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Hopf bifurcation for two types of Liénard systems ☆

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A B S T R A C T

In this paper we study the maximal number of limit cycles in Hopf bifurcations for two types of Liénard systems and obtain an upper bound of the number. In some cases the upper bound is the least, called the Hopf cyclicity.

1. Introduction and main results

The problem of limit cycle bifurcations of Liénard equation of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

or its equivalent form

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x),$$

where $F(x) = \int_0^x f(x)\,dx$, has been extensively considered, see [1–23] for instance. In [2,3,17,23,22] the number of local limit cycles were obtained in Hopf bifurcation when f and g are polynomials of certain degrees. In [16], the authors considered a system of the form

$$\dot{x} = y, \qquad \dot{y} = -x - \epsilon g_m(x) - \epsilon f_n(x) y,$$

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where f_n and g_m are polynomials of degrees n and m respectively, and ϵ is a small parameter. A number $\widetilde{H}_{n,m}$ was introduced in [16] for the above system which is the maximal number of limit cycles bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$. By using the averaging theory of order 3, the authors [16] obtained

$$\widetilde{H}_{n,m} \geqslant \left[\frac{n+m-1}{2}\right].$$

Consider a more general system of the form

$$\dot{x} = p(y) - F(x, a), \qquad \dot{y} = -g(x), \tag{1.1}$$

where F, g and p are C^∞ functions near the origin with

$$g(0) = 0, \qquad g'(0) > 0, \qquad p(0) = 0, \qquad p'(0) > 0, \qquad F(0, a) = 0, \quad a \in \mathbb{R}^n. \tag{1.2}$$

Let

$$G(x) = \int_0^x g(x)\,\mathrm{d}x, \qquad F\big(\alpha(x), a\big) - F(x, a) = \sum_{i\geqslant 1} B_i(a)x^i, \qquad a = (a_1, \ldots, a_n)$$

where $\alpha(x) = -x + \mathrm{O}(x^2)$ satisfies $G(\alpha(x)) \equiv G(x)$ for $|x| \ll 1$.

From [7] we have the following result.

Theorem 1. *(See [7].) Let* (1.2) *hold. If there exists $k \geqslant 1$ such that*

$$F\big(\alpha(x), a\big) \equiv F(x, a) \quad \text{when } B_{2j+1} = 0,\ j = 0, \ldots, k \tag{1.3}$$

for all $a \in \mathbb{R}^n$, then

(1) *the origin is a focus of order at most k of Eq.* (1.1) *unless it is a center,*
(2) *if $B_{2j+1} = \mathrm{O}(|B_1, B_3, \ldots, B_{2k+1}|)$ for $j \geqslant k+1$, then for any $N > 0$ there exists a neighborhood U_N of the origin such that Eq.* (1.1) *has at most k limit cycles in U_N for all $|B_1| + |B_3| + \cdots + |B_{2k+1}| \leqslant N$,*
(3) *suppose* (1.3) *holds for some $1 \leqslant k \leqslant n-1$. If further*

$$B_{2j+1}(a_0) = 0, \quad j = 0, \ldots, k, \qquad \operatorname{rank} \frac{\partial(B_1, \ldots, B_{2k+1})}{\partial(a_1, \ldots, a_n)}\bigg|_{a=a_0} = k+1,$$

then Eq. (1.1) *has at most k limit cycles near the origin for all a near a_0, and has k limit cycles near the origin for some a near a_0 small. In other words, Eq.* (1.1) *has Hopf cyclicity k at the origin.*

The above theorem has many applications to various Liénard systems and certain models from biomathematics, see [7,9,10,20] and [21]. For example, Jiang et al. [10] considered the system

$$\dot{x} = y - \sum_{i=0}^{n} a_i x^{2i+1}, \qquad \dot{y} = -x\big(x^2 - 1\big)$$

and proved that it has Hopf cyclicity n at the points $A(1, y_0)$ and $B(-1, -y_0)$ each, where $y_0 = \sum_{i=0}^{n} a_i$. Hence, the maximal number of its small-amplitude limit cycles is $2n$.

