(x,t) - b(t) = \mathcal{W}(x, \sigma_{b(t)} g) \quad \text{for } t \in \mathbb{R}. \tag{6}$$

An APTW is called *regular* if there exists $\rho > 0$ such that

$$b'(t) \geqslant \rho \quad \text{for } t \in \mathbb{R}.$$

The *average speed* of an APTW U is defined by

$$c := \lim_{T\to\infty} \frac{U(0,t+T)-U(0,t)}{T} \quad \text{uniformly in } t\in\mathbb{R}, \tag{7}$$

provided that this limit exists. It is shown in [21] (see also [22,23]) that any APTW U has average speed c, and $c>0$ if U is regular.

In this and the next subsections, also in Sections 4 and 5 we impose the following additional conditions on g and f

$$\sup_{y\in\mathbb{R}} g(y) = \tan\alpha > 0, \qquad f_0 := \inf_{|p|\leqslant \tan\alpha} f(p) > 0. \tag{8}$$

These conditions are imposed only to shorten the statement and shorten the proof. In fact, other cases rather than (8) can be studied similarly.

Existence, uniqueness and stability of APTW (*see* [21]):

(i) under the conditions (8) and

$$\int_{-\tan\alpha}^{\tan\alpha} \frac{a(p)}{f(p)}\,dp < 2, \tag{9}$$

the problem (E)–(NBC) has a regular APTW. The APTW is unique up to time shift;

(ii) the APTW is asymptotically stable.

We denote this APTW by $U^\varepsilon(x,t)$ and denote its speed by c^ε, in order to emphasize their dependence on ε. U^ε is uniquely determined by the normalization condition

$$U^\varepsilon(0,0)=0. \tag{10}$$

Furthermore, as is shown in [21], we have

$$\left|U_x^\varepsilon(x,t)\right| \leqslant G, \qquad U_t^\varepsilon(x,t) > 0, \quad (x,t)\in[-1,1]\times\mathbb{R}. \tag{11}$$

In case (8) holds, the hypothesis (9) can be interpreted that the front propagation occurs if and only if the driving force f (or A in (1)) is bigger than a certain threshold value, and that this threshold value depends only on $\sup g$. Furthermore, hypothesis (9) is sharp in the sense that (E)–(NBC) has no APTW if a and f are even and if the opposite inequality of (9) holds (see [21–23] for details).

2.3. Homogenization limit of U^ε

Assume that g is almost periodic in this subsection. Our second aim in this paper is to study the homogenization limits of U^ε and c^ε. One can expect that, as $\varepsilon\to 0$, U^ε converges to a classical traveling wave $\varphi(x)+ct$ of (E). Substituting this form into (E) gives

$$c = a(\varphi')\varphi'' + f(\varphi'). \tag{12}$$

We impose the following boundary conditions:

$$\varphi'(-1)=\tan\theta_-, \qquad \varphi'(1)=\tan\theta_+, \tag{13}$$

where $\theta_\pm$ denote the slope angles at the endpoints of the graph of $\varphi(x)$. Putting $\psi=\varphi'$ and integrating (12) yields

$$2=\int_{\tan\theta_-}^{\tan\theta_+}\frac{a(\psi)}{c-f(\psi)}\,d\psi. \tag{14}$$

This gives the relation between the speed c and the angles $\theta_\pm$. Thus, if we can find the endpoint angles $\theta_\pm$, then (14) determines c uniquely. Once c is determined, the profile $\varphi(x)$ can be determined by solving (12)–(13).

In order to make the above argument rigorous and complete, the first thing we have to do is to show that the limit profile does exist. We then have to determine the angles $\theta_\pm$. As we will see in Sections 4 and 5, the limit angles $\theta_\pm$ have surprisingly simple expressions:

$$\theta_-=\alpha, \qquad \theta_+=-\alpha.$$

Our results are the following:

Theorem 2.6 *(Homogenization limits of U^ε and c^ε). Assume that* (8) *and* (9) *hold. Then*:

(i) *c^ε converges to c_0 as $\varepsilon\to 0$, where the constant $c_0=c_0(\alpha)$ is uniquely determined by*

$$2=\int_{-\tan\alpha}^{\tan\alpha}\frac{a(p)}{f(p)-c_0}\,dp. \tag{15}$$

(ii) *$U^\varepsilon(x,t)$ converges to $\varphi_0(x)+c_0t$ in $C^{2,1}_{\mathrm{loc}}((-1,1)\times\mathbb{R})$ as $\varepsilon\to 0$, where $\varphi_0(x)$ is the solution of* (12)–(13) *with $c=c_0$, $\theta_-=\alpha$, $\theta_+=-\alpha$, and φ_0 satisfies the normalization condition $\varphi_0(0)=0$.*

From the formula (15), we easily obtain:

Corollary 2.7. *Assume that* (8) *and* (9) *hold. Then the limit speed $c_0(\alpha)$ satisfies*

$$c_0>0, \qquad \frac{\partial c_0}{\partial\alpha}<0. \tag{16}$$

Theorem 2.8 *(Estimate of the speed). Assume that* (8) *and* (9) *hold. Let c_0 and c^ε be as in Theorem* 2.6. *Then for any small $\delta>0$ there exists a constant $M_\delta>0$ depending only on g and δ such that*

$$c_0\leqslant c^\varepsilon<c_0+P_1M_\delta\sqrt{\varepsilon}+P_2\delta \quad \textit{for sufficiently small } \varepsilon>0, \tag{17}$$

where P_1 and P_2 are positive constants dependent on g but independent of ε and δ.

In the special case where g is periodic, we have a much simpler estimate

$$c_0\leqslant c^\varepsilon<c_0+P\sqrt{\varepsilon} \tag{18}$$

for some positive P dependent on g but independent of ε and δ.

The constant M_δ in (17) coincides with the one defined in Definition 2.4. In the non-periodic case, a simple estimate like (18) does not hold in general. The difference comes from the fact that in the case of almost periodic g, the supremum $\sup g=\tan\alpha$ are not necessarily attained.

Theorem 2.6(i) follows from the conclusions in Section 4.1 and Theorem 2.8, and Theorem 2.8 is proved in Section 5.1. Theorem 2.6(ii) is proved in Sections 5.2 and 5.3. Corollary 2.7 follows from Lemma 4.1.

We close this section by summarizing the relations among u^ε, η^δ, U^ε and $\varphi_0 + c_0 t$.

Proposition 2.9. *Assume that* (8) *and* (9) *hold.*

(i) *Under all the hypotheses of Theorem* 2.3 *we have*

$$\lim_{s\to\infty}\lim_{\varepsilon\to 0}\big[u^\varepsilon(x,t+s)-u^\varepsilon(0,s)\big]=\lim_{s\to\infty}\big[\eta^0(x,t+s)-\eta^0(0,s)\big]=\varphi_0(x)+c_0t, \tag{19}$$

for $(x,t)\in(-1,1)\times\mathbb{R}$.

(ii) *Assume that g is almost periodic. Let* $\{s_n\}$ *be any sequence satisfying* $s_n\to\infty$ $(n\to\infty)$ *and*

$$\sigma_{u^\varepsilon(0,s_n)}g\to g \quad \text{in } L^\infty(\mathbb{R}). \tag{20}$$

Then, for $(x,t)\in(-1,1)\times\mathbb{R}$,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\big[u^\varepsilon(x,t+s_n)-u^\varepsilon(0,s_n)\big]=\lim_{\varepsilon\to 0}U^\varepsilon(x,t)=\varphi_0(x)+c_0t. \tag{21}$$

We remark that the choice of the sequence $\{s_n\}$ in (ii) is possible. In fact, we will prove in Section 4.1 that, under the hypotheses (8) and (9), problem (E)–(LBC)$_0$ has a classical traveling wave $\varphi_0(x)+c_0t$ with $c_0>0$. Hence any time-global solution u^ε of (E)–(NBC) moves to infinity as $t\to\infty$ since $\varphi_0(x)+c_0t$ is a lower solution of (E)–(NBC). Therefore, for any increasing sequence $\{b_n\}$ satisfying $b_0>\|u^\varepsilon(\cdot,0)\|_C$ and $\sigma_{b_n}g\to g$ $(n\to\infty)$ in $L^\infty(\mathbb{R})$, there exists a time sequence $\{s_n\}$ such that $b_n=u^\varepsilon(0,s_n)$, so (20) holds.

Proof of Proposition 2.9. The two equalities in (19) are proved in Theorems 2.3 and 4.3, respectively.

In [21] we used the renormalization method to construct the APTW U^ε. Roughly speaking, for any given $\varepsilon>0$, $\{u^\varepsilon(x,t+s_n)-u^\varepsilon(0,s_n)\}$ is bounded in $C^{2+\mu,1+\mu/2}_{\mathrm{loc}}([-1,1]\times(-s_n,\infty))$ for some $\mu\in(0,1)$ (see also Lemma 3.7 in the next section), so a subsequence of which converges to an entire solution U^ε. The entire solution satisfies the normalization condition $U^\varepsilon(0,0)=0$, and its uniqueness can be proved by comparison principle. So we have

$$\lim_{n\to\infty}\big[u^\varepsilon(x,t+s_n)-u^\varepsilon(0,s_n)\big]=U^\varepsilon(x,t),\quad (x,t)\in[-1,1]\times\mathbb{R}.$$

This proves the first equality in (21) (see details in [21], see also [22,25]). The second equality of (21) is the conclusion of Theorem 2.6. This completes the proof. □

3. Homogenization limits of time-global solutions of (E)–(NBC)

In this section we study the homogenization limit of u^ε and prove Theorem 2.3. In Sections 3.2–3.4, we always assume that the hypotheses of Theorem 2.3 hold.

