# LYZ matrices and SL( $n$ ) contravariant valuations on polytopes 

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#### Abstract

All SL( $n$ ) contravariant symmetric matrix valued valuations on convex polytopes in $\mathbb{R}^{n}$ are completely classified without any continuity assumptions. The general Lutwak-Yang-Zhang matrix is shown to be essentially the unique such valuation.


## 1 Introduction

Let $\left\{e_{1}, \ldots, e_{n}\right\}$ be the standard basis of $\mathbb{R}^{n}$. We write $x=\left(x_{1}, \ldots, x_{n}\right)$ for the corresponding coordinates. Let $\mathcal{P}_{(0)}^{n}$ denote the space of convex polytopes containing the origin in their interiors in $\mathbb{R}^{n}$. For $P \in \mathcal{P}_{(0)}^{n}$, the Lutwak-Yang-Zhang (LYZ) matrix $\mathrm{L}(P)$ of $P$ is the $(n \times n)$-matrix with coefficients (see [38])

$$
\mathrm{L}_{i j}(P)=\sum_{u \in \mathcal{N}(P)} \frac{a_{P}(u)}{h_{P}(u)} u_{i} u_{j}
$$

where $\mathcal{N}(P)$ denotes the set of all outer unit normals of facets of $P$ and where $a_{P}(u)$ is the ( $n-1$ )-dimensional volume of the facet with unit normal $u \in S^{n-1}$ and $h_{P}(u)=\max \{x \cdot u$ : $x \in P\}$ is the support function of $P$.

[^0]For a general convex body (compact convex subset with nonempty interior) $K \subset \mathbb{R}^{n}$ that contains the origin in its interior, an approximation (with respect to the Hausdorff metric) allows us to define the LYZ matrix $\mathrm{L}(K)$ by an integral involving the $L_{2}$ surface area measure of $K$ (see [38]), i.e.,

$$
\mathrm{L}_{i j}(K)=\int_{S^{n-1}} u_{i} u_{j} d S_{2}(K, u)
$$

Here, the measure $S_{2}(K, \cdot)$ is absolutely continuous with respect to the classical surface area measure $S_{K}$ and has the Radon-Nikodym derivative

$$
\frac{d S_{2}(K, \cdot)}{d S_{K}}=\frac{1}{h_{K}} .
$$

Therefore, with the standard inner product $x \cdot y$ for $x, y \in \mathbb{R}^{n}$, it generates the LYZ ellipsoid $\Gamma_{-2}(K)$ of $K$ by

$$
\Gamma_{-2}(K)=\sqrt{V(K)} E_{\mathrm{L}(K)}
$$

where $V(K)$ denotes the $n$-dimensional volume of $K$ and $E_{A}=\left\{x \in \mathbb{R}^{n}: x \cdot A x \leq 1\right\}$.
The John ellipsoid is a fundamental tool in convex geometry and Banach space geometry $[4,5,20,40,44,50]$. For each convex body $K$, its John ellipsoid is the unique ellipsoid of maximal volume contained in $K$. In 2005, Lutwak, Yang and Zhang [40] extended the classical John ellipsoid to the $L_{p}$ John ellipsoid in the framework of the $L_{p}$-Brunn-Minkowski theory. Indeed, the $L_{2}$ John ellipsoid is just the LYZ ellipsoid $\Gamma_{-2}(K)$ which is in some sence dual to the classical Legendre ellipsoid $\Gamma_{2}(K)$ of classical mechanics: the Legendre ellipsoid is an object of the dual Brunn-Minkowski theory, while the LYZ ellipsoid is the corresponding object of the classical Brunn-Minkowski theory (see [38]). Moreover, Lutwak, Yang and Zhang [39] proved that $\Gamma_{-2}(K) \subset \Gamma_{2}(K)$ which can be viewed as a geometrical analogue of the Cramér-Rao inequality (see $[7,45]$ ). For more information on the LYZ ellipsoid, its applications, and its connection to the Fisher information from information theory, see $[9$, 29, 38, 39].

The LYZ matrix defines a matrix valued valuation. A function $\mu$ defined on a lattice $(\mathcal{L}, \vee, \wedge)$ and taking values in an abelian semigroup is called a valuation if

$$
\begin{equation*}
\mu(P \vee Q)+\mu(P \wedge Q)=\mu(P)+\mu(Q) \tag{1.1}
\end{equation*}
$$

for all $P, Q \in \mathcal{L}$. A function $\mu$ defined on some subset $\mathcal{L}_{0}$ of $\mathcal{L}$ is called a valuation on $\mathcal{L}_{0}$ if (1.1) holds whenever $P, Q, P \vee Q, P \wedge Q \in \mathcal{L}_{0}$. Valuations on the space of convex bodies are a classical concept going back to Dehn's solution of Hilbert's Third Problem. Ever since Hadwiger [17] proved his now classical characterization of the quermassintegrals (elementary mixed volumes), the classification of valuations on the space of convex bodies and related spaces has been an important subject in geometry. For detailed information and an historical
account, see $[8,22,46]$. See also $[1-3,10-16,21,23-33,35,42,43]$ for some of the more recent contributions.

In 2003, Ludwig [29] established the first characterization of the moment matrix and the LYZ matrix. Let $\mathbb{M}^{n}$ denote the set of symmetric $(n \times n)$-matrices over $\mathbb{R}^{n}$. A function from a topological space with values in a Euclidean space is called measurable if it is Borel measurable. A function $\mu: \mathcal{P}_{(0)}^{n} \rightarrow \mathbb{M}^{n}$ is called $\operatorname{GL}(n)$ contravariant if there exists a $q \in \mathbb{R}$ such that

$$
\mu(\phi P)=|\operatorname{det} \phi|^{q} \phi^{-t} \mu(P) \phi^{-1}
$$

for every $P \in \mathcal{P}_{(0)}^{n}$ and every $\phi \in \mathrm{GL}(n)$. Here $\operatorname{det} \phi$ denotes the determinant of $\phi$ and $\phi^{t}$ denotes the transpose of $\phi$. For a convex polytope $P \subset \mathbb{R}^{n}$, the moment matrix $\mathrm{M}(P)$ of $P$ is the $(n \times n)$-matrix with coefficients

$$
\mathrm{M}_{i j}(P)=\int_{P} x_{i} x_{j} d x
$$

Theorem 1.1 ( [29]). Let $n \geq 3$. A function $\mu: \mathcal{P}_{(0)}^{n} \rightarrow \mathbb{M}^{n}$ is a measurable $\operatorname{GL}(n)$ contravariant valuation if and only if there exist constants $c_{1}, c_{2} \in \mathbb{R}$ such that

$$
\mu(P)=c_{1} \mathrm{~L}(P)+c_{2} \mathrm{M}\left(P^{*}\right)
$$

for every $P \in \mathcal{P}_{(0)}^{n}$, where $P^{*}$ denotes the polar body of $P$.
Haberl and Parapatits [15] established a classification of tensor valuations without any homogeneity assumptions (see $[3,6,18,19,33,34,37,49]$ for more information on matrix and tensor valuations). Note that Ludwig's definitions of GL( $n$ ) contravariance turn out to coincide with the corresponding definitions of tensors.

Recently, Ludwig and Reitzner [36] showed that the continuity assumptions can be removed in the characterization of $\mathrm{SL}(n)$ invariant valuations on polytopes in $\mathcal{P}_{0}^{n}$, the space of convex polytopes containing the origin in $\mathbb{R}^{n}$.

In 2019, the first author [41] obtained a characterization of the moment matrix on $\mathcal{P}_{0}^{n}$ without any continuity assumptions. Let $\mathcal{P}^{n}$ denote the space of convex polytopes in $\mathbb{R}^{n}$. Here, let $\mathcal{Q}^{n}$ be either $\mathcal{P}_{0}^{n}$ or $\mathcal{P}^{n}$. A function $\mu: \mathcal{Q}^{n} \rightarrow \mathbb{M}^{n}$ is called $\operatorname{SL}(n)$ equivariant if $\mu(\phi P)=\phi \mu(P) \phi^{t}$ for every $P \in \mathcal{Q}^{n}$ and every $\phi \in \operatorname{SL}(n)$. Correspondingly, a function $\mu: \mathcal{Q}^{n} \rightarrow \mathbb{M}^{n}$ is called $\operatorname{SL}(n)$ contravariant if

$$
\mu(\phi P)=\phi^{-t} \mu(P) \phi^{-1}
$$

for every $P \in \mathcal{Q}^{n}$ and every $\phi \in \operatorname{SL}(n)$. A different notation for this identity is

$$
\mu(\phi P)=\phi^{-t} \cdot \mu(P)
$$

(see Section 2 for the definition of $\cdot$ ).