In recent years non-smooth systems were studied widely, see Coll et al. [4], Küpper and Moritz [12], Leine [13], Zou et al. [24]. Liu and Han [15] considered the following non-smooth Liénard system

$$\dot{x} = p(y) - F(x,a), \qquad \dot{y} = -g(x), \tag{1.4}$$

where $a \in R^n$,

$$F(x,a) = \begin{cases} F^+(x,a), & x > 0, \\ F^-(x,a), & x \leqslant 0, \end{cases} \qquad g(x) = \begin{cases} g^+(x), & x > 0, \\ g^-(x), & x \leqslant 0. \end{cases}$$

$F^\pm$ and $g^\pm$ are all C^∞ functions and satisfy

$$F^\pm(0,a) = 0, \qquad g^\pm(0) = 0, \qquad p(0) = 0, \qquad \big(g^\pm\big)'(0) = g_1^\pm > 0, \tag{1.5}$$

$$p'(0) = p_0 > 0, \qquad \big(F_x^\pm(0,a_0)\big)^2 - 4p_0 g_1^\pm < 0, \quad a_0 \in R^n. \tag{1.6}$$

Let

$$G(x) = \int_0^x g(u)\,\mathrm{d}u, \qquad \alpha(x) = -\big(\sqrt{g_1^+}/\sqrt{g_1^-}\big)x + \mathrm{O}\big(x^2\big),$$

where $\alpha(x)$ satisfies $G(\alpha(x)) \equiv G(x)$ for $|x| \ll 1$. Suppose formally for $0 < x \ll 1$

$$F\big(\alpha(x),a\big) - F(x,a) = F^-\big(\alpha(x),a\big) - F^+(x,a) = \sum_{i \geqslant 1} B_i(a)x^i.$$

Theorem 2. *(See [15].) Let* (1.5) *and* (1.6) *hold. Then for the displacement function of* (1.4), *we have formally*

$$d(r,a) = \sum_{i \geqslant 1} d_i(a)r^i \quad \textit{for } |a - a_0| \textit{ small},$$

where

$$d_1(a) = B_1 N_1^*(a), \qquad d_i(a) = B_i N_i^*(a) + \mathrm{O}\big(|B_1, \ldots, B_{i-1}|\big),$$

with $N_i^* \in C^\infty$ *and* $N_i^*(a_0) > 0$ *for* $i \geqslant 1$.

A polynomial system of the form

$$\dot{x} = y p_m(x) - q_n(x), \qquad \dot{y} = -g(x) p_m(x) \tag{1.7}$$

was introduced in Jiang and Han [9], where

$$p_m(x) = 1 + \sum_{i=1}^m b_i x^i, \qquad q_n(x) = \sum_{i=0}^n a_i x^i \tag{1.8}$$

with a_i and b_i being parameters. Obviously, on the region $p_m(x) > 0$ the system (1.7) is equivalent to the Liénard system

$$\dot{x} = y - \frac{q_n(x)}{p_m(x)}, \qquad \dot{y} = -g(x). \tag{1.9}$$

It is clear that the system (1.4) has a singular point with index +1 at $(0, a_0)$. By applying Theorem 1 it was proved in [9] that the system (1.7) or (1.9) has Hopf cyclicity $[\frac{m+n-1}{2}]$ at the point $(0, a_0)$ if (1.8) holds and $g(0) = 0$, $g'(0) > 0$, and $g(-x) = -g(x)$ for $|x|$ small.

Based on a result of Petrov [1], Han [7] and Christopher and Lynch [3] separately proved the following theorem.

Theorem 3. *The Liénard system*

$$\dot{x} = y - q_n(x), \qquad \dot{y} = -x(x+1) \tag{1.10}$$

has Hopf cyclicity $[\frac{2n-1}{3}]$ *at the point* $(0, a_0)$.

In this paper we give a new proof to the above theorem by using Theorem 1 without using the result of Petrov [1]. Then based on the idea in the new proof we obtained the following theorem which is one of the main results of the paper.

Theorem 4. *Let* (1.8) *hold. If* $g(x) = x(x+1)$ *in* (1.7) *or* (1.9), *then for Eq.* (1.7) *or* (1.9) *an upper bound of the maximum number of limit cycles in a neighborhood of the point* $C(0, a_0)$ *is* $[\frac{4n+2m-4}{3}] - [\frac{n-m}{3}]$ *as* $n \geqslant m$ *or* $[\frac{4m+2n-4}{3}] - [\frac{m-n}{3}]$ *as* $m > n$.

From the following theorem one can see that the upper bound obtained in the above theorem is the Hopf cyclicity at the point $C(0, a_0)$ in the case of $m = n$.

Theorem 5. *Let* (1.8) *hold with* $m = n$. *If* $g(x) = x(x+1)$ *in* (1.7) *or* (1.9), *then Eq.* (1.7) *or* (1.9) *has Hopf cyclicity* $2n - 2$ $(= [\frac{6n-4}{3}])$ *at the point* $C(0, a_0)$ *for* $n = 1, 2, 3, 4$.

The conclusion in Theorem 5 is an improvement to Theorem 4 in the case of $m = n \leqslant 4$.