3.1. Global existence

We first present basic existence results for (E)–(NBC), as well as various a priori estimates of the solution. All the results except Proposition 3.2 were proved in [21] (see also [25] for similar results).

Our main existence result is the following.

Theorem 3.1. *(See [21].) Let* $0<\lambda<1$. *For any* $u_0^\varepsilon \in C^1_{\mathrm{com}} \cap C^{1+\lambda}([-1,1])$, *there exists a time-global classical solution* $u^\varepsilon(x,t)$ *of* (E)–(NBC) *with initial data* u_0^ε. *Moreover,* $u^\varepsilon \in C^{2+\mu,1+\mu/2}([-1,1]\times[\tau,T])$ *for any* $0<\tau<T$, *where* $\mu\in(0,1)$ *is a constant depending on* $\|g\|_{C^1}$ *and* $\|(u_0^\varepsilon)_x\|_C$ *but independent of* τ, T *and* u^ε. *If, in addition,* $u_0^\varepsilon \in C^2([-1,1])$, *then* u_{xx}^ε *and* u_t^ε *are continuous up to* $t=0$.

A consequence of this theorem is the following result.

Proposition 3.2. *Given* $\delta \geqslant 0$. *Let* $0<\lambda<1$ *and* $\eta_0^\delta \in C^1_{\mathrm{com},\delta} \cap C^{1+\lambda}([-1,1])$. *Then there exists a time-global classical solution* $\eta^\delta(x,t)$ *of* (E)–(LBC)$_\delta$ *with initial data* η_0^δ. *If, in addition,* $\eta_0^\delta \in C^2([-1,1])$, *then* η_{xx}^δ *and* η_t^δ *are continuous up to* $t=0$.

To prove Theorem 3.1, [21] derived various a priori estimate for the solution u^ε. First, using maximum principle one easily get

Lemma 3.3 *(A priori gradient bound for* u^ε*). Let* u^ε *be a classical solution of* (E)–(NBC) *with initial data* $u_0^\varepsilon \in C^1([-1,1])$ *on the interval* $[0,T]$. *Then*

$$\left|u_x^\varepsilon(x,t)\right| \leqslant G_1 := \max\left\{\left\|\left(u_0^\varepsilon\right)'\right\|_C, G\right\} \quad \text{for } (x,t)\in[-1,1]\times[0,T]. \tag{22}$$

To prove Theorem 3.1 and to study the homogenization problems below, we need the concepts of upper and lower solutions and comparison principles.

Definition 3.4. A function $u^- \in C^{2,1}([-1,1]\times[0,T])$ is called a *lower solution* of (E)–(NBC) on the interval $[0,T]$ if

$$\begin{cases} u_t^- \leqslant a\left(u_x^-\right)u_{xx}^- + f\left(u_x^-\right), & (x,t)\in(-1,1)\times(0,T], \\ u_x^-(-1,t) \geqslant g\left(u^-(-1,t)/\varepsilon\right), & t\in(0,T), \\ u_x^-(1,t) \leqslant -g\left(u^-(1,t)/\varepsilon\right), & t\in(0,T). \end{cases}$$

A function $u^+ \in C^{2,1}([-1,1]\times[0,T])$ is called an *upper solution* of (E)–(NBC) if the reversed inequalities hold.

The following proposition follows easily from the maximum principle:

Proposition 3.5 *(Comparison principle). Let* u^- *and* u^+ *be a lower and an upper solution of* (E)–(NBC) *on the interval* $[0,T]$, *respectively. Suppose that* $u^-(x,0)\leqslant u^+(x,0)$ *for* $x\in[-1,1]$. *Then*

$$u^-(x,t) \leqslant u^+(x,t) \quad \text{for } (x,t)\in[-1,1]\times[0,T].$$

Furthermore, if $u^-(x,0)\not\equiv u^+(x,0)$ *then*

$$u^-(x,t) < u^+(x,t) \quad \text{for } (x,t)\in[-1,1]\times(0,T].$$

Applying this comparison principle to

$$u^\pm(x,t) := \pm\left(\frac{G}{2}x^2 + M_0 t + \left\|u_0^\varepsilon\right\|_C\right) \quad \text{on } [-1,1]\times[0,T],$$

with

$$M_0 := \max_{|p| \leqslant G} \{ a(p)G + |f(p)| \}, \tag{23}$$

we have the following result.

Lemma 3.6 *(The growth bound of u). Let u^ε be a classical solution of* (E)–(NBC) *with initial data $u_0^\varepsilon \in C^1([-1,1])$ on the interval* $[0,T]$. *Then*

$$|u^\varepsilon(x,t)| \leqslant M_0 t + \|u_0^\varepsilon\|_C + G \quad \text{for } (x,t) \in [-1,1] \times [0,T]. \tag{24}$$

In [21] we also derived the following uniform Hölder estimates of u^ε.

Lemma 3.7. *Let u^ε be the classical solution of* (E)–(NBC) *on the time interval* $[0,T]$ *with initial data $u_0^\varepsilon \in C^1_{\mathrm{com}}$. There exists a constant $\mu \in (0,1)$ depending on ε, $\|g\|_{C^1}$ and $\|(u_0^\varepsilon)'\|_C$ but independent of τ, T and u^ε, such that if $u^\varepsilon \in C^{2+\mu,1+\mu/2}([-1,1] \times (0,T])$, then for any $\tau \in (0,T)$ we have*

$$\|u_x^\varepsilon, u_{xx}^\varepsilon, u_t^\varepsilon\|_{C^{\mu,\mu/2}([-1,1]\times[\tau,T])} \leqslant C_\tau, \tag{25}$$

$$\|u^\varepsilon\|_{C^{\mu,\mu/2}([-1,1]\times[\tau,T])} \leqslant C_\tau (1 + T + \|u_0^\varepsilon\|_C), \tag{26}$$

where C_τ is a constant dependent on ε, τ, $\|g\|_{C^2}$ and $\|u_0^\varepsilon\|_{C^1}$, but independent of u^ε and T.

This lemma can be proved by the Hölder estimates for quasilinear parabolic equations [28] and the a priori estimates for linear parabolic equations [18]. The proof is essentially identical to those of [25, Lemmas 3.15 and 3.17] (see details in [21]). Note that this lemma can be used to prove Theorem 3.1, but it cannot be used to study the homogenization in Theorems 2.3 and 2.6, since both μ and C_τ depend on ε. In Sections 3.4 and 5.2, we will use some interior estimates to give the homogenization limits.

Remark 3.8. It is easily seen that similar estimates as (25) and (26) hold for the solution η^δ of the problem (E)–(LBC)$_\delta$. Moreover, in this case μ depends on $\alpha - \delta$ and $\|\eta_0^\delta\|_{C^1}$ but not on τ, T and η^δ, while C_τ depends on $\tau, \alpha - \delta$ and $\|\eta_0^\delta\|_{C^1}$ but not on T and η^δ.

3.2. Lower solution

Lemma 3.9. *For any $t_0 \geqslant 0$, suppose that $C_-(t_0)$ satisfies*

$$\eta^0(x,t_0) + C_-(t_0) \leqslant u^\varepsilon(x,t_0), \quad x \in [-1,1].$$

Then

$$\eta^0(x,t+t_0) + C_-(t_0) \leqslant u^\varepsilon(x,t+t_0) \quad \text{for } x \in [-1,1],\ t > 0.$$

Proof. $\eta^0(x,t)$ satisfies (E) in $[-1,1] \times (0,\infty)$ and

$$\eta_x^0(-1,t) = \tan\alpha \geqslant g(\eta^0(-1,t)/\varepsilon), \qquad \eta_x^0(1,t) = -\tan\alpha \leqslant -g(\eta^0(1,t)/\varepsilon).$$

Thus η^0 is a lower solution of (E)–(NBC). The conclusion then follows from comparison principle. $\square$

3.3. Upper solution

To construct a suitable upper solution, we require that g has property (H). In other words,

(H)* For any $\delta > 0$, there exist $M_\delta > 0$ and $\{y_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ such that $y_n \to \pm\infty$ $(n \to \pm\infty)$ and that $0 < y_{n+1} - y_n < M_\delta$, $g(y_n) \geqslant \tan\alpha^\delta = \tan(\alpha - \delta)$.