Theorem 1.2 ( [41]). Let $n \geq 3$. A function $\mu: \mathcal{P}_{0}^{n} \rightarrow \mathbb{M}^{n}$ is an $\operatorname{SL}(n)$ equivariant valuation if and only if there exists a constant $c \in \mathbb{R}$ such that

$$
\mu(P)=c \mathrm{M}(P)
$$

for every $P \in \mathcal{P}_{0}^{n}$.
We extend the LYZ matrix from convex ploytopes containing the origin in their interiors to arbitrary convex ploytopes. Solutions $f: \mathbb{R} \rightarrow \mathbb{R}$ of Cauchy's functional equation

$$
f(a+b)=f(a)+f(b), \quad \text { for } a, b \in \mathbb{R}
$$

play an important role in this paper. For a solution of Cauchy's functional equation $\zeta: \mathbb{R} \rightarrow$ $\mathbb{R}$, the general LYZ matrix $\mathrm{L}_{\zeta}(P)$ of $P \in \mathcal{P}^{n}$ is defined by

$$
\mathrm{L}_{\zeta, i j}(P)=\sum_{u \in \mathcal{N}(P) \backslash\left\{h_{P}=0\right\}} \frac{\zeta\left(a_{P}(u) h_{P}(u)\right)}{h_{P}^{2}(u)} u_{i} u_{j} .
$$

The aim of this paper is to obtain a complete classification of $\operatorname{SL}(n)$ contravariant matrix valuations on polytopes without any continuity assumptions. We are able to extend Ludwig's result to $\mathcal{P}_{0}^{n}$ without any homogeneity assumptions or any continuity assumptions.

Theorem 1.3. Let $n \geq 3$. A function $\mu: \mathcal{P}_{0}^{n} \rightarrow \mathbb{M}^{n}$ is an $\mathrm{SL}(n)$ contravariant valuation if and only if there exists a solution of Cauchy's functional equation $\zeta: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$
\mu(P)=\mathrm{L}_{\zeta}(P)
$$

for every $P \in \mathcal{P}_{0}^{n}$.
Similar to the classification of convex body valuations by Schuster and Wannerer [47] and Wannerer [48], we further extend this result to $\mathcal{P}^{n}$.

Theorem 1.4. Let $n \geq 3$. A function $\mu: \mathcal{P}^{n} \rightarrow \mathbb{M}^{n}$ is an $\operatorname{SL}(n)$ contravariant valuation if and only if there exist solutions of Cauchy's functional equation $\zeta_{1}, \zeta_{2}: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$
\mu(P)=\mathrm{L}_{\zeta_{1}}(P)+\mathrm{L}_{\zeta_{2}}([0, P])
$$

for every $P \in \mathcal{P}^{n}$, where $[0, P]$ denotes the convex hull of the origin and $P$.
We remark that the characterization of $\mathrm{SL}(2)$ contravariant symmetric matrix valued valuations on convex polytopes in $\mathbb{R}^{2}$ is still an open question.

## 2 Notation and preliminary results

We work in the $n$-dimensional Euclidean space $\mathbb{R}^{n}$ with the standard basis $\left\{e_{1}, \ldots, e_{n}\right\}$. We write a vector $x \in \mathbb{R}^{n}$ in coordinates by $x=\left(x_{1}, \ldots, x_{n}\right)$. The standard inner product will be written as $x \cdot y$ for vectors $x, y \in \mathbb{R}^{n}$.

For $A=\left(a_{i j}\right) \in \mathbb{M}^{n}$, we use the tensor representation, namely,

$$
A=\sum_{1 \leq i \leq j \leq n} a_{i j} e_{i} \otimes e_{j},
$$

and write $a_{i j}=A\left(e_{i}, e_{j}\right)$. Moreover, for every $\phi \in \operatorname{GL}(n)$ and $y_{1}, y_{2} \in \mathbb{R}^{n}$, we define

$$
(\phi \cdot A)\left(y_{1}, y_{2}\right)=A\left(\phi^{t} y_{1}, \phi^{t} y_{2}\right)
$$

which coincides with the action $\phi A \phi^{t}$ in Ludwig [21-23] in the following way

$$
\phi \cdot A=\sum_{1 \leq i \leq j \leq n} a_{i j}\left(\phi e_{i}\right) \otimes\left(\phi e_{j}\right)=\sum_{1 \leq i \leq j \leq n} a_{i j} \phi\left(e_{i} \otimes e_{j}\right) \phi^{t}=\phi A \phi^{t} .
$$

The affine hull, the relative interior and the dimension of a given set in $\mathbb{R}^{n}$ are denoted by aff, relint and dim, respectively. Denote by $\left[v_{1}, \ldots, v_{k}\right]$ the convex hull of $v_{1}, \ldots, v_{k} \in \mathbb{R}^{n}$. A convex polytope is the convex hull of finitely many points in $\mathbb{R}^{n}$. Two basic classes of polytopes are the $k$-dimensional standard simplex $T^{k}=\left[0, e_{1}, \ldots, e_{k}\right]$ and one of their $(k-1)$ dimensional facets $\widetilde{T}^{k}=\left[e_{1}, \ldots, e_{k}\right]$. For $i=1, \ldots, n$, let $\mathcal{T}^{i}$ be the set of $i$-dimensional simplices with one vertex at the origin and, let $\widetilde{\mathcal{T}}^{i}$ denote the set of $(i-1)$-dimensional simplices $T \subset \mathbb{R}^{n}$ with $0 \notin$ aff $T$. Indeed, every polytope can be triangulated into simplices. We define a triangulation of a $k$-dimensional polytope $P$ into simplices as a set of $k$-dimensional simplices $\left\{T_{1}, \ldots, T_{r}\right\}$ which have pairwise disjoint interiors, with $P=\cup T_{i}$ and with the property that for arbitrary $1 \leq i_{1}<\cdots<i_{j} \leq r$ the intersections $T_{i_{1}} \cap \cdots \cap T_{i_{j}}$ are again simplices.

Let $\mathcal{Q}^{n}$ be either $\mathcal{P}_{0}^{n}$ or $\mathcal{P}^{n}$ and $\mathcal{A}$ be an abelian group. The following inclusion-exclusion principle on valuations will be required (see [22] and [42, Theorem 3.1 and Lemma 3.3]).

Lemma 2.1. Let $\mu: \mathcal{Q}^{n} \rightarrow \mathcal{A}$ be a valuation. Then

$$
\mu\left(P_{1} \cup \cdots \cup P_{k}\right)=\sum_{\varnothing \neq S \subseteq\{1,2, \ldots, k\}}(-1)^{|S|-1} \mu\left(\bigcap_{i \in S} P_{i}\right)
$$

for all $k \in \mathbb{N}$ and $P_{1}, \ldots, P_{k} \in \mathcal{Q}^{n}$ with $P_{1} \cup \cdots \cup P_{k} \in \mathcal{Q}^{n}$.
A valuation on $\mathcal{Q}^{n}$ is called simple if it vanishes on every lower dimensional $P \in \mathcal{Q}^{n}$. Using triangulations of polytopes, a simple valuation is determined by its values on $n$-dimensional
simplices with one vertex at the origin (see [42, Lemma 3.4]). Furthermore, since these simplices are $\operatorname{SL}(n)$ image of dilated standard simplices, we only need to consider $s T^{n}$ for $s>0$. Similarly, it also suffices to consider $s \widetilde{T}^{k}$ for $s>0$ and $k=1, \ldots, n$ to determine a valuation on the space of polytopes that do not contain the origin in their affine hull.

Next, we mention a series of triangulations that will be used by several times in this paper. Let $\lambda \in(0,1)$ and denote by $H$ the hyperplane through the origin with unit normal vector $(1-\lambda) e_{1}-\lambda e_{2}$. Set

$$
H^{+}=\left\{x \in \mathbb{R}^{n}: x \cdot\left((1-\lambda) e_{1}-\lambda e_{2}\right) \geq 0\right\} \text { and } H^{-}=\left\{x \in \mathbb{R}^{n}: x \cdot\left((1-\lambda) e_{1}-\lambda e_{2}\right) \leq 0\right\} .
$$

Obviously, $H^{+}$and $H^{-}$are the two halfspaces bounded by $H$. This hyperplane induces the series of triangulations of $T^{i}$ as well as $\widetilde{T}^{i}$ for $i=2, \ldots, n$. There are two representations corresponding to these triangulations due to the following definitions.