As another interesting application of Theorem 2, we study the Hopf bifurcation of the following non-smooth Liénard system

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x), \tag{1.11}$$

where

$$F(x) = \begin{cases} \sum_{i=1}^{n} a_i^+ x^i, & x > 0, \\ \sum_{i=1}^{m} a_i^- x^i, & x \leqslant 0, \end{cases} \qquad g(x) = x(x+1). \tag{1.12}$$

We consider a_i^+ and a_i^- as parameters. It is clear that (1.11) has a singular point at the origin.

Another main result is the following theorem.

Theorem 6. *Let* (1.12) *hold. Then the Hopf cyclicity of Eq.* (1.11) *at the origin is* $[\frac{3m+2n-1}{3}]$ *as* $m \geqslant n$ *or* $[\frac{3n+2m-1}{3}]$ *as* $n > m$.

2. Preliminary lemmas

For $g(x) = x(x+1)$, we suppose

$$G(x) = \int_0^x g(x)\,\mathrm{d}x = \frac{x^2}{2} + \frac{x^3}{3}. \tag{2.1}$$

From (2.1) we have

$$\frac{G(\alpha(x)) - G(x)}{\alpha - x} = \frac{1}{2}(\alpha + x) + \frac{1}{3}\left(\alpha^2 + \alpha x + x^2\right),$$

which gives

$$\alpha(x) = \frac{-2x - 3 + \sqrt{-12x^2 - 12x + 9}}{4} = -x - \frac{2}{3}x^2 - \left(\frac{2}{3}\right)^2 x^3 - 2\left(\frac{2}{3}\right)^3 x^4 + O\left(x^5\right) \tag{2.2}$$

since $G(\alpha(x)) = G(x)$ for $|x| \ll 1$. Let

$$I_{i,j}(x) = \alpha^i(x)x^j - \alpha^j(x)x^i,$$

and let $I_i(x)$ denote $I_{i,0}(x)$ for short, i.e. $I_i(x) = \alpha^i(x) - x^i$. We have the following lemma.

Lemma 1. *For any integer $n > 0$, we have*

$$I_{3n}(x) = -\sum_{i=1}^{n} C_n^i \left(\frac{3}{2}\right)^i I_{3n-i}(x).$$

Proof. When $n = 1$, since $G(\alpha) = G(x)$ we get from (2.1) that

$$I_3(x) = \alpha^3 - x^3 = -\frac{3}{2}\left(\alpha^2 - x^2\right) = -\frac{3}{2} I_2(x). \tag{2.3}$$

Thus, the conclusion is true for $n = 1$.

Assume that the conclusion is true for $n = k$. That is,

$$I_{3k}(x) = -\sum_{i=1}^{k} C_k^i \left(\frac{3}{2}\right)^i I_{3k-i}(x). \tag{2.4}$$

Let us prove that the conclusion is also true for $n = k+1$.

Multiplication of (2.4) by $\alpha^j + x^j$ gives

$$I_{3k}(x)\left(\alpha^j + x^j\right) = -\sum_{i=1}^{k} C_k^i \left(\frac{3}{2}\right)^i I_{3k-i}(x)\left(\alpha^j + x^j\right), \quad j = 2, 3,$$

or

$$I_{3k+j}(x)+I_{3k,j}(x)=-\sum_{i=1}^{k}C_k^i\left(\frac{3}{2}\right)^i\big(I_{3k-i+j}(x)+I_{3k-i,j}(x)\big),\quad j=2,3. \tag{2.5}$$

Multiplying both sides of (2.3) by α^m+x^m, we get

$$I_3(x)\big(\alpha^m+x^m\big)=-\frac{3}{2}I_2(x)\big(\alpha^m+x^m\big),$$

which yields

$$I_{m+3}(x)+\frac{3}{2}I_{m+2}(x)=I_{m,3}(x)+\frac{3}{2}I_{m,2}(x), \tag{2.6}$$

where m is a positive integer. Then (2.5) and (2.6) together imply

$$\begin{aligned}
&I_{3k+3}(x)+\frac{3}{2}I_{3k+2}(x)\\
&\quad=\frac{1}{2}\left[I_{3k+3}(x)+I_{3k,3}(x)+\frac{3}{2}\big(I_{3k+2}(x)+I_{3k,2}(x)\big)\right]\\
&\quad=-\frac{1}{2}\sum_{i=1}^{k}C_k^i\left(\frac{3}{2}\right)^i\left[\big(I_{3k-i+3}(x)+I_{3k-i,3}(x)\big)+\frac{3}{2}\big(I_{3k-i+2}(x)+I_{3k-i,2}(x)\big)\right]\\
&\quad=-\frac{1}{2}\sum_{i=1}^{k}C_k^i\left(\frac{3}{2}\right)^i\left[\left(I_{3k-i+3}(x)+\frac{3}{2}I_{3k-i+2}(x)\right)+\left(I_{3k-i,3}(x)+\frac{3}{2}I_{3k-i,2}(x)\right)\right]\\
&\quad=-\sum_{i=1}^{k}C_k^i\left(\frac{3}{2}\right)^i\left(I_{3k-i+3}(x)+\frac{3}{2}I_{3k-i+2}(x)\right).
\end{aligned}$$