Denote

$$\eta_0^\delta(x) := \eta_0^0(x) + \frac{1}{2}\big[\tan\alpha - \tan\alpha^\delta\big]x^2. \tag{27}$$

Lemma 3.10. *For $0 \leqslant \delta \ll \alpha$, let η^δ be the time-global solution of* (E)–(LBC)$_\delta$ *with initial data η_0^δ.*

(i) *If η_0^0 satisfies* (i) *of Theorem* 2.3, *then*

$$\eta_t^\delta(x,t) \geqslant 0 \quad \textit{for } (x,t) \in [-1,1] \times [0,\infty). \tag{28}$$

(ii) *There exists D depending on $G, \alpha, \|(\eta_0^0)'\|_C$ and $\|(\eta_0^0)''\|_C$ but not on δ such that*

$$\big|\eta_{xx}^\delta(x,t)\big| \leqslant D \quad \textit{for } (x,t) \in [-1,1] \times [0,\infty). \tag{29}$$

Proof. By Lemma 3.3 we have

$$\big|\eta_x^\delta(x,t)\big| \leqslant \max\big\{\big\|\big(\eta_0^\delta\big)'\big\|_C, G\big\} \leqslant K_1 := \big\|\big(\eta_0^0\big)'\big\|_C + \tan\alpha + G$$

for $(x,t) \in [-1,1] \times [0,\infty)$. Denote $\zeta := \eta_t^\delta$, then ζ solves

$$\begin{cases} \zeta_t = a\big(\eta_x^\delta\big)\zeta_{xx} + \big[a'\big(\eta_x^\delta\big)\eta_{xx}^\delta + f'\big(\eta_x^\delta\big)\big]\zeta_x, & (x,t) \in (-1,1) \times (0,\infty), \\ \zeta_x(\pm 1, t) = 0, & t \in (0,\infty), \\ \zeta(x,0) = a\big(\big(\eta_0^\delta\big)'\big)\big(\eta_0^\delta\big)'' + f\big(\big(\eta_0^\delta\big)'\big), & x \in [-1,1]. \end{cases}$$

When $\delta \ll \alpha$, by (i) of Theorem 2.3 we have

$$\eta_t^\delta(x,0) = \zeta(x,0) = a\big(\big(\eta_0^\delta\big)'\big)\big(\eta_0^\delta\big)'' + f\big(\big(\eta_0^\delta\big)'\big) \geqslant 0, \quad x \in [-1,1]. \tag{30}$$

So the comparison principle implies that (28) holds and

$$\big|\zeta(x,t)\big| \leqslant K_2 := \max_{0\leqslant\delta\leqslant\alpha} \big\|a\big(\big(\eta_0^\delta\big)'\big)\big(\eta_0^\delta\big)'' + f\big(\big(\eta_0^\delta\big)'\big)\big\|_C.$$

Hence, for $(x,t) \in [-1,1] \times [0,\infty)$,

$$\big|\eta_{xx}^\delta(x,t)\big| = \left|\frac{\eta_t^\delta - f(\eta_x^\delta)}{a(\eta_x^\delta)}\right| \leqslant D := \Big(K_2 + \max_{|p|\leqslant K_1}\big|f(p)\big|\Big)\Big(\min_{|p|\leqslant K_1} a(p)\Big)^{-1}.$$

Clearly, K_2 and D depend on $G, \alpha, \|(\eta_0^0)'\|_C$ and $\|(\eta_0^0)''\|_C$, but not on δ. $\square$

For any $t_0 \geqslant 0$, let $C_+^\delta(t_0)$ be a constant satisfying $u^\varepsilon(x,t_0) \leqslant \eta^\delta(x,t_0) + C_+^\delta(t_0)$ for $x \in [-1,1]$. We define

$$u^{+,\delta}(x,t) := \eta^{\delta}(x,t) + \psi^{\delta}(x,t) + C_{+}^{\delta}(t_0) \tag{31}$$

for $x \in [-1,1]$ and $t \geqslant 0$, where

$$\psi^{\delta}(x,t) := L\sqrt{\varepsilon}\left(1 - e^{-\frac{B\pi^2 t}{16}} \cos\frac{\pi x}{4}\right), \tag{32}$$

B and L are defined as the following. Denote

$$B_1 := \max_{|p|\leqslant K_1+1,\, |r|\leqslant D} \left|a'(p)r + f'(p)\right|, \qquad B_2 := \max_{|p|\leqslant K_1+1} a(p) \quad \text{and} \quad B := \frac{4}{\pi}B_1 + B_2,$$

where K_1 and D are those in the proof of the previous lemma. So B depends on α, G, $\|(\eta_0^0)'\|_C$ and $\|(\eta_0^0)''\|_C$, but not on δ. L is defined by

$$L = \frac{8}{\sqrt{2}\pi} e^{\frac{B\pi^2}{16}} \left[M_\delta + \max_{0\leqslant\delta\leqslant\alpha} \frac{f(\tan\alpha^\delta)}{2a(\tan\alpha^\delta)} + \frac{D}{2} + 1\right], \tag{33}$$

which depends on G, $\|(\eta_0^0)'\|_C$, $\|(\eta_0^0)''\|_C$ and α, and depends on δ only through M_δ, where M_δ is the number in (H)*.

Lemma 3.11. *Assume that g has property* (H). *Given $\delta > 0$ small. If $\varepsilon > 0$ is sufficiently small, then*

$$u^{\varepsilon}(x,t) \leqslant u^{+,\delta}(x,t) \quad \textit{for } x \in [-1,1],\ t \in [0,1]. \tag{34}$$

Proof. For simplicity, in this proof we write u^+, ψ, η and C_+ instead of $u^{+,\delta}$, ψ^δ, η^δ and C_+^δ, respectively.

To prove (34), it suffices to show that

$$u_t^+ \geqslant a\big(u_x^+\big)u_{xx}^+ + f\big(u_x^+\big), \quad x \in [-1,1],\ 0 \leqslant t \leqslant 1, \tag{35}$$

and

$$u^{\varepsilon}(\pm 1, t) \leqslant u^+(\pm 1, t), \quad t \in [0,1]. \tag{36}$$

First, it is easily seen that

$$\begin{aligned} u_t^+ - a\big(u_x^+\big)u_{xx}^+ - f\big(u_x^+\big) &= \eta_t + \psi_t - a(\eta_x + \psi_x)(\eta_{xx} + \psi_{xx}) - f(\eta_x + \psi_x) \\ &= a(\eta_x)\eta_{xx} + f(\eta_x) + B\psi_{xx} - a(\eta_x + \psi_x)(\eta_{xx} + \psi_{xx}) - f(\eta_x + \psi_x) \\ &= -\psi_x\big[a'(\eta_x + \theta_1\psi_x)\eta_{xx} + f'(\eta_x + \theta_2\psi_x)\big] + \big[B - a(\eta_x + \psi_x)\big]\psi_{xx} \end{aligned}$$

for some $\theta_1, \theta_2 \in [0,1]$, and so $|\eta_x + \theta_1\psi_x|, |\eta_x + \theta_2\psi_x| \leqslant K_1 + 1$ for sufficiently small $\varepsilon > 0$. Hence,

$$u_t^+ - a\big(u_x^+\big)u_{xx}^+ - f\big(u_x^+\big) \geqslant \left[B - \frac{4}{\pi}B_1 - B_2\right]\frac{\sqrt{2}\pi^2}{32} L\sqrt{\varepsilon} e^{-\frac{B\pi^2 t}{16}} = 0.$$

Next we show (36) by indirect method. Define

$$\tau_0 = \sup\{\tau > 0 \mid u^\varepsilon(\pm 1, t) < u^+(\pm 1, t) \text{ for } t \in [0, \tau]\} > 0$$

and suppose $\tau_0 < 1$. Then we may assume that $u^\varepsilon(-1, \tau_0) = u^+(-1, \tau_0)$ since the other case where $u^\varepsilon(1, \tau_0) = u^+(1, \tau_0)$ can be treated similarly. Note that $u^\varepsilon(x, t) \leqslant u^+(x, t)$ for $(x, t) \in [-1, 1] \times [0, \tau_0]$ by the comparison theorem.

Since g has property (H), we have $g((\varepsilon y_n)/\varepsilon) \geqslant \tan \alpha^\delta$ by (H)*. For any $n \in \mathbb{Z}$ we define

$$\chi_n(x) = \varepsilon y_n + \int_{-1}^{x} p(x)\, dx, \quad x \in [-1, 1],$$

where $p(x)$ is the unique solution of the initial-value problem

$$p'(x) = -\frac{f(p)}{a(p)} \quad (x \geqslant -1), \qquad p(-1) = \tan \alpha^\delta.$$

It is easily seen that

$$\chi_n(-1) = \varepsilon y_n, \qquad \chi_n'(-1) = \tan \alpha^\delta, \qquad \chi_n''(-1) = -\frac{f(\tan \alpha^\delta)}{a(\tan \alpha^\delta)}, \tag{37}$$

the graph of χ_n is a curve intersecting $x = -1$ at $(-1, \varepsilon y_n)$ with contacting angle $\frac{\pi}{2} - \alpha^\delta$. Hence χ_n can be regarded as a stationary upper solution of (E) on the interval $I := [-1, -1 + \sqrt{\varepsilon}\,]$. In other words, if

$$u^\varepsilon(-1 + \sqrt{\varepsilon}, t) \leqslant \chi_n(-1 + \sqrt{\varepsilon}\,) \quad \text{for } t \in [0, \tau_0],$$

then

$$u^\varepsilon(x, t) \leqslant \chi_n(x) \quad \text{for } x \in I,\ t \in [0, \tau_0]. \tag{38}$$

By the definition of χ_n we have

$$\chi_n(-1 + \sqrt{\varepsilon}\,) = \chi_n(-1) + \sqrt{\varepsilon} \tan \alpha^\delta - \frac{f(\tan \alpha^\delta)}{2a(\tan \alpha^\delta)} \varepsilon + O\left(\varepsilon^{3/2}\right) \tag{39}$$

and that

$$\left|\chi_{n+1}(-1 + \sqrt{\varepsilon}\,) - \chi_n(-1 + \sqrt{\varepsilon}\,)\right| = \varepsilon(y_{n+1} - y_n) < \varepsilon M_\delta.$$

Since $\chi_n(-1 + \sqrt{\varepsilon}\,) \to \pm\infty$ as $n \to \pm\infty$, there exists an integer N such that

$$\chi_N(-1 + \sqrt{\varepsilon}\,) - \varepsilon M_\delta \leqslant u^+(-1 + \sqrt{\varepsilon}, \tau_0) < \chi_N(-1 + \sqrt{\varepsilon}\,). \tag{40}$$

By (28) we have

$$u_t^+(x, t) = \eta_t(x, t) + \psi_t(x, t) > 0 \quad \text{for all } (x, t) \in [-1, 1] \times [0, 1].$$

Consequently, for all $t \in [0, \tau_0]$ we have

$$u^\varepsilon(-1+\sqrt{\varepsilon}, t) \leqslant u^+(-1+\sqrt{\varepsilon}, t) \leqslant u^+(-1+\sqrt{\varepsilon}, \tau_0) < \chi_N(-1+\sqrt{\varepsilon}). \tag{41}$$

We remark that this is the unique place where we use the conditions (28) and so is the unique place to use the hypothesis (i) of Theorem 2.3.