Definition 2.1. For $\lambda \in(0,1)$, define the linear transform $\phi_{1} \in \operatorname{SL}(n)$ by

$$
\phi_{1} e_{1}=\lambda e_{1}+(1-\lambda) e_{2}, \quad \phi_{1} e_{2}=e_{2}, \quad \phi_{1} e_{n}=e_{n} / \lambda, \phi_{1} e_{j}=e_{j}, \text { where } j \neq 1,2, n
$$

and $\psi_{1} \in \mathrm{SL}(n)$ by
$\psi_{1} e_{1}=e_{1}, \psi_{1} e_{2}=\lambda e_{1}+(1-\lambda) e_{2}, \psi_{1} e_{n}=e_{n} /(1-\lambda), \psi_{1} e_{j}=e_{j}$, where $j \neq 1,2, n$.
It is clear that

$$
T^{i} \cap H^{+}=\psi_{1} T^{i}, T^{i} \cap H^{-}=\phi_{1} T^{i} \text { and } T^{i} \cap H=\phi_{1} T^{i-1} .
$$

Similarly,

$$
\widetilde{T}^{i} \cap H^{+}=\psi_{1} \widetilde{T}^{i}, \widetilde{T}^{i} \cap H^{-}=\phi_{1} \widetilde{T}^{i} \text { and } \widetilde{T}^{i} \cap H=\phi_{1} \widetilde{T}^{i-1}
$$

for $i=2, \ldots, n-1$.
Definition 2.2. For $\lambda \in(0,1)$, define the linear transform $\phi_{2} \in \mathrm{GL}(n)$ by

$$
\phi_{2} e_{1}=\lambda e_{1}+(1-\lambda) e_{2}, \quad \phi_{2} e_{2}=e_{2}, \quad \phi_{2} e_{j}=e_{j}, \text { where } j=3, \ldots, n,
$$

and $\psi_{2} \in \mathrm{GL}(n)$ by

$$
\psi_{2} e_{1}=e_{1}, \psi_{2} e_{2}=\lambda e_{1}+(1-\lambda) e_{2}, \psi_{2} e_{j}=e_{j}, \text { where } j=3, \ldots, n
$$

It is clear that

$$
s T^{n} \cap H^{+}=\psi_{2} s T^{n}, s T^{n} \cap H^{-}=\phi_{2} s T^{n}, \text { and } s T^{n} \cap H=\phi_{2} s T^{i-1}
$$

for every $s>0$. Similarly,

$$
s \widetilde{T}^{n} \cap H^{+}=\psi_{2} s \widetilde{T}^{n}, s \widetilde{T}^{n} \cap H^{-}=\phi_{2} s \widetilde{T}^{n}, \text { and } s \widetilde{T}^{n} \cap H=\phi_{2} s \widetilde{T}^{n-1},
$$

for every $s>0$.
Finally, we have several reduction steps for $\mathrm{SL}(n)$ contravariant functions towards the classification.

Lemma 2.2. Let $n \geq 2, s>0$, and $\mu: \mathcal{P}_{0}^{n} \rightarrow \mathbb{M}^{n}$ be an $\operatorname{SL}(n)$ contravariant function. Then, for $k=0, \ldots, n-1$, the coefficients $\mu\left(s T^{k}\right)\left(e_{i}, e_{j}\right)=0$ for all $i=1, \ldots, n-1$ and $j=1, \ldots, n$.

Proof. Let $i=1, \ldots, n-1$, we consider $\rho_{i} \in \operatorname{SL}(n)$ such that

$$
\rho_{i} e_{n}=e_{n}-e_{i}, \quad \rho_{i} e_{j}=e_{j}, \text { where } j=1, \ldots, n-1
$$

Since $\rho_{i}$ fixes $s T^{k}$, the $\operatorname{SL}(n)$ contravariance of $\mu$ gives

$$
\begin{aligned}
\mu\left(s T^{k}\right)\left(e_{j}, e_{n}\right) & =\mu\left(\rho_{i} s T^{k}\right)\left(e_{j}, e_{n}\right)=\mu\left(s T^{k}\right)\left(\rho_{i}^{-1} e_{j}, \rho_{i}^{-1} e_{n}\right) \\
& =\mu\left(s T^{k}\right)\left(e_{j}, e_{n}+e_{i}\right)=\mu\left(s T^{k}\right)\left(e_{j}, e_{n}\right)+\mu\left(T^{k}\right)\left(e_{j}, e_{i}\right)
\end{aligned}
$$

which implies

$$
\begin{equation*}
\mu\left(s T^{k}\right)\left(e_{i}, e_{j}\right)=0 \tag{2.1}
\end{equation*}
$$

for $i, j=1, \ldots, n-1$. The $\mathrm{SL}(n)$ contravariance of $\mu$ also gives

$$
\begin{align*}
\mu\left(s T^{k}\right)\left(e_{n}, e_{n}\right) & =\mu\left(\rho_{i} s T^{k}\right)\left(e_{n}, e_{n}\right)=\mu\left(s T^{k}\right)\left(\rho_{i}^{-1} e_{n}, \rho_{i}^{-1} e_{n}\right) \\
& =\mu\left(s T^{k}\right)\left(e_{n}+e_{i}, e_{n}+e_{i}\right)  \tag{2.2}\\
& =\mu\left(s T^{k}\right)\left(e_{n}, e_{n}\right)+\mu\left(s T^{k}\right)\left(e_{i}, e_{i}\right)+2 \mu\left(s T^{k}\right)\left(e_{i}, e_{n}\right) .
\end{align*}
$$

Combining with (2.1) and (2.2), we obtain

$$
\mu\left(s T^{k}\right)\left(e_{i}, e_{n}\right)=0
$$

for $i=1, \ldots, n-1$.

Lemma 2.3. Let $n \geq 2, s>0$, and $\mu: \mathcal{P}_{0}^{n} \rightarrow \mathbb{M}^{n}$ be an $\operatorname{SL}(n)$ contravariant function. Then, $\mu\left(s T^{k}\right)=0$ for $k=0, \ldots, n-2$.

Proof. By Lemma 2.2, it remains to show that $\mu\left(s T^{k}\right)\left(e_{n}, e_{n}\right)=0$. For $k=0, \ldots, n-2$, we consider $\sigma \in \mathrm{SL}(n)$ such that

$$
\sigma e_{n-1}=-e_{n}, \quad \sigma e_{n}=e_{n-1}, \quad \sigma e_{j}=e_{j}, \text { where } j=1, \ldots, n-2
$$

Since $\sigma$ fixes $s T^{k}$, Lemma 2.2 and the $\operatorname{SL}(n)$ contravariance of $\mu$ give

$$
\begin{aligned}
0 & =\mu\left(s T^{k}\right)\left(e_{n-1}, e_{n-1}\right)=\mu\left(\sigma s T^{k}\right)\left(e_{n-1}, e_{n-1}\right) \\
& =\mu\left(s T^{k}\right)\left(\sigma^{-1} e_{n-1}, \sigma^{-1} e_{n-1}\right)=\mu\left(s T^{k}\right)\left(e_{n}, e_{n}\right)
\end{aligned}
$$

This completes the proof.

Lemma 2.4. Let $n \geq 3, s>0$, and $\mu: \mathcal{P}_{0}^{n} \rightarrow \mathbb{M}^{n}$ be an $\operatorname{SL}(n)$ contravariant function. Then, all the coefficients $\mu\left(s T^{n}\right)\left(e_{i}, e_{i}\right)$ are equal for $i=1, \ldots, n$.

Proof. For $l=0, \ldots, n-3$, we consider the permutation $\theta_{l}$ such that

$$
\theta_{l} e_{l+1}=e_{l+3}, \theta_{l} e_{l+2}=e_{l+1}, \theta_{l} e_{l+3}=e_{l+2}, \theta_{l} e_{j}=e_{j}, \text { where } j \neq l+1, l+2, l+3
$$

Since $\theta_{l}$ fixes $s T^{n}$, the $\mathrm{SL}(n)$ contravariance of $\mu$ gives

$$
\mu\left(s T^{n}\right)\left(e_{l+1}, e_{l+1}\right)=\mu\left(s T^{n}\right)\left(e_{l+2}, e_{l+2}\right)=\mu\left(s T^{n}\right)\left(e_{l+3}, e_{l+3}\right)
$$

Repeating the above for different $l$, we obtain the desired result.

Lemma 2.5. Let $n \geq 3, s>0$, and $\mu: \mathcal{P}_{0}^{n} \rightarrow \mathbb{M}^{n}$ be an $\operatorname{SL}(n)$ contravariant function. Then, all the coefficients $\mu\left(s T^{n}\right)\left(e_{i}, e_{j}\right)$ are equal for $1 \leq i<j \leq n$.