By (38) we have

$$u^\varepsilon(x,t) \leqslant \chi_N(x) \quad \text{for } x \in I,\ t \in [0, \tau_0].$$

Especially,

$$u^+(-1, \tau_0) = u^\varepsilon(-1, \tau_0) \leqslant \chi_N(-1). \tag{42}$$

On the other hand,

$$\eta(-1+\sqrt{\varepsilon}, \tau_0) - \eta(-1, \tau_0) = \sqrt{\varepsilon}\tan\alpha^\delta + \frac{\eta_{xx}(-1,\tau_0)}{2}\varepsilon + O\big(\varepsilon^{3/2}\big),$$
$$\psi(-1+\sqrt{\varepsilon}, \tau_0) - \psi(-1, \tau_0) = -\frac{\sqrt{2}\pi}{8} L e^{-\frac{B\pi^2}{16}\tau_0}\varepsilon + O\big(\varepsilon^{3/2}\big).$$

By our assumption $\tau_0 < 1$ we have

$$u^+(-1+\sqrt{\varepsilon}, \tau_0) - u^+(-1, \tau_0) \leqslant \sqrt{\varepsilon}\tan\alpha^\delta + \frac{D}{2}\varepsilon - \frac{\sqrt{2}\pi}{8} L e^{-\frac{B\pi^2}{16}}\varepsilon + O\big(\varepsilon^{3/2}\big). \tag{43}$$

Combining (43) with (39) and (40), we obtain

$$\begin{aligned} u^+(-1,\tau_0) - \chi_N(-1) &\geqslant u^+(-1+\sqrt{\varepsilon}, \tau_0) - \chi_N(-1+\sqrt{\varepsilon}) \\ &\quad + \left(L\frac{\sqrt{2}\pi}{8}e^{-\frac{B\pi^2}{16}} - \frac{D}{2} - \frac{f(\tan\alpha^\delta)}{2a(\tan\alpha^\delta)}\right)\varepsilon + O\big(\varepsilon^{3/2}\big) \\ &\geqslant \varepsilon\left(L\frac{\sqrt{2}\pi}{8}e^{-\frac{B\pi^2}{16}} - \frac{D}{2} - \frac{f(\tan\alpha^\delta)}{2a(\tan\alpha^\delta)} - M_\delta\right) + O\big(\varepsilon^{3/2}\big) \\ &> 0 \end{aligned}$$

by the definition of L. This contradicts (42), and so we have $\tau_0 \geqslant 1$. The lemma is proved. □

Conclusion (34) implies that

$$u^\varepsilon(x,1) \leqslant \eta^\delta(x,1) + L\sqrt{\varepsilon} + C_+^\delta,$$

for some C_+^δ. From time $t = 1$, we can consider the next time interval $t \in [1,2]$ as above and to obtain

$$\begin{aligned} u^\varepsilon(x, 1+t) &\leqslant \eta^\delta(x, 1+t) + L\sqrt{\varepsilon} + C_+^\delta + \psi^\delta(x,t) \\ &\leqslant \eta^\delta(x, 1+t) + 2L\sqrt{\varepsilon} + C_+^\delta, \quad (x,t) \in [-1,1]\times[0,1]. \end{aligned}$$

Repeating this process we have the following conclusion.

Corollary 3.12. *Assume that g has property* (H). *Given small $\delta > 0$. If $\varepsilon > 0$ is sufficiently small, then*

$$u^\varepsilon(x,t) \leqslant \eta^\delta(x,t) + (t+1)L\sqrt{\varepsilon} + C_+^\delta \quad \textit{for } t > 0. \tag{44}$$

3.4. Homogenization limit

Proof of Theorem 2.3. By hypothesis (ii) of Theorem 2.3 we have

$$d(\varepsilon) := \|u_0^\varepsilon - \eta_0^0\|_C \to 0 \quad \text{as } \varepsilon \to 0.$$

Now we compare three solutions of different problems. The first one is the time-global solution u^ε of the problem (E)–(NBC) with initial data u_0^ε; The second one is the time-global solution η^0 of the problem (E)–(LBC)$_0$ with initial data η_0^0; The third one is the time-global solution η^δ of the problem (E)–(LBC)$_\delta$ with initial data η_0^δ given by (27). $\delta > 0$ is small such that (30) and (28) hold by the assumption (i) of Theorem 2.3. Since

$$\eta_0^0(x) - d(\varepsilon) \leqslant u_0^\varepsilon(x) \quad \text{for } x \in [-1,1],$$

we have by Lemma 3.9

$$\eta^0(x,t) - d(\varepsilon) \leqslant u^\varepsilon(x,t) \quad \text{for } x \in [-1,1],\ t > 0. \tag{45}$$

Since

$$u_0^\varepsilon(x) \leqslant \eta_0^0(x) + d(\varepsilon) \leqslant \eta_0^\delta(x) + d(\varepsilon) \quad \text{for } x \in [-1,1],$$

we have by Lemma 3.11 and Corollary 3.12,

$$u^\varepsilon(x,t) \leqslant \eta^\delta(x,t) + (t+1)L\sqrt{\varepsilon} + d(\varepsilon) \quad \text{for } x \in [-1,1],\ t > 0.$$

Therefore,

$$\eta^0(x,t) - d(\varepsilon) \leqslant u^\varepsilon(x,t) \leqslant \eta^\delta(x,t) + (t+1)L\sqrt{\varepsilon} + d(\varepsilon) \tag{46}$$

for any $x \in [-1,1]$, $t > 0$.

Choose $\varepsilon_0 > 0$ sufficiently small and consider the set $\{u^\varepsilon(x,t)\}_{\varepsilon \in (0,\varepsilon_0)}$. Applying the interior Hölder estimates for quasilinear parabolic equations [28, Theorem 2.3] to (E), we see that there exists a constant $\tilde{\mu} \in (0,1)$ independent of ε such that for any fixed $\varrho \in (0,1)$ and $\tau, T \in \mathbb{R}$ with $T > \tau > 0$, we have

$$\|u_x^\varepsilon\|_{C^{\tilde{\mu},\tilde{\mu}/2}([-1+\varrho,1-\varrho]\times[\tau,T])} \leqslant C,$$

where C is a positive constant depending on ϱ and τ but independent of ε and T. Therefore, Lemma 3.6 and the interior a priori estimates for linear parabolic equations [18, Theorem 8.11.1] imply that there exists a constant $\widetilde{C} = \widetilde{C}(\varrho, \tau, T)$ independent of ε satisfying

$$\|u^\varepsilon\|_{C^{2+\tilde{\mu},1+\tilde{\mu}/2}([-1+\varrho,1-\varrho]\times[\tau,T])} \leqslant \widetilde{C}.$$

Hence, using Cantor's diagonal argument, we can find a subsequence $\{u^{\varepsilon_j}\}_j$ which converges to a function $u^* \in C^{2+\tilde{\mu},1+\tilde{\mu}/2}((-1,1)\times(0,\infty))$ in the topology of $C^{2,1}_{\mathrm{loc}}((-1,1)\times(0,\infty))$.

By (46) and by the definition of $d(\varepsilon)$ we have

$$\eta^0(x,t) \leqslant \lim_{j\to\infty} u^{\varepsilon_j}(x,t) = u^*(x,t) \leqslant \eta^\delta(x,t), \quad x \in (-1,1),\ t > 0.$$

Now we take limit as $\delta \to 0$ and obtain

$$u^*(x,t) = \eta^0(x,t) \quad \text{for } -1 < x < 1,\ t > 0.$$

Here we used the fact that $\eta^\delta \to \eta^0$ as $\delta \to 0$. This fact follows from the continuously dependence of η^δ on the boundary conditions $(\mathrm{LBC})_\delta$ and on the initial data (27).

Finally, for any convergent sequence $\{u^{\varepsilon_j}\}$, it converges to the same limit η^0 in $C^{2,1}_{\mathrm{loc}}((-1,1) \times (0,\infty))$. This proves Theorem 2.3. □

4. Problem (E)–(LBC)$_\delta$

In this section we always assume that (8) and (9) hold.

4.1. Classical traveling waves

As we have mentioned above, classical traveling waves of (E)–(LBC)$_\delta$ with the form $u(x,t) = \varphi(x) + ct$ play important roles in our study. They are solutions of

$$c = a(\varphi_x)\varphi_{xx} + f(\varphi_x), \quad x \in [-1,1], \tag{47}$$

with

$$\varphi_x(-1) = \tan\alpha^\delta, \qquad \varphi_x(1) = -\tan\alpha^\delta. \tag{48}$$

Without loss of generality, we impose a normalization condition on φ:

$$\varphi(0) = 0. \tag{49}$$

We want to find solutions (c,φ) of (47)–(48)–(49) with $c \geqslant 0$. The following notation will be used

$$\mathcal{I}(\delta,c) := \int_{-\tan\alpha^\delta}^{\tan\alpha^\delta} \frac{a(p)}{f(p)-c}\,dp$$

if $f(p) \neq c$ for $|p| \leqslant \tan\alpha^\delta$.

Lemma 4.1. *Assume that* (8) *and* (9) *hold. Then for any small* $\delta \geqslant 0$*, problem* (47)–(48)–(49) *has a unique solution* $(c_\delta, \varphi_\delta)$*. Moreover,* φ_δ *satisfies*

$$\left|\varphi_\delta'(x)\right| \leqslant \tan\alpha^\delta \leqslant G, \qquad \varphi_\delta''(x) < 0 \quad \text{and} \quad \left|\varphi_\delta''(x)\right| \leqslant Q(\alpha) \tag{50}$$

for $x \in [-1, 1]$. The speed $c_\delta > 0$ is uniquely determined by $\mathcal{I}(\delta, c_\delta) = 2$. Furthermore, c_δ is strictly increasing in δ:

$$\frac{\partial c_\delta}{\partial \delta} > 0. \tag{51}$$

Proof. For sufficiently small $\delta \geqslant 0$, by (8) and (9) we have $\alpha^\delta > 0$ and

$$f_\delta := \min_{|p| \leqslant \tan\alpha^\delta} f(p) \geqslant f_0 > 0 \quad \text{and} \quad \mathcal{I}(\delta, 0) < 2.$$

Set $\psi(x) = \varphi'(x)$, then (47) is converted into

$$\psi' = \frac{c - f(\psi)}{a(\psi)}. \tag{52}$$

For each $c \geqslant 0$, we use $\psi(x; c)$ to denote the solution of (52) under the initial data

$$\psi(-1; c) = \tan\alpha^\delta,$$

and use $\varphi(x; c)$ to denote the corresponding solution of (47) which satisfies

$$\varphi_x(-1; c) = \tan\alpha^\delta, \qquad \varphi(0; c) = 0.$$

For any $0 \leqslant c < f_\delta$, we see that

$$\psi'(x; c) = \frac{c - f(\psi(x; c))}{a(\psi(x; c))} < 0$$

as long as $|\psi| \leqslant \tan\alpha^\delta$. Thus, when $|\psi(x; c)| \leqslant \tan\alpha^\delta$, ψ is implicitly defined by

$$x + 1 = \int_{\tan\alpha^\delta}^{\psi(x;c)} \frac{a(p)\,dp}{c - f(p)}. \tag{53}$$

There exists a unique $x_+(c) > -1$ such that $\psi(x_+(c); c) = -\tan\alpha^\delta$ and

$$x_+(c) + 1 = \int_{-\tan\alpha^\delta}^{\tan\alpha^\delta} \frac{a(p)\,dp}{f(p) - c} = \mathcal{I}(\delta, c). \tag{54}$$

Therefore, $x_+(c)$ is continuous and strictly monotone increasing in $c \in [0, f_\delta)$. By our assumption $\mathcal{I}(\delta, 0) < 2$ we have $x_+(0) < 1$. On the other hand, choose $c_1 := f(\tan\alpha^\delta) > 0$, then $\psi(x; c_1) \equiv \tan\alpha^\delta > 0$. Hence there exists a unique $c = c_\delta \in (0, c_1)$ satisfying $x_+(c_\delta) = 1$. This implies that $\psi(1; c_\delta) = \varphi_x(1; c_\delta) = -\tan\alpha^\delta$. For this c_δ, the function $\psi(x; c_\delta)$ given by Eq. (53) corresponds to a function $\varphi_\delta(x) := \varphi(x; c_\delta)$. c_δ and φ_δ clearly satisfy (50) and (51). This completes the proof. □

Corollary 4.2. *There holds*

$$c_\delta = c_0 + k_0\delta + o(\delta) \quad \text{as } \delta \to 0, \tag{55}$$

where c_0 is the constant defined by $\mathcal{I}(0, c_0) = 2$ and $k_0 = k_0(\alpha)$ is a positive constant.

Proof. Differentiating $\mathcal{I}(\delta, c_\delta) = 2$ by δ yields

$$\frac{\partial c_\delta}{\partial \delta} \int_{-\tan\alpha^\delta}^{\tan\alpha^\delta} \frac{a(p)}{(f(p) - c_\delta)^2}\, dp - \frac{a(\tan\alpha^\delta)\sec^2\alpha^\delta}{f(\tan\alpha^\delta) - c_\delta} - \frac{a(-\tan\alpha^\delta)\sec^2\alpha^\delta}{f(-\tan\alpha^\delta) - c_\delta} = 0.$$

Hence,

$$\begin{aligned} k_0 &:= \frac{\partial c_\delta}{\partial \delta}\bigg|_{\delta=0} \\ &= \left(\frac{a(\tan\alpha)\sec^2\alpha}{f(\tan\alpha) - c_0} + \frac{a(-\tan\alpha)\sec^2\alpha}{f(-\tan\alpha) - c_0}\right)\left(\int_{-\tan\alpha}^{\tan\alpha} \frac{a(p)}{(f(p) - c_0)^2}\, dp\right)^{-1} > 0. \end{aligned}$$

Thus we obtain (55). □

4.2. Asymptotic limit of η^δ

We show that the classical traveling wave in the previous subsection can also be obtained by taking asymptotic limit for η^δ.

Theorem 4.3. *Assume that* (8) *and* (9) *hold. For a small $\delta \geqslant 0$, if $\eta_0^\delta \in C^1_{\mathrm{com},\delta} \cap C^2([-1,1])$ satisfies $a((\eta_0^\delta)')(\eta_0^\delta)'' + f((\eta_0^\delta)') > 0$, then*

$$\lim_{s\to\infty}\big[\eta^\delta(x, t+s) - \eta^\delta(0, s)\big] = \varphi_\delta(x) + c_\delta t$$

in $C^{2,1}_{\mathrm{loc}}([-1,1] \times (0,\infty))$.

Proof. For any sequence $s_n \nearrow \infty$, define

$$\eta_n^\delta(x,t) := \eta^\delta(x, t+s_n) - \eta^\delta(0, s_n).$$

Then $\eta_n^\delta(0,0) = 0$. We now prove that η_n^δ converges to the entire solution $\varphi_\delta(x) + c_\delta t$ of (E)–(LBC)$_\delta$.

First we construct an entire solution. For any $T_1 > 1$, by Lemma 3.7 and Remark 3.8 we have

$$\|\eta^\delta\|_{C^{2+\mu,1+\mu/2}([-1,1]\times[1,T_1])} \leqslant C(1 + T_1), \tag{56}$$

where μ and C both depend on α, δ and $\|\eta_0^\delta\|_{C^1}$, but not on T_1 and η^δ.

For any $T > 0$, choose n sufficiently large such that $s_n > 1 + T$, by Lemma 3.7 and Remark 3.8 again we have

$$\|\eta_n^\delta\|_{C^{2+\mu,1+\mu/2}([-1,1]\times[-T,T])} \leqslant C(1 + T),$$

where μ and C both depend on α, δ and $\|\eta_0^\delta\|_{C^1}$, but not on n, T and η_n^δ. Hence there exists $\{n_i\} \subset \{n\}$ and $\bar{\eta}^\delta \in C_{\text{loc}}^{2+\mu,1+\mu/2}([-1,1]\times\mathbb{R})$ such that

$$\eta_{n_i}^\delta \to \bar{\eta}^\delta \quad \text{in } C_{\text{loc}}^{2,1}\big([-1,1]\times\mathbb{R}\big), \text{ as } i \to \infty.$$

Then $\bar{\eta}^\delta(x,t)$ with $\bar{\eta}^\delta(0,0)=0$ is an entire solution of (E)–(LBC)$_\delta$.

Next, we explain the uniqueness of the entire solution. There are several ways to prove this uniqueness. For example, a similar way as proving the uniqueness of the entire solution in [25], or similar argument as Lemmas 5.7, 5.8 and Corollary 5.9 below, or using the maximum principle and the backward uniqueness of parabolic equations (cf. [8]). We omit the detail here. The uniqueness result implies that

$$\bar{\eta}^\delta(x,t) \equiv \varphi_\delta(x) + c_\delta t \quad \text{for all } (x,t) \in [-1,1]\times\mathbb{R},$$

since $\varphi_\delta(x)+c_\delta t$ is also an entire solution of (E)–(LBC)$_\delta$ satisfying $\varphi_\delta(x)+c_\delta t = 0$ at $(x,t)=(0,0)$.

Now we can conclude that η_n^δ converges to the unique entire solution $\varphi_\delta(x)+c_\delta t$ of (E)–(LBC)$_\delta$. Since $\{s_n\}$ is arbitrarily chosen, we complete the proof. □

5. Homogenization limit of the almost periodic traveling wave

Let U^ε be the APTW of (E)–(NBC) and c^ε be its average speed. In this section we study their homogenization limits and prove the conclusions in Section 2.3 under the hypotheses (8), (9) and that g is almost periodic. Part of our approach is similar as that in [22] in some sense, we give details below for the readers' convenience.