Proof. Applying all these permutations $\theta_{l}$ defined in Lemma 2.4 within $e_{1}, \ldots, e_{n}$. Since these permutations fix $s T^{n}$, the $\mathrm{SL}(n)$ contravariance of $\mu$ implies the lemma.

## 3 The general LYZ matrix

Lemma 3.1. Let $\zeta: \mathbb{R} \rightarrow \mathbb{R}$ be a solution of Cauchy's functional equation. The general LYZ matrix operator $L_{\zeta}: \mathcal{P}^{n} \rightarrow \mathbb{M}^{n}$ is a simple valuation.

Proof. In order to prove that $\mathrm{L}_{\zeta}$ is a valuation, we need to show that

$$
\begin{equation*}
\mathrm{L}_{\zeta}(P \cup Q)+\mathrm{L}_{\zeta}(P \cap Q)=\mathrm{L}_{\zeta}(P)+\mathrm{L}_{\zeta}(Q) \tag{3.1}
\end{equation*}
$$

for all $P, Q \in \mathcal{P}^{n}$ with $P \cup Q \in \mathcal{P}^{n}$. We distinguish three sets of unit vectors:

$$
\begin{aligned}
& I_{1}:=\left\{u \in S^{n-1}: h_{P}(u)<h_{Q}(u)\right\}, \\
& I_{2}:=\left\{u \in S^{n-1}: h_{P}(u)=h_{Q}(u)\right\}, \\
& I_{3}:=\left\{u \in S^{n-1}: h_{P}(u)>h_{Q}(u)\right\} .
\end{aligned}
$$

Note that the sets $I_{1}, I_{3}$ are open and that $h_{P \cup Q}=\max \left\{h_{P}, h_{Q}\right\}$ and $h_{P \cap Q}=\min \left\{h_{P}, h_{Q}\right\}$ if $P \cup Q$ is convex. For $u \in S^{n-1}$, let $H(K, u)$ denote the support plane, i.e.,

$$
H(K, u)=\left\{x \in \mathbb{R}^{n}: x \cdot u=h_{K}(u)\right\},
$$

and $F(K, u):=H(K, u) \cap K$. It is well known that (see [46])

$$
\partial h_{K}(u)=F(K, u)
$$

for every $u \in S^{n-1}$. For $u \in I_{1}$, we have

$$
a_{P \cup Q}(u)=a_{Q}(u), h_{P \cup Q}(u)=h_{Q}(u), a_{P \cap Q}(u)=a_{P}(u), h_{P \cap Q}(u)=h_{P}(u) .
$$

Analogous for $I_{3}$. Note that

$$
\begin{aligned}
& \left(\mathcal{N}(P \cup Q) \backslash\left\{h_{P \cup Q}=0\right\}\right) \cap I_{1}=\left(\mathcal{N}(Q) \backslash\left\{h_{Q}=0\right\}\right) \cap I_{1}, \\
& \left(\mathcal{N}(P \cap Q) \backslash\left\{h_{P \cap Q}=0\right\}\right) \cap I_{1}=\left(\mathcal{N}(P) \backslash\left\{h_{P}=0\right\}\right) \cap I_{1}, \\
& \left(\mathcal{N}(P \cup Q) \backslash\left\{h_{P \cup Q}=0\right\}\right) \cap I_{3}=\left(\mathcal{N}(P) \backslash\left\{h_{P}=0\right\}\right) \cap I_{3}, \\
& \left(\mathcal{N}(P \cap Q) \backslash\left\{h_{P \cap Q}=0\right\}\right) \cap I_{3}=\left(\mathcal{N}(Q) \backslash\left\{h_{Q}=0\right\}\right) \cap I_{3} .
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
& \sum_{u \in\left(\mathcal{N}(P \cup Q) \backslash\left\{h_{P \cup Q}=0\right\}\right) \cap I_{1}} \frac{\zeta\left(a_{P \cup Q}(u) h_{P \cup Q}(u)\right)}{h_{P \cup Q}^{2}(u)} u_{i} u_{j}+\sum_{u \in\left(\mathcal{N}(P \cap Q) \backslash\left\{h_{P \cap Q}=0\right\}\right) \cap I_{1}} \frac{\zeta\left(a_{P \cap Q}(u) h_{P \cap Q}(u)\right)}{h_{P \cap Q}^{2}(u)} u_{i} u_{j} \\
& +\sum_{u \in\left(\mathcal{N}(P \cup Q) \backslash\left\{h_{P \cup Q}=0\right\}\right) \cap I_{3}} \frac{\zeta\left(a_{P \cup Q}(u) h_{P \cup Q}(u)\right)}{h_{P \cup Q}^{2}(u)} u_{i} u_{j}+\sum_{u \in\left(\mathcal{N}(P \cap Q) \backslash\left\{h_{P \cap Q}=0\right\}\right) \cap I_{3}} \frac{\zeta\left(a_{P \cap Q}(u) h_{P \cap Q}(u)\right)}{h_{P \cap Q}^{2}(u)} u_{i} u_{j} \\
& =\sum_{u \in\left(\mathcal{N}(P) \backslash\left\{h_{P}=0\right\}\right) \cap I_{1}} \frac{\zeta\left(a_{P}(u) h_{P}(u)\right)}{h_{P}^{2}(u)} u_{i} u_{j}+\sum_{u \in\left(\mathcal{N}(Q) \backslash\left\{h_{Q}=0\right\}\right) \cap I_{1}} \frac{\zeta\left(a_{Q}(u) h_{Q}(u)\right)}{h_{Q}^{2}(u)} u_{i} u_{j} \\
& +\sum_{u \in\left(\mathcal{N}(P) \backslash\left\{h_{P}=0\right\}\right) \cap I_{3}} \frac{\zeta\left(a_{P}(u) h_{P}(u)\right)}{h_{P}^{2}(u)} u_{i} u_{j}+\sum_{u \in\left(\mathcal{N}(Q) \backslash\left\{h_{Q}=0\right\}\right) \cap I_{3}} \frac{\zeta\left(a_{Q}(u) h_{Q}(u)\right)}{h_{Q}^{2}(u)} u_{i} u_{j}
\end{aligned}
$$

for all $1 \leq i, j \leq n$. It follows that (3.1) is equivalent to

$$
\begin{aligned}
& \sum_{u \in\left(\mathcal{N}(P \cup Q) \backslash\left\{h_{P \cup Q}=0\right\}\right) \cap I_{2}} \frac{\zeta\left(a_{P \cup Q}(u) h_{P \cup Q}(u)\right)}{h_{P \cup Q}^{2}(u)} u_{i} u_{j}+\sum_{u \in\left(\mathcal{N}(P \cap Q) \backslash\left\{h_{P \cap Q}=0\right\}\right) \cap I_{2}} \frac{\zeta\left(a_{P \cap Q}(u) h_{P \cap Q}(u)\right)}{h_{P \cap Q}^{2}(u)} u_{i} u_{j} \\
& =\sum_{u \in\left(\mathcal{N}(P) \backslash\left\{h_{P}=0\right\}\right) \cap I_{2}} \frac{\zeta\left(a_{P}(u) h_{P}(u)\right)}{h_{P}^{2}(u)} u_{i} u_{j}+\sum_{u \in\left(\mathcal{N}(Q) \backslash\left\{h_{Q}=0\right\}\right) \cap I_{2}} \frac{\zeta\left(a_{Q}(u) h_{Q}(u)\right)}{h_{Q}^{2}(u)} u_{i} u_{j}
\end{aligned}
$$

for all $1 \leq i, j \leq n$.

Fix $u \in S^{n-1}$. Since $P \mapsto a_{P}(u), P \in \mathcal{P}^{n}$ is a valuation, we have

$$
\begin{equation*}
a_{P \cup Q}(u)+a_{P \cap Q}(u)=a_{P}(u)+a_{Q}(u) \tag{3.2}
\end{equation*}
$$

for all $P, Q \in \mathcal{P}^{n}$ with $P \cup Q \in \mathcal{P}^{n}$. Note that

$$
\begin{equation*}
h_{P \cup Q}(u)=h_{P \cap Q}(u)=h_{P}(u)=h_{Q}(u) \tag{3.3}
\end{equation*}
$$

for $u \in I_{2}$. Then,

$$
a_{P \cup Q}(u) h_{P \cup Q}(u)+a_{P \cap Q}(u) h_{P \cap Q}(u)=a_{P}(u) h_{P}(u)+a_{Q}(u) h_{Q}(u)
$$

for $u \in I_{2}$. Since $\zeta$ is a solution of Cauchy's functional equation, we obtain

$$
\begin{equation*}
\frac{\zeta\left(a_{P \cup Q}(u) h_{P \cup Q}(u)\right)}{h_{P \cup Q}^{2}(u)}+\frac{\zeta\left(a_{P \cap Q}(u) h_{P \cap Q}(u)\right)}{h_{P \cap Q}^{2}(u)}=\frac{\zeta\left(a_{P}(u) h_{P}(u)\right)}{h_{P}^{2}(u)}+\frac{\zeta\left(a_{Q}(u) h_{Q}(u)\right)}{h_{Q}^{2}(u)} \tag{3.4}
\end{equation*}
$$

for $u \in I_{2}$, where $P, Q \in \mathcal{P}^{n}$ with $P \cup Q \in \mathcal{P}^{n}$. It follows from (3.2) that

$$
\begin{equation*}
\mathcal{N}(P \cup Q) \cup \mathcal{N}(P \cap Q)=\mathcal{N}(P) \cup \mathcal{N}(Q) . \tag{3.5}
\end{equation*}
$$

Combining with (3.3), (3.4) and (3.5), we obtain the desired valuation property.
Next, we will show that the general LYZ matrix operator is simple via the following three cases.