5.1. Homogenization limit of the speed

Lemma 5.1. *For any $t_0 \in \mathbb{R}$, if $C_\pm(t_0)$ satisfy*

$$\varphi_0(x) + C_-(t_0) \leqslant U^\varepsilon(x,t_0) \leqslant \varphi_\delta(x) + C_+(t_0),$$

then, for sufficiently small $\varepsilon > 0$,

$$\varphi_0(x) + C_-(t_0) + c_0 t \leqslant U^\varepsilon(x,t_0+t) \leqslant \varphi_\delta(x) + C_+(t_0) + c_\delta t + (t+1)L\sqrt{\varepsilon}, \tag{57}$$

for $t > 0$, where L is the constant given by (33) *and it depends on δ and α.*

Proof. The proof is similar as those in Sections 3.2 and 3.3. □

Proof of Theorem 2.8. By (11) and (50) and by the normalization conditions

$$U^\varepsilon(0,0)=0, \qquad \varphi_\delta(0)=0,$$

we have

$$\varphi_0(x) - 2G \leqslant U^\varepsilon(x,0) \leqslant \varphi_\delta(x) + 2G, \quad x \in [-1,1].$$

So Lemma 5.1 implies that

$$\varphi_0(x) - 2G + c_0 t \leqslant U^\varepsilon(x,t) \leqslant \varphi_\delta(x) + 2G + c_\delta t + (t+1)L\sqrt{\varepsilon},$$

where L is given by (33). It depends on G and α, depends on δ only through M_δ. Thus

$$c_0 - \frac{2G}{t} \leqslant \frac{U^\varepsilon(0,t) - U^\varepsilon(0,0)}{t} \leqslant c_\delta + L\sqrt{\varepsilon} + \frac{2G + L\sqrt{\varepsilon}}{t}.$$

By the definition of c^ε we have

$$c_0 \leqslant c^\varepsilon \leqslant c_\delta + L\sqrt{\varepsilon} \leqslant c_\delta + C_1 M_\delta \sqrt{\varepsilon} + C_2\sqrt{\varepsilon}. \tag{58}$$

The last inequality follows from (33), where both C_1 and C_2 depend only on G, α, but do not depend on ε and δ.

When $\delta \to 0$, one can choose M_δ large enough such that $C_1 M_\delta \geqslant C_2$. On the other hand, by Corollary 4.2, we have $c_\delta = c_0 + k_0\delta + o(\delta)$ as $\delta \to 0$. Thus, taking $P_1 := 2C_1$ and $P_2 := 2k_0$, we obtain (17). □

Corollary 5.2. $\lim_{\varepsilon\to 0} c^\varepsilon = c_0$.

Proof. Fix a small $\delta > 0$. For any sequence $\{\varepsilon_i\}_i$ with $\varepsilon_i \to 0$, we see from (17) that there exists a subsequence of $\{\varepsilon_i\}_i$, again denoted by $\{\varepsilon_i\}_i$, such that $c^{\varepsilon_i} \to c^*$ for some $c^* \in [c_0, c_0 + P_2\delta]$. Since δ is independent of ε and can be chosen as small as possible, we have $c^* = c_0$. Consequently, $c^\varepsilon \to c_0$ as $\varepsilon \to 0$. □

Remark 5.3. When g is L_0-periodic, we can choose $M_\delta = L_0$ and choose $\varphi_0(x) + c_0 t + \psi(x,t)$ as an upper solution, as we did in the proof of Lemma 3.11. So we can get $c^\varepsilon < c_0 + P\sqrt{\varepsilon}$ (cf. [25]).

5.2. Homogenization limit of the profile

We now study the homogenization limit of the profile of U^ε (Theorem 2.6 (ii)). At first glance this result seems a consequence of Theorem 2.3. More precisely, one may expect that there exists $\tau > 0$ such that, for any $\varepsilon > 0$, $t_0 \in \mathbb{R}$, there holds

$$U^\varepsilon(x, t_\varepsilon) \to \varphi_0(x) + C(t_0), \quad \text{as } \varepsilon \to 0, \tag{59}$$

for some $t_\varepsilon \in [t_0, t_0 + \tau]$. Then it follows from Theorem 2.3 that

$$U^\varepsilon(x, t + \tilde{t}) \to \varphi_0(x) + C(t_0) + c_0 t \quad \text{as } \varepsilon \to 0, \text{ for } t > 0.$$

The problem is that (59) is not obvious, and in some sense, it is as difficult as proving Theorem 2.6(ii). So Theorem 2.3 cannot be applied here directly.

Since U^ε is an APTW rather than the solution u^ε of an initial–boundary problem, we expect to show that the homogenization limit of U^ε is a classical traveling wave. First we derive an estimate of $U^\varepsilon(x,t)$ in an independent way from ε. In a similar way as establishing Lemma 3.6 we have

Lemma 5.4. *For M_0 defined by* (23) (*which is independent of ε*), *there holds*

$$-M_0 t - 3G \leqslant U^\varepsilon(x, t + t_0) - U^\varepsilon(\tilde{x}, t_0) \leqslant M_0 t + 3G \tag{60}$$

for all x, $\tilde{x} \in [-1, 1]$, $t \geqslant 0$ and $t_0 \in \mathbb{R}$. Especially, when $\tilde{x} = 0$, $t_0 = 0$ we have

$$\left\| U^\varepsilon(\cdot, t) \right\|_C \leqslant M_0|t| + 3G \quad \textit{for all } t \in \mathbb{R}. \tag{61}$$

Proof of Theorem 2.6(ii). The first part of this proof is similar as the proof of Theorem 2.3 in Section 3.4. By the above lemma and the fact $|U_x^\varepsilon| \leqslant G$, applying the interior Hölder estimates for quasilinear parabolic equations [28, Theorem 2.3] to (E), we see that there exists a constant $\tilde{\mu} \in (0,1)$ independent of ε such that for any fixed $\varrho \in (0,1)$ and $T > 0$, we have

$$\left\| U_x^\varepsilon \right\|_{C^{\tilde{\mu},\tilde{\mu}/2}([-1+\varrho,1-\varrho]\times[-T,T])} \leqslant C_\varrho,$$

where C_ϱ is a positive constant depending on ϱ but independent of ε and T. Therefore, (61) and the interior a priori estimates for linear parabolic equations [18, Theorem 8.11.1] imply that there exists a constant $\widetilde{C} = \widetilde{C}(\varrho, T)$ independent of ε satisfying

$$\left\| U^\varepsilon \right\|_{C^{2+\tilde{\mu},1+\tilde{\mu}/2}([-1+\varrho,1-\varrho]\times[-T,T])} \leqslant \widetilde{C}.$$

Hence, we can find a subsequence $\{U^{\varepsilon_j}\}_j$ which converges to a function $\overline{U}$ in $C^{2,1}_{\mathrm{loc}}((-1,1)\times\mathbb{R})$. Define

$$\overline{U}(\pm 1, t) = \lim_{x\to\pm 1} \overline{U}(x,t), \qquad \overline{U}_x(\pm 1, t) = \lim_{x\to\pm 1} \overline{U}_x(x,t),$$

then $\overline{U} \in C^{2,1}((-1,1)\times\mathbb{R}) \cap C^{1,0}([-1,1]\times\mathbb{R})$. Moreover, by (11),

$$\left|\overline{U}_x(x,t)\right| \leqslant G \quad \text{for } (x,t) \in [-1,1]\times\mathbb{R}.$$

In order to complete the proof we need some lemmas on the limit $\overline{U}$.

Lemma 5.5. *Let $\overline{U}(x,t)$ be as above.*

(i) *Suppose that*

$$\overline{U}(x,t_0) \geqslant \varphi_0(x) + c_0 t_0 + C_-, \quad x \in [-1,1]$$

for some $t_0 \in \mathbb{R}$ and some constant C_-. Then

$$\overline{U}(x,t) \geqslant \varphi_0(x) + c_0 t + C_-, \quad x \in [-1,1],\ t \geqslant t_0.$$

(ii) *Suppose that*

$$\overline{U}(x,t_0) \leqslant \varphi_0(x) + c_0 t_0 + C_+, \quad x \in [-1,1]$$

for some $t_0 \in \mathbb{R}$ and some constant C_+. Then

$$\overline{U}(x,t) \leqslant \varphi_0(x) + c_0 t + C_+, \quad x \in [-1,1],\ t \geqslant t_0.$$

Proof. (i) Fix a small $\sigma > 0$. Then, for sufficiently large j,

$$U^{\varepsilon_j}(x,t_0) \geqslant \varphi_0(x) + c_0 t_0 + C_- - \sigma, \quad x \in [-1+\sigma, 1-\sigma].$$

Since $|U_x^{\varepsilon_j}| \leqslant G$ and since $|\varphi_0'| \leqslant G$, the above inequality implies that

$$U^{\varepsilon_j}(x,t_0) \geqslant \varphi_0(x) + c_0 t_0 + C_- - \sigma - 2G\sigma, \quad x \in [-1,1].$$

As we have seen in the proof of Lemma 3.9, $\varphi_0(x)+c_0t+C$ is a lower solution of (E)–(NBC) for any C. Hence,

$$U^{\varepsilon_j}(x,t) \geqslant \varphi_0(x) + c_0t + C_- - \sigma - 2G\sigma, \quad x \in [-1,1],\ t \geqslant t_0.$$

Letting $j \to \infty$, we obtain

$$\overline{U}(x,t) \geqslant \varphi_0(x) + c_0t + C_- - (2G+1)\sigma, \quad x \in [-1,1],\ t \geqslant t_0.$$

This implies the conclusion of (i) since $\sigma > 0$ can be chosen arbitrarily small.