If $\operatorname{dim} P \leq n-2$, it is clear that $\mathrm{L}_{\zeta}(P)=0$ as $\mathcal{N}(P)=\varnothing$.
If $\operatorname{dim} P=n-1$ and $0 \in \operatorname{aff} P$, then $h_{P}( \pm u)=0$, where $\pm u$ are the outer unit normals of $P$. Thus, $\mathcal{N}(P) \backslash\left\{h_{P}=0\right\}=\varnothing$. By the definition of the general LYZ matrix, we obtain $\mathrm{L}_{\zeta}(P)=0$.

If $\operatorname{dim} P=n-1$ and $0 \notin$ aff $P$, then $h_{P}(-u)=-h_{P}(u)$ and $a_{P}(-u)=a_{P}(u)$. Here $\mathcal{N}(P)=\{ \pm u\}$. Hence, $a_{P}(-u) h_{P}(-u)=-a_{P}(u) h_{P}(u)$. Since $\zeta: \mathbb{R} \rightarrow \mathbb{R}$ is a solution of Cauchy's functional equation,

$$
\begin{aligned}
\mathrm{L}_{\zeta, i j}(P) & =\sum_{u \in \mathcal{N}(P) \backslash\left\{h_{P}=0\right\}} \frac{\zeta\left(a_{P}(u) h_{P}(u)\right)}{h_{P}^{2}(u)} u_{i} u_{j} \\
& =\frac{\zeta\left(a_{P}(u) h_{P}(u)\right)}{h_{P}^{2}(u)} u_{i} u_{j}+\frac{\zeta\left(a_{P}(-u) h_{P}(-u)\right)}{h_{P}^{2}(-u)}\left(-u_{i}\right)\left(-u_{j}\right)=0
\end{aligned}
$$

for all $1 \leq i, j \leq n$. We conclude that $\mathrm{L}_{\zeta}(P)=0$.

Lemma 3.2. Let $\zeta: \mathbb{R} \rightarrow \mathbb{R}$ be a solution of Cauchy's functional equation. The general LYZ matrix operator $L_{\zeta}: \mathcal{P}^{n} \rightarrow \mathbb{M}^{n}$ is $\mathrm{SL}(n)$ contravariant.

Proof. Let $\phi \in \mathrm{SL}(n)$. Note that

$$
u \in \mathcal{N}(P) \backslash\left\{h_{P}=0\right\} \quad \Leftrightarrow \quad \tilde{u} \in \mathcal{N}(\phi P) \backslash\left\{h_{\phi P}=0\right\}
$$

with

$$
\begin{equation*}
\tilde{u}:=\left\|\phi^{-t} u\right\|^{-1} \phi^{-t} u \tag{3.6}
\end{equation*}
$$

and that

$$
\begin{equation*}
h_{\phi P}(\tilde{u})=h_{P}\left(\phi^{t} \tilde{u}\right)=\left\|\phi^{-t} u\right\|^{-1} h_{P}(u) . \tag{3.7}
\end{equation*}
$$

Furthermore,

$$
\begin{equation*}
a_{\phi P}(\tilde{u})=\left\|\phi^{-t} u\right\| a_{P}(u) . \tag{3.8}
\end{equation*}
$$

Applying (3.6), (3.7), (3.8) and the definition of the general LYZ matrix, we obtain

$$
\begin{aligned}
\mathrm{L}_{\zeta}(\phi P) & =\sum_{\tilde{u} \in \mathcal{N}(\phi P) \backslash\left\{h_{\phi P}=0\right\}} \frac{\zeta\left(a_{\phi P}(\tilde{u}) h_{P}(\tilde{u})\right)}{h_{\phi P}^{2}(\tilde{u})} \tilde{u} \otimes \tilde{u} \\
& =\sum_{u \in \mathcal{N}(P) \backslash\left\{h_{P}=0\right\}} \frac{\zeta\left(\left\|\phi^{-t} u\right\| a_{P}(u)\left\|\phi^{-t} u\right\|^{-1} h_{P}(u)\right)}{\left\|\phi^{-t} u\right\|^{-2} h_{P}^{2}(u)}\left(\left\|\phi^{-t} u\right\|^{-1} \phi^{-t} u\right) \otimes\left(\left\|\phi^{-t} u\right\|^{-1} \phi^{-t} u\right) \\
& =\sum_{u \in \mathcal{N}(P) \backslash\left\{h_{P}=0\right\}} \frac{\zeta\left(a_{P}(u) h_{P}(u)\right)}{h_{P}^{2}(u)} \phi^{-t}(u \otimes u) \phi^{-1} \\
& =\phi^{-t} \cdot \mathrm{~L}_{\zeta}(P)
\end{aligned}
$$

Thus, we have finished the proof of the $\mathrm{SL}(n)$ contravariance of the general LYZ matrix operator.

## 4 Main results on $\mathcal{P}_{0}^{n}$

Lemma 4.1. Let $n \geq 3$ and $\mu: \mathcal{P}_{0}^{n} \rightarrow \mathbb{M}^{n}$ be an $\operatorname{SL}(n)$ contravariant valuation. Then $\mu$ is simple.

Proof. Due to Lemma 2.3, it suffices to show that $\mu\left(T^{n-1}\right)=0$. We use the triangulation in Definition 2.1. Since $\mu$ is a valuation, we have

$$
\mu\left(T^{n-1}\right)+\mu\left(\phi_{1} T^{n-2}\right)=\mu\left(\phi_{1} T^{n-1}\right)+\mu\left(\psi_{1} T^{n-1}\right) .
$$

Note that $\phi_{1}, \psi_{1} \in \operatorname{SL}(n)$, the $\operatorname{SL}(n)$ contravariance of $\mu$ gives

$$
\mu\left(T^{n-1}\right)+\phi_{1}^{-t} \cdot \mu\left(T^{n-2}\right)=\phi_{1}^{-t} \cdot \mu\left(T^{n-1}\right)+\psi_{1}^{-t} \cdot \mu\left(T^{n-1}\right)
$$

By Lemma 2.4, it follows that

$$
\mu\left(T^{n-1}\right)=\phi_{1}^{-t} \cdot \mu\left(T^{n-1}\right)+\psi_{1}^{-t} \cdot \mu\left(T^{n-1}\right)
$$

Thus,

$$
\begin{aligned}
\mu\left(T^{n-1}\right)\left(e_{n}, e_{n}\right) & =\phi_{1}^{-t} \cdot \mu\left(T^{n-1}\right)\left(e_{n}, e_{n}\right)+\psi_{1}^{-t} \cdot \mu\left(T^{n-1}\right)\left(e_{n}, e_{n}\right) \\
& =\mu\left(T^{n-1}\right)\left(\lambda e_{n}, \lambda e_{n}\right)+\mu\left(T^{n-1}\right)\left((1-\lambda) e_{n},(1-\lambda) e_{n}\right) \\
& =\left(1-2 \lambda+2 \lambda^{2}\right) \mu\left(T^{n-1}\right)\left(e_{n}, e_{n}\right) .
\end{aligned}
$$

It means that

$$
2 \lambda(1-\lambda) \mu\left(T^{n-1}\right)\left(e_{n}, e_{n}\right)=0
$$

for $0<\lambda<1$. Then, we must have $\mu\left(T^{n-1}\right)\left(e_{n}, e_{n}\right)=0$. Combing with Lemma 2.2, we obtain $\mu\left(T^{n-1}\right)=0$.