(ii) Fix a small $\sigma > 0$. Then, for sufficiently large j,

$$U^{\varepsilon_j}(x,t_0) \leqslant \varphi_0(x) + c_0t_0 + C_+ + \sigma, \quad x \in [-1+\sigma, 1-\sigma].$$

Since $|U_x^{\varepsilon_j}| \leqslant G$ and since $|\varphi_0'| \leqslant G$, the above inequality implies that

$$U^{\varepsilon_j}(x,t_0) \leqslant \varphi_0(x) + c_0t_0 + C_+ + (2G+1)\sigma, \quad x \in [-1,1].$$

Furthermore, by Corollary 4.2 and the definitions of φ_δ and c_δ in Section 4.1, there exists a constant $\delta = \delta(\sigma)$ with $\delta(\sigma) \to 0$ as $\sigma \to 0$ such that $|\varphi_0(x) + c_0t_0 - (\varphi_\delta(x) + c_\delta t_0)| \leqslant \sigma$ for all $x \in [-1,1]$. Hence we have

$$U^{\varepsilon_j}(x,t_0) \leqslant \varphi_\delta(x) + c_\delta t_0 + C_+ + 2(G+1)\sigma, \quad x \in [-1,1].$$

Therefore, arguing as in the proof of Lemma 3.11, we obtain

$$U^{\varepsilon_j}(x,t) \leqslant \varphi_\delta(x) + c_\delta t + C_+ + 2(G+1)\sigma + L\sqrt{\varepsilon_j}, \quad x \in [-1,1],\ t \in [t_0, t_0+1],$$

where L is the constant defined in (33) which depends on δ and α. Letting $j \to \infty$, we see that

$$\overline{U}(x,t) \leqslant \varphi_\delta(x) + c_\delta t + C_+ + 2(G+1)\sigma, \quad x \in [-1,1],\ t \in [t_0, t_0+1].$$

Since $\sigma > 0$ can be chosen arbitrarily small,

$$\overline{U}(x,t) \leqslant \varphi_0(x) + c_0t + C_+, \quad x \in [-1,1],\ t \in [t_0, t_0+1].$$

Repeating this argument, we obtain the desired conclusion. □

Lemma 5.6. *For $t \in \mathbb{R}$ we have*

$$\overline{U}_x(x,t) \leqslant \tan\alpha \quad \text{for } x \in [-1,0], \qquad \overline{U}_x(x,t) \geqslant -\tan\alpha \quad \text{for } x \in [0,1].$$

When a and f are even functions, this lemma is obvious. In fact, the uniqueness of APTW implies that U^ε is symmetric in x if a and f are even functions. So $U_x^\varepsilon(0,t) \equiv 0$. The conclusion of the lemma then follows from maximum principle. If a and f are not even, the proof is complicated and we postpone it to the next subsection.

Lemma 5.7. *There exist constants $C_-^* \leqslant C_+^*$ such that the following hold*:

(i) *For any $t \in \mathbb{R}$,*

$$\varphi_0(x) + c_0 t + C_-^* \leqslant \overline{U}(x,t) \leqslant \varphi_0(x) + c_0 t + C_+^*, \quad x \in [-1,1]. \tag{62}$$

(ii) *There exists a sequence $\{t_n\}_n$ with $t_n \to -\infty$ such that $\overline{U}(x, t+t_n) - c_0 t_n$ converges to a function $\overline{V}(x,t)$ in $C^{2,1}_{\mathrm{loc}}((-1,1)\times\mathbb{R})$. Furthermore, for any $t \in \mathbb{R}$, the limit $\overline{V}$ satisfies that*

$$\varphi_0(x) + c_0 t + C_-^* \leqslant \overline{V}(x,t) \leqslant \varphi_0(x) + c_0 t + C_+^*, \quad x \in [-1,1] \tag{63}$$

and that

$$\min_{x\in[-1,1]} \bigl\{\overline{V}(x,t) - \bigl(\varphi_0(x) + c_0 t + C_-^*\bigr)\bigr\} = 0, \tag{64}$$

$$\max_{x\in[-1,1]} \bigl\{\overline{V}(x,t) - \bigl(\varphi_0(x) + c_0 t + C_+^*\bigr)\bigr\} = 0. \tag{65}$$

Proof. (i) Define

$$C_-(t) = \max\bigl\{C \in \mathbb{R} \bigm| \overline{U}(x,t) \geqslant \varphi_0(x) + c_0 t + C \text{ for } x \in [-1,1]\bigr\},$$
$$C_+(t) = \min\bigl\{C \in \mathbb{R} \bigm| \overline{U}(x,t) \leqslant \varphi_0(x) + c_0 t + C \text{ for } x \in [-1,1]\bigr\}.$$

Then, for each $t \in \mathbb{R}$, we see that

$$\varphi_0(x) + c_0 t + C_-(t) \leqslant \overline{U}(x,t) \leqslant \varphi_0(x) + c_0 t + C_+(t), \quad x \in [-1,1] \tag{66}$$

and that

$$\min_{x\in[-1,1]} \bigl\{\overline{U}(x,t) - \bigl(\varphi_0(x) + c_0 t + C_-(t)\bigr)\bigr\} = 0, \tag{67}$$

$$\max_{x\in[-1,1]} \bigl\{\overline{U}(x,t) - \bigl(\varphi_0(x) + c_0 t + C_+(t)\bigr)\bigr\} = 0. \tag{68}$$

In view of them together with the fact that $|\overline{U}_x| \leqslant G$, $|\varphi_0'| \leqslant G$, we have

$$0 \leqslant C_+(t) - C_-(t) \leqslant 4G$$

for all $t \in \mathbb{R}$. Furthermore, by Lemma 5.5, $C_-(t)$ is nondecreasing in t and $C_+(t)$ is nonincreasing in t. Therefore, the limits

$$C_-^* = \lim_{t\to-\infty} C_-(t), \qquad C_+^* = \lim_{t\to-\infty} C_+(t)$$

exist and are finite. Thus we obtain (62).

(ii) By (62),

$$\varphi_0(x) + c_0 t + C_-^* \leqslant \overline{U}(x, t+s) - c_0 s \leqslant \varphi_0(x) + c_0 t + C_+^*$$

for all $(x,t) \in [-1,1] \times \mathbb{R}$ and $s \in \mathbb{R}$. Since $\overline{U}$ satisfies (E) and since $|\overline{U}_x| \leqslant G$, arguing as previously, we can find a sequence $\{t_n\}_n$ with $t_n \to -\infty$ as $n \to \infty$ such that $\{\overline{U}(x,t+t_n) - c_0 t_n\}_n$ converges to a function $\overline{V}(x,t)$ as $n \to \infty$ in the topology of $C^{2,1}_{\mathrm{loc}}((-1,1) \times \mathbb{R})$. Clearly the limit $\overline{V}$ satisfies (63). Furthermore, replacing t by $t+t_n$ in (67), (68) and letting $n \to \infty$, we obtain (64) and (65). $\square$

Lemma 5.8. *Let $C^*_\pm$ be as in the previous lemma. Then $C^*_- = C^*_+$.*

Proof. Suppose $C^*_- < C^*_+$ and set $W(x,t) = \overline{V}(x,t) - c_0 t$ for $(x,t) \in [-1,1] \times \mathbb{R}$. Note that $W(x,t)$ is a bounded solution of

$$W_t = a(W_x)W_{xx} + f(W_x) - c_0, \quad x \in (-1,1) \times \mathbb{R} \tag{69}$$

and that $\varphi_0(x) + C^*_\pm$ are stationary solutions of the same equation. Therefore, by (63), (64), (65) and the comparison theorem, we see that for every $t \in \mathbb{R}$, $W(\cdot,t) - (\varphi_0 + C^*_-)$ attains its minimum 0 at a boundary point while $W(\cdot,t) - (\varphi_0 + C^*_+)$ attains its maximum 0 at the other boundary point. In view of them, we may assume without loss of generality that

$$W(-1,t) = \varphi_0(-1) + C^*_-, \qquad W(1,t) = \varphi_0(1) + C^*_+ \tag{70}$$

for all $t \in \mathbb{R}$.