Lemma 4.2. Let $n \geq 3, s>0$, and $\mu: \mathcal{P}_{0}^{n} \rightarrow \mathbb{M}^{n}$ be an $\mathrm{SL}(n)$ contravariant valuation. Then, all the coefficients $\mu\left(s T^{n}\right)\left(e_{i}, e_{j}\right)$ are equal for $i, j=1, \ldots, n$.

Proof. For $s>0$, since $\mu$ is a valuation, we have

$$
\mu\left(s T^{n}\right)+\mu\left(\phi_{2} s T^{n-1}\right)=\mu\left(\phi_{2} s T^{n}\right)+\mu\left(\psi_{2} s T^{n}\right)
$$

Note that $\phi_{2} / \lambda^{\frac{1}{n}}, \psi_{2} /(1-\lambda)^{\frac{1}{n}} \in \operatorname{SL}(n)$, Lemma 4.1 and the $\operatorname{SL}(n)$ contravariance of $\mu$ give

$$
\mu\left(s T^{n}\right)=\left(\lambda^{\frac{1}{n}} \phi_{2}^{-t}\right) \cdot \mu\left(\lambda^{\frac{1}{n}} s T^{n}\right)+\left((1-\lambda)^{\frac{1}{n}} \psi_{2}^{-t}\right) \cdot \mu\left((1-\lambda)^{\frac{1}{n}} s T^{n}\right)
$$

By Lemma 2.4, it follows that

$$
\begin{align*}
& \mu\left(s T^{n}\right)\left(e_{1}, e_{1}\right) \\
= & \lambda^{\frac{2}{n}} \mu\left(\lambda^{\frac{1}{n}} s T^{n}\right)\left(\frac{1}{\lambda} e_{1}-\frac{1-\lambda}{\lambda} e_{2}, \frac{1}{\lambda} e_{1}-\frac{1-\lambda}{\lambda} e_{2}\right)+(1-\lambda)^{\frac{2}{n}} \mu\left((1-\lambda)^{\frac{1}{n}} s T^{n}\right)\left(e_{1}, e_{1}\right)  \tag{4.1}\\
= & \lambda^{\frac{2}{n}-2}\left[\left(2-2 \lambda+\lambda^{2}\right) \mu\left(\lambda^{\frac{1}{n}} s T^{n}\right)\left(e_{1}, e_{1}\right)-(2-2 \lambda) \mu\left(\lambda^{\frac{1}{n}} s T^{n}\right)\left(e_{1}, e_{2}\right)\right] \\
& +(1-\lambda)^{\frac{2}{n}} \mu\left((1-\lambda)^{\frac{1}{n}} s T^{n}\right)\left(e_{1}, e_{1}\right) .
\end{align*}
$$

Also,

$$
\begin{equation*}
\mu\left(s T^{n}\right)\left(e_{n}, e_{n}\right)=\lambda^{\frac{2}{n}} \mu\left(\lambda^{\frac{1}{n}} s T^{n}\right)\left(e_{n}, e_{n}\right)+(1-\lambda)^{\frac{2}{n}} \mu\left((1-\lambda)^{\frac{1}{n}} s T^{n}\right)\left(e_{n}, e_{n}\right) \tag{4.2}
\end{equation*}
$$

Lemma 2.4 implies that (4.2) is equivalent to

$$
\begin{equation*}
\mu\left(s T^{n}\right)\left(e_{1}, e_{1}\right)=\lambda^{\frac{2}{n}} \mu\left(\lambda^{\frac{1}{n}} s T^{n}\right)\left(e_{1}, e_{1}\right)+(1-\lambda)^{\frac{2}{n}} \mu\left((1-\lambda)^{\frac{1}{n}} s T^{n}\right)\left(e_{1}, e_{1}\right) . \tag{4.3}
\end{equation*}
$$

Plug (4.3) into (4.1) and get

$$
2(1-\lambda)\left(\mu\left(\lambda^{\frac{1}{n}} s T^{n}\right)\left(e_{1}, e_{1}\right)-\mu\left(\lambda^{\frac{1}{n}} s T^{n}\right)\left(e_{1}, e_{2}\right)\right)=0
$$

Since $0<\lambda<1$, we must have

$$
\mu\left(\lambda^{\frac{1}{n}} s T^{n}\right)\left(e_{1}, e_{1}\right)=\mu\left(\lambda^{\frac{1}{n}} s T^{n}\right)\left(e_{1}, e_{2}\right)
$$

Applying Lemmas 2.4 and 2.5, we complete the proof.

Proof of Theorem 1.3. Lemmas 3.1 and 3.2 imply that the general LYZ matrix operator $L_{\zeta}$ is an $\mathrm{SL}(n)$ contravariant valuation.

Conversely, let $\mu: \mathcal{P}_{0}^{n} \rightarrow \mathbb{M}^{n}$ be an $\operatorname{SL}(n)$ contravariant valuation. Let $s>0$. By using the triangulation in Definition 2.2, we have

$$
\mu\left(s^{\frac{1}{n}} T^{n}\right)+\mu\left(\phi_{2} s^{\frac{1}{n}} T^{n-1}\right)=\mu\left(\phi_{2} s^{\frac{1}{n}} T^{n}\right)+\mu\left(\psi_{2} s^{\frac{1}{n}} T^{n}\right)
$$

Note that $\phi_{2} / \lambda^{\frac{1}{n}}, \psi_{2} /(1-\lambda)^{\frac{1}{n}} \in \operatorname{SL}(n)$, Lemma 4.1 and the $\operatorname{SL}(n)$ contravariance of $\mu$ give

$$
\mu\left(s^{\frac{1}{n}} T^{n}\right)=\left(\lambda^{\frac{1}{n}} \phi_{2}^{-t}\right) \cdot \mu\left((\lambda s)^{\frac{1}{n}} T^{n}\right)+\left((1-\lambda)^{\frac{1}{n}} \psi_{2}^{-t}\right) \cdot \mu\left(((1-\lambda) s)^{\frac{1}{n}} T^{n}\right)
$$

Thus,

$$
\begin{equation*}
\mu\left(s^{\frac{1}{n}} T^{n}\right)\left(e_{n}, e_{n}\right)=\lambda^{\frac{2}{n}} \mu\left((\lambda s)^{\frac{1}{n}} T^{n}\right)\left(e_{n}, e_{n}\right)+(1-\lambda)^{\frac{2}{n}} \mu\left(((1-\lambda) s)^{\frac{1}{n}} T^{n}\right)\left(e_{n}, e_{n}\right) \tag{4.4}
\end{equation*}
$$

Setting $x=\lambda s$ and $y=(1-\lambda) s$, then (4.4) can be rewritten as

$$
(x+y)^{\frac{2}{n}} \mu\left((x+y)^{\frac{1}{n}} T^{n}\right)\left(e_{n}, e_{n}\right)=x^{\frac{2}{n}} \mu\left(x^{\frac{1}{n}} T^{n}\right)\left(e_{n}, e_{n}\right)+y^{\frac{2}{n}} \mu\left(y^{\frac{1}{n}} T^{n}\right)\left(e_{n}, e_{n}\right)
$$

Therefore,

$$
\mu\left(x^{\frac{1}{n}} T^{n}\right)\left(e_{n}, e_{n}\right)=x^{-\frac{2}{n}} \zeta_{0}(x)
$$

where $\zeta_{0}: \mathbb{R} \rightarrow \mathbb{R}$ is a solution of Cauchy's functional equation. Thus,

$$
\mu\left(s^{\frac{1}{n}} T^{n}\right)\left(e_{n}, e_{n}\right)=\frac{\zeta_{0}(s)}{s^{\frac{2}{n}}}
$$

Lemma 4.2 implies that

$$
\mu\left(s^{\frac{1}{n}} T^{n}\right)=\frac{\zeta_{0}(s)}{s^{\frac{2}{n}}} \mathbf{1},
$$

where $\mathbf{1}$ denotes the $(n \times n)$-matrix where every element is one.
Let $\zeta: \mathbb{R} \rightarrow \mathbb{R}$ be a solution of Cauchy's functional equation. By the definition of the general LYZ matrix, we have

$$
\mathrm{L}_{\zeta}\left(s^{\frac{1}{n}} T^{n}\right)=\frac{n \zeta(s / n!)}{s^{\frac{2}{n}}} \mathbf{1}
$$

Setting

$$
\zeta(s / n!)=\zeta_{0}(s),
$$

we obtain

$$
\mu\left(s^{\frac{1}{n}} T^{n}\right)=\mathrm{L}_{\zeta}\left(s^{\frac{1}{n}} T^{n}\right)
$$

Therefore,

$$
\mu(T)=\mathrm{L}_{\zeta}(T)
$$

for each $T \in \mathcal{T}^{n}$. Finally, we dissect $P \in \mathcal{P}_{0}^{n}$ into simplices with one vertex at the origin. Since $\mu$ is simple and by the inclusion-exclusion principle, we obtain

$$
\mu(P)=\mathrm{L}_{\zeta}(P)
$$

for each $P \in \mathcal{P}_{0}^{n}$.