Applying Proposition in [24, appendix] to (69)–(70), we can construct a Lyapunov function of the form

$$J[W] := \int_{-1}^{1} \Psi(W, W_x)\, dx,$$

where

$$\Psi(W,p) = \int_0^p \left(\int_0^q \frac{a(r)\, dr}{f(r) - c_0} \right) dq - W.$$

Indeed,

$$\frac{d}{dt} J\big[W(\cdot,t)\big] = -\int_{-1}^{1} \frac{W_t^2}{f(W_x) - c_0}\, dx \leqslant 0.$$

Since $J[W(\cdot,t)]$ is bounded in $t \in \mathbb{R}$, a standard dynamical systems theory shows that the ω-limit set of W is non-empty and is contained in the set of stationary solutions. The uniqueness of the ω-limit point can be shown by the same zero-number argument as in [23], or it also follows from the result in [30]. (The result in [23] is given for semilinear equations, but the proof is virtually the same for a quasilinear equation.) Consequently, $W(x,t)$ converges to a stationary solution $\varphi_0(x - x_\infty) + C_\infty$ of (69)–(70) as $t \to +\infty$ in C^2 sense. Therefore, by Lemma 5.6,

$$\varphi_0'(-1 - x_\infty) \leqslant \tan\alpha = \varphi_0'(-1), \qquad \varphi_0'(1 - x_\infty) \geqslant -\tan\alpha = \varphi_0'(1).$$

Furthermore, (70) implies

$$\varphi_0(-1-x_\infty)+C_\infty=\varphi_0(-1)+C_-^*, \qquad \varphi_0(1-x_\infty)+C_\infty=\varphi_0(1)+C_+^*.$$

These contradict the Hopf boundary lemma. □

Corollary 5.9. $\overline{U}(x,t)=\varphi_0(x)+c_0t$.

Proof. It follows from Lemmas 5.7 and 5.8 that $\overline{U}(x,t)\equiv\varphi_0(x)+c_0t+C_+^*$. On the other hand, since $U^\varepsilon(0,0)=0$ for any $\varepsilon>0$, we have $\overline{U}(0,0)=0$. In view of this and the fact $\varphi_0(0)=0$, we see that the constant C_+^* must be zero. □

Proof of Theorem 2.6(ii). (Continued.) From the above arguments, for any sequence $\{\varepsilon_j\}_j$ with $\varepsilon_j\to 0$ as $j\to\infty$, we can find a subsequence of $\{U^{\varepsilon_j}\}_j$ which converges to the same limit $\varphi_0(x)+c_0t$ in $C^{2,1}_{\mathrm{loc}}((-1,1)\times\mathbb{R})$. Therefore, we conclude that

$$\lim_{\varepsilon\to 0}\left\|U^\varepsilon(x,t)-\varphi_0(x)-c_0t\right\|_{C^{2,1}_{\mathrm{loc}}((-1,1)\times\mathbb{R})}=0.$$

The theorem is proved. □

5.3. Proof of Lemma 5.6

We need an estimate for U_x^ε near the boundaries $x=\pm1$. Since $U_t^\varepsilon>0$ and since $|U_x^\varepsilon|\leqslant G$, we have

$$U_{xx}^\varepsilon(x,t)\geqslant-\rho_1:=-\max_{|p|\leqslant G}\frac{f(p)}{a(p)}\quad\text{for } x\in[-1,1],\ t\in\mathbb{R}, \tag{71}$$

Lemma 5.10. *There exist constants $\varrho\in(0,1)$ and $\varepsilon_0>0$ such that if $\varepsilon\in(0,\varepsilon_0)$ then*

$$U_x^\varepsilon(1-\varrho,t)\leqslant 0\leqslant U_x^\varepsilon(-1+\varrho,t)\quad \textit{for all } t\in\mathbb{R}.$$

Proof. We only prove the first inequality since the other can be treated similarly.

Since we now assume that g is almost periodic, we can find a constant $M>0$ and a sequence $\{y_n\}_{n\in\mathbb{Z}}$ with $y_n\to\pm\infty$ as $n\to\pm\infty$ such that

$$0<y_{n+1}-y_n\leqslant M,\qquad g\big((\varepsilon y_n)/\varepsilon\big)=\frac{3}{4}\tan\alpha \tag{72}$$

for all $n\in\mathbb{Z}$. Let $\{t_n\}_{n\in\mathbb{Z}}$ be a sequence satisfying $t_n\to\pm\infty$ as $n\to\pm\infty$ and $U^\varepsilon(1,t_n)=\varepsilon y_n$ for all $n\in\mathbb{Z}$. Then we have

$$U_x^\varepsilon(1,t_n)=-g\left(\frac{U^\varepsilon(1,t_n)}{\varepsilon}\right)=-g(y_n)=-\frac{3}{4}\tan\alpha.$$

Fix $n\in\mathbb{Z}$. By (71), for $x\in(-1,1)$,

$$U_x^\varepsilon(x,t_n)\leqslant U_x^\varepsilon(1,t_n)+\rho_1(1-x)\leqslant-\frac{3}{4}\tan\alpha+\rho_1(1-x).$$

Letting $\varrho = \min\{1, (4\rho_1)^{-1}\tan\alpha\}$, we see that

$$U_x^\varepsilon(x, t_n) \leqslant -\frac{1}{2}\tan\alpha \quad \text{for } x \in [1-\varrho, 1].$$

Hence,

$$U^\varepsilon(1-\varrho, t_n) \geqslant \varepsilon y_n + \frac{\tan\alpha}{2}\varrho.$$

Let l be a line segment with slope $-\frac{1}{4}\tan\alpha$ through $(1, \varepsilon y_{n+1})$ and let $(1-\varrho, y_+)$ be the intersection point between l and the line $x = 1-\varrho$. Then

$$y_+ = \varepsilon y_{n+1} + \frac{\tan\alpha}{4}\varrho.$$

Therefore, by (72),

$$y_+ - U^\varepsilon(1-\varrho, t_n) \leqslant \varepsilon M - \frac{\tan\alpha}{4}\varrho \leqslant 0$$

provided

$$\varepsilon \leqslant \varepsilon_0 := \frac{\varrho\tan\alpha}{4M}.$$

Hence if $0 < \varepsilon \leqslant \varepsilon_0$ then the graph of $U^\varepsilon(\cdot, t_n)$ intersects the line l at one point in the region $\{x \geqslant 1-\varrho\}$. We denote by x_+ the x-coordinate of the intersection. Then, since $U_t^\varepsilon > 0$, the graph of $U^\varepsilon(\cdot, t)$ must intersect l in the region $\{x \geqslant x_+\}$ for each $t \in [t_n, t_{n+1})$. Therefore, for each $t \in [t_n, t_{n+1})$ there exists $x_0(t) \in [x_+, 1]$ satisfying $U_x^\varepsilon(x_0(t), t) \leqslant -\frac{1}{4}\tan\alpha$. Thus we obtain

$$U_x^\varepsilon(1-\varrho, t) \leqslant U_x^\varepsilon\big(x_0(t), t\big) + \rho_1\big[x_0(t) - (1-\varrho)\big] \leqslant 0$$

if $0 < \varepsilon \leqslant \varepsilon_0$. $\square$

Lemma 5.11. *Let ϱ and ε_0 be as in the previous lemma. Then for any $\varepsilon \in (0, \varepsilon_0)$,*

$$U_x^\varepsilon(x, t) \leqslant \tan\alpha \quad \textit{for } -1 \leqslant x \leqslant 1-\varrho,\ t \in \mathbb{R},$$
$$U_x^\varepsilon(x, t) \geqslant -\tan\alpha \quad \textit{for } -1+\varrho \leqslant x \leqslant 1,\ t \in \mathbb{R}.$$

Proof. We only prove the first inequality since the second one can be treated similarly. Fix $\varepsilon \in (0, \varepsilon_0)$. First we note that $w = U_x^\varepsilon$ satisfies the linear parabolic equation of the form

$$w_t = a(x,t)w_{xx} + b(x,t)w_x \tag{73}$$

with $a(x,t) > 0$.

Suppose that

$$G_- := \sup\big\{U_x^\varepsilon(x,t) \;\big|\; (x,t) \in [-1, 1-\varrho] \times \mathbb{R}\big\} > \tan\alpha.$$

By the maximum principle for (73) in combination with the fact that $U_x^\varepsilon \leqslant \tan\alpha$ on $x = -1$, $x = 1 - \varrho$, there exists a sequence $\{(x_n, t_n)\}_n$ with $t_n \to -\infty$ $(n \to \infty)$ such that $U_x^\varepsilon(x_n, t_n) \to G_-$ as $n \to \infty$.

Set

$$b_n := U^\varepsilon(0, t_n), \qquad U_n(x,t) := U^\varepsilon(x, t + t_n) - b_n, \qquad g_n := \sigma_{b_n} g.$$

Then each U_n $(n \in \mathbb{N})$ is the unique entire solution of (E)–(NBC) with g replaced by g_n, and $U_n(0,0) = 0$. Therefore, taking a subsequence if necessary, we see that

$$U_n \xrightarrow{n\to\infty} U^* \quad \text{in } C^{2,1}_{\mathrm{loc}}\big([-1,1] \times \mathbb{R}\big), \qquad g_n \xrightarrow{n\to\infty} h^* \quad \text{in } L^\infty(\mathbb{R})$$

for some $h^* \in \mathcal{H}_g$ and some entire solution U^* of (E)–(NBC) with g replaced by h^*. Assume that $x_n \to x^* \in [-1, 1 - \varrho]$, then

$$U_x^*\big(x^*, 0\big) = G_- = \sup\big\{U_x^* \,\big|\, (x,t) \in [-1, 1-\varrho] \times \mathbb{R}\big\}.$$

By the maximum principle for U_x^*, we have $x^* = -1$ or $x^* = 1 - \varrho$. This is however impossible since $U_x^*(-1, t) \leqslant \tan\alpha < G_-$ and $U_x^*(1 - \varrho, t) \leqslant 0 < G_-$. The lemma is proved. □

Lemma 5.6 is an immediate consequence of the above lemma.

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