## 5 Main results on $\mathcal{P}^{n}$

Since all the steps also work on $\widetilde{\mathcal{T}}^{k}$ for $k=1, \ldots, n$, including reductions in Lemmas 2.2-2.4 and triangulations in Definitions 2.1 and 2.2, we similarly have the following Lemma.

Lemma 5.1. Let $n \geq 3$ and $\mu: \mathcal{P}^{n} \rightarrow \mathbb{M}^{n}$ be an $\operatorname{SL}(n)$ contravariant valuation. Then $\mu(P)=0$ for every $P \in \mathcal{P}^{n}$ with $\operatorname{dim} P \leq n-2$.

Next, we determine such valuations on every $P \in \mathcal{P}^{n}$ with $\operatorname{dim} P \leq n-1$ and $0 \in \operatorname{aff} P$.
Lemma 5.2. Let $n \geq 3$ and $\mu: \mathcal{P}^{n} \rightarrow \mathbb{M}^{n}$ be an $\mathrm{SL}(n)$ contravariant valuation. Then $\mu(P)=0$ for every $P \in \mathcal{P}^{n}$ with $\operatorname{dim} P \leq n-1$ and $0 \in$ aff $P$.

Proof. Let $P \in \mathcal{P}^{n}$ with $\operatorname{dim} P \leq n-1$ and $0 \in \operatorname{aff} P$. The case for $0 \in P$ is already included in Lemma 4.1. It suffices to consider such polytopes that do not contain the origin. Let $F_{1}, \ldots, F_{r}$ be the facets of $P$ visible from the origin, i.e. $P \cap \operatorname{relint}\left[0, F_{i}\right]=\varnothing$. Since $\mu$ is a valuation, the inclusion-exclusion principle yields

$$
\begin{aligned}
0= & \mu([0, P]) \\
= & \mu(P)+\sum_{i=1}^{r} \mu(\underbrace{\left[0, F_{i}\right]}_{\in \mathcal{P}_{0}^{n}})-\sum_{j=2}^{r}(-1)^{j} \sum_{1 \leq i_{1}<\cdots<i_{j} \leq r} \mu(\underbrace{\left[0, F_{i_{1}}\right] \cap \cdots \cap\left[0, F_{i_{j}}\right]}_{\operatorname{dim} \leq n-2}) \\
& -\sum_{i=1}^{r} \mu(\underbrace{\left[0, F_{i}\right] \cap P}_{\operatorname{dim}=n-2})+\sum_{j=2}^{r}(-1)^{j} \sum_{1 \leq i_{1}<\cdots<i_{j} \leq r} \mu(\underbrace{\left[0, F_{i_{1}}\right] \cap \cdots \cap\left[0, F_{i_{j}}\right] \cap P}_{\operatorname{dim} \leq n-3}) \\
= & \mu(P),
\end{aligned}
$$

where the steps follow from Lemmas 4.1 and 5.1.

Now, we have the following characterization on the set of $(n-1)$-dimensional polytopes that do not contain the origin in their affine hull.

Lemma 5.3. Let $n \geq 3$ and $\mu: \mathcal{P}^{n} \rightarrow \mathbb{M}^{n}$ be an $\mathrm{SL}(n)$ contravariant valuation. Then there exists a solution of Cauchy's functional equation $\zeta: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$
\mu(P)=\mathrm{L}_{\zeta}([0, P])
$$

for every $P \in \mathcal{P}^{n}$ with $\operatorname{dim} P=n-1$ and $0 \notin$ aff $P$.
Proof. First, it suffices to consider $s^{\frac{1}{n}} \widetilde{T}^{n}$ for $s>0$. We use the dissection in Definition 2.2. Since $\mu$ is a valuation, we have

$$
\mu\left(s^{\frac{1}{n}} \widetilde{T}^{n}\right)+\mu\left(\phi_{2} s^{\frac{1}{n}} \widetilde{T}^{n-1}\right)=\mu\left(\phi_{2} s^{\frac{1}{n}} \widetilde{T}^{n}\right)+\mu\left(\psi_{2} s^{\frac{1}{n}} \widetilde{T}^{n}\right)
$$

Note that $\phi_{2} / \lambda^{\frac{1}{n}}, \psi_{2} /(1-\lambda)^{\frac{1}{n}} \in \operatorname{SL}(n)$, Lemma 5.1 and the $\operatorname{SL}(n)$ contravariance of $\mu$ give

$$
\mu\left(s^{\frac{1}{n}} \widetilde{T}^{n}\right)=\left(\lambda^{\frac{1}{n}} \phi_{2}^{-t}\right) \cdot \mu\left((\lambda s)^{\frac{1}{n}} \widetilde{T}^{n}\right)+\left((1-\lambda)^{\frac{1}{n}} \psi_{2}^{-t}\right) \cdot \mu\left(((1-\lambda) s)^{\frac{1}{n}} \widetilde{T}^{n}\right)
$$

Using a similar argument as in the proof of Theorem 1.2, we obtain that there exists a solution of Cauchy's functional equation $\zeta_{2}: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
\mu\left(s^{\frac{1}{n}} \widetilde{T}^{n}\right)=\mathrm{L}_{\zeta_{2}}\left(\left[0, s^{\frac{1}{n}} \widetilde{T}^{n-1}\right]\right) \tag{5.1}
\end{equation*}
$$

Second, let $P$ be an $(n-1)$-dimensional polytope with $0 \notin$ aff $P$. Triangulate $P$ into simplices $T_{1}, \ldots, T_{r}$. Using the inclusion-exclusion principle, Lemma 5.2, (5.1), and Lemma 3.1, we get

$$
\mu(P)=\sum_{i=1}^{r} \mu\left(T_{i}\right)=\sum_{i=1}^{r} \mathrm{~L}_{\zeta_{2}}\left(\left[0, T_{i}\right]\right)=\mathrm{L}_{\zeta_{2}}([0, P])
$$

Proof of Theorem 1.4. Let $\zeta_{1}, \zeta_{2}: \mathbb{R} \rightarrow \mathbb{R}$ be the solutions of Cauchy's functional equation. Set

$$
\mu(P)=\mathrm{L}_{\zeta_{1}}(P)+\mathrm{L}_{\zeta_{2}}([0, P])
$$

for every $P \in \mathcal{P}^{n}$. For $P, Q \in \mathcal{P}^{n}$ with $P \cup Q \in \mathcal{P}^{n}$, we have $[0, P \cup Q]=[0, P] \cup[0, Q]$ and $[0, P \cap Q]=[0, P] \cap[0, Q]$. Note that $[0, \phi P]=\phi[0, P]$ for every $\phi \in \operatorname{SL}(n)$. Lemmas 3.1 and 3.2 show that $\mu$ is an $\mathrm{SL}(n)$ contravariant valuation on $\mathcal{P}^{n}$.

It remains to show the reverse statement. Let $\mu: \mathcal{P}^{n} \rightarrow \mathbb{M}^{n}$ be an $\operatorname{SL}(n)$ contravariant valuation. By Theorem 1.2 and Lemmas 5.1-5.3, we can assume that $P \in \mathcal{P}^{n} \backslash \mathcal{P}_{0}^{n}$ with
$\operatorname{dim} P=n$. Let $F_{1}, \ldots, F_{r}$ be the facets of $P$ visible from the origin. Since $\mu$ is a valuation, the inclusion-exclusion principle yields that there exist solutions of Cauchy's functional equation $\tilde{\zeta}_{1}, \zeta_{2}: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$
\begin{aligned}
\mathrm{L}_{\tilde{\zeta}_{1}}([0, P])= & \mu([0, P]) \\
= & \mu(P)+\sum_{i=1}^{r} \mu(\underbrace{\left[0, F_{i}\right]}_{\in \mathcal{P}_{0}^{n}})-\sum_{j=2}^{r}(-1)^{j} \sum_{1 \leq i_{1}<\cdots<i_{j} \leq r} \mu(\underbrace{\left[0, F_{i_{1}}\right] \cap \cdots \cap\left[0, F_{i_{j}}\right]}_{\operatorname{dim} \leq n-2}) \\
& -\sum_{i=1}^{r} \mu(\underbrace{\left[0, F_{i}\right] \cap P}_{=F_{i}})+\sum_{j=2}^{r}(-1)^{j} \sum_{1 \leq i_{1}<\cdots<i_{j} \leq r} \mu(\underbrace{\left[0, F_{i_{1}}\right] \cap \cdots \cap\left[0, F_{i_{j}}\right] \cap P}_{\operatorname{dim} \leq n-3}) \\
= & \mu(P)+\sum_{i=1}^{r} \mu\left(\left[0, F_{i}\right]\right)-\sum_{i=1}^{r} \mu\left(F_{i}\right) \\
= & \mu(P)+\sum_{i=1}^{r} \mathrm{~L}_{\tilde{\zeta}_{1}}\left(\left[0, F_{i}\right]\right)-\sum_{i=1}^{r} \mathrm{~L}_{\zeta_{2}}\left(\left[0, F_{i}\right]\right) .
\end{aligned}
$$

Since $\mathrm{L}_{\tilde{\zeta}_{1}}$ is a simple valuation, we have $\sum_{i=1}^{r} \mathrm{~L}_{\tilde{\zeta}_{1}}\left(\left[0, F_{i}\right]\right)=\mathrm{L}_{\tilde{\zeta}_{1}}([0, P])-\mathrm{L}_{\tilde{\zeta}_{1}}(P)$ and so is $\mathrm{L}_{\zeta_{2}}$. Finally, we finish the proof by setting $\zeta_{1}=\tilde{\zeta}_{1}-\zeta_{2}$.

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## References

[1] S. Alesker, Continuous rotation invariant valuations on convex sets, Ann. of Math. (2), 149 (3) (1999), 977-1005.
[2] S. Alesker, Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture, Geom. Funct. Anal., 11 (2) (2001), 244-272.
[3] S. Alesker, A. Bernig, and F. E. Schuster, Harmonic analysis of translation invariant valuations, Geom. Funct. Anal., 21 (4) (2011), 751-773.
[4] K. Ball, Volume ratios and a reverse isoperimetric inequality, J. London Math. Soc. (2), 44 (2) (1991), 351-359.
[5] K. Ball, Ellipsoids of maximal volume in convex bodies, Geom. Dedicata, 41 (2) (1992), 241-250.
[6] A. Bernig and D. Hug, Kinematic formulas for tensor valuations, J. Reine Angew. Math., 736 (2018), 141-191.
[7] H. Cramér, Mathematical methods of statistics, reprint of the 1946 original, Princeton University Press, Princeton, NJ, 1999.
[8] P. M. Gruber, Convex and discrete geometry, Springer, Berlin, 2007.
[9] O. G. Guleryuz, E. Lutwak, D. Yang, and G. Zhang, Information-theoretic inequalities for contoured probability distributions, IEEE Trans. Inform. Theory 48 (8) (2002), 23772383.
[10] C. Haberl, Blaschke valuations, Amer. J. Math., 133 (3) (2011), 717-751.
[11] C. Haberl, Minkowski valuations intertwining with the special linear group, J. Eur. Math. Soc., 14 (5) (2012), 1565-1597.
[12] C. Haberl and M. Ludwig, A characterization of $L_{p}$ intersection bodies, Int. Math. Res. Not. 2006 (2006), Article ID 10548, 29 pages.
[13] C. Haberl and L. Parapatits, The centro-affine Hadwiger theorem, J. Amer. Math. Soc., 27 (3) (2014), 685-705.
[14] C. Haberl and L. Parapatits, Valuations and surface area measures, J. Reine Angew. Math., 687 (2014), 225-245.
[15] C. Haberl and L. Parapatits, Moments and valuations, Amer. J. Math., 138 (6) (2016), 1575-1603.
[16] C. Haberl and L. Parapatits, Centro-affine tensor valuations, Adv. Math., 316 (2017), 806-865.
[17] H. Hadwiger, Vorlesungen über Inhalt, Oberfläche und Isoperimetrie, Springer, Berlin, 1957.
[18] D. Hug and R. Schneider, Local tensor valuations, Geom. Funct. Anal., 24 (5) (2014), 1516-1564.
[19] D. Hug, R. Schneider, and R. Schuster, Integral geometry of tensor valuations, Adv. Appl. Math., 41 (4) (2008), 482-509.
[20] F. John, Extremum problems with inequalities as subsidiary conditions, in: Studies and Essays Presented to R. Courant on his 60th Birthday, Interscience Publishers, Inc., New York, 1948, 187-204.
[21] D. A. Klain, Star valuations and dual mixed volumes, Adv. Math., 121 (1) (1996), 80-101.
[22] D. A. Klain and G. C. Rota, Introduction to geometric probability, Cambridge University Press, Cambridge, 1997.
[23] J. Li and G. Leng, $L_{p}$ Minkowski valuations on polytopes, Adv. Math. 299 (2016), 139173.
[24] J. Li and D. Ma, Laplace transforms and valuations, J. Funct. Anal. 272 (2) (2017), 738-758.
[25] J. Li, S. Yuan, and G. Leng, $L_{p}$-Blaschke valuations, Trans. Amer. Math. Soc., 367 (5) (2015), 3161-3187.
[26] M. Ludwig, Moment vectors of polytopes, Rend. Circ. Mat. Pale. (2) Suppl., 70 (2002), 123-138.
[27] M. Ludwig, Projection bodies and valuations, Adv. Math., 172 (2) (2002), 158-168.
[28] M. Ludwig, Valuations on ploytopes containing the origin in their interiors, Adv. Math., 170 (2) (2002), 239-256.
[29] M. Ludwig, Ellipsoids and matrix-valued valuations, Duke Math. J., 119 (1) (2003), 159-188.
[30] M. Ludwig, Minkowski valuations, Trans. Amer. Math. Soc., 357 (10) (2005), 4191-4213.
[31] M. Ludwig, Intersection bodies and valuations, Amer. J. Math., 128 (6) (2006), 14091428.
[32] M. Ludwig, Minkowski areas and valuations, J. Differential Geom., 86 (1) (2010), 133161.
[33] M. Ludwig, Fisher information and matrix-valued valuations, Adv. Math., 226 (3) (2011), 2700-2711.
[34] M. Ludwig, Covariance matrices and valuations, Adv. Appl. Math., 51 (3) (2013), 359366.
[35] M. Ludwig and M. Reitzner, A classification of $\mathrm{SL}(n)$ invariant valuations, Ann. of Math. (2), 172 (2) (2010), 1219-1267.
[36] M. Ludwig and M. Reitzner, SL (n) invariant valuations on polytopes, Discrete Comput. Geom., 57 (3) (2017), 571-581.
[37] M. Ludwig and L. Silverstein, Tensor valuations on lattice polytopes, Adv. Math., 319 (2017), 76-110.
[38] E. Lutwak, D. Yang, and G. Zhang, A new ellipsoid associated with convex bodies, Duke. Math. J., 104 (3) (2000), 375-390.
[39] E. Lutwak, D. Yang, and G. Zhang, The Cramer-Rao inequality for star bodies, Duke Math. J., 112 (1) (2002), 59-81.
[40] E. Lutwak, D. Yang, and G. Zhang, L $L_{p}$ John ellipsoids, Proc. London Math. Soc. (3), 90 (2) (2005), 497-520.
[41] D. Ma, Moment matrices and SL(n) equivariant valuations on polytopes, Int. Math. Res. Not., doi: 10.1093/imrn/rnz137.
[42] L. Parapatits, SL (n)-contravariant $L_{p}$-Minkowski valuations, Trans. Amer. Math. Soc., 366 (3) (2014), 1195-1211.
[43] L. Parapatits, SL(n)-covariant $L_{p}$-Minkowski valuations, J. London Math. Soc. (2), 89 (2) (2014), 397-414.
[44] G. Pisier, The volume of convex bodies and Banach space geometry, Cambridge University Press, Cambridge, 1989.
[45] C. R. Rao, Information and the accuracy attainable in the estimation of statistical parameters, Bull. Calcutta Math. Soc., 37 (1945), 81-91.
[46] R. Schneider, Convex bodies: the Brunn-Minkowski theory, 2nd expanded edition, Cambridge University Press, Cambridge, 2014.
[47] F. E. Schuster and T. Wannerer, GL(n) contravariant Minkowski valuations, Trans. Amer. Math. Soc., 364 (2) (2012), 815-826.
[48] T. Wannerer, GL(n) equivariant Minkowski valuations, Indiana Univ. Math. J., 60 (5) (2011), 1655-1672.
[49] C. Zeng and D. Ma, SL(n) covariant vector valuations on ploytopes, Trans. Amer. Math. Soc., 370 (12) (2018), 8999-9023.
[50] D. Zou and G. Xiong, Orlicz-John ellipsoids, Adv. Math., 265 (2014), 132-168.


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