Hence

$$I_{3k+3}(x)=-\big(1+C_k^1\big)\frac{3}{2}I_{3k+2}(x)-\sum_{i=2}^{k}\big(C_k^{i-1}+C_k^i\big)\left(\frac{3}{2}\right)^i I_{3k+3-i}(x)-\left(\frac{3}{2}\right)^{k+1}I_{2k+2}(x),$$

which can be written as

$$I_{3k+3}(x)=-\sum_{i=1}^{k+1}C_{k+1}^i\left(\frac{3}{2}\right)^i I_{3k+3-i}(x).$$

Therefore the conclusion is true for $n=k+1$, and the proof is completed. □

We introduce a new variable θ by

$$x=(-1-\sqrt{3}\sin\theta+\cos\theta)/2\equiv\xi(\theta)\quad\text{for } |\theta|\ll 1.$$

Then it follows that

$$\begin{aligned}-12x^2-12x+9&=-3\big[(2x+1)^2-4\big]=-3\big(3\sin^2\theta+\cos^2\theta-2\sqrt{3}\sin\theta\cos\theta-4\big)\\&=-3\big(-\sin^2\theta-3\cos^2\theta-2\sqrt{3}\sin\theta\cos\theta\big)\\&=3(\sin\theta+\sqrt{3}\cos\theta)^2.\end{aligned}$$

Substituting the above into (2.2) yields that

$$\alpha(x)=(-1+\sqrt{3}\sin\theta+\cos\theta)/2=\xi(-\theta).$$

Suppose that $I_n(x)=I_n(\xi(\theta))\equiv\widetilde{I}_n(\theta)$. Then we have $\widetilde{I}_n(\theta)=[\xi(-\theta)]^n-[\xi(\theta)]^n$. Thus the periodic function $\widetilde{I}_n$ is odd in θ. Further, for its Fourier expansion we have the following lemma.

Lemma 2. *For any positive integer n, the function $\widetilde{I}_n(\theta)$ has the following Fourier expansion*

$$\widetilde{I}_n(\theta)=\sum_{i\in S(n)}b_{n,i}\sin i\theta,$$

where $S(n)=\{k\mid k\neq 0\pmod 3,\ 1\leqslant k\leqslant n\}$ and $b_{n,i}$ are coefficients independent of θ with $b_{n,n}=2^{-n+2}\sin\frac{n\pi}{3}$.

Proof. For any integer $n\geqslant 1$, by the definition of $\widetilde{I}_n$ we have

$$\begin{aligned}\widetilde{I}_n(\theta)&=(-1+\sqrt{3}\sin\theta+\cos\theta)^n/2^n-(-1-\sqrt{3}\sin\theta+\cos\theta)^n/2^n\\&=2^{-n}\sum_{j=0}^{n}C_n^j\big[1-(-1)^j\big](\sqrt{3}\sin\theta)^j(-1+\cos\theta)^{n-j}\\&=2^{-n+1}\sum_{j=1,\ j\text{ odd}}^{n}C_n^j(\sqrt{3})^j\sin\theta\big(1-\cos^2\theta\big)^{\frac{j-1}{2}}(-1+\cos\theta)^{n-j}.\end{aligned}$$

Then using the expansions of $(1-\cos^2\theta)^{\frac{j-1}{2}}$ in $\cos^2\theta$ and $(-1+\cos\theta)^{n-j}$ in $\cos\theta$ for j odd we further have

$$\begin{aligned}\widetilde{I}_n(\theta)&=2^{-n+1}\sum_{j=1,\ j\text{ odd}}^{n}C_n^j(\sqrt{3})^j\sin\theta\sum_{i=0}^{n-1}\tilde{b}_{j,i}\cos^i\theta\\&=2^{-n+1}\sum_{i=0}^{n-1}\sum_{j=1,\ j\text{ odd}}^{n}\tilde{b}_{j,i}C_n^j(\sqrt{3})^j\sin\theta\cos^i\theta\\&=2^{-n+1}\sum_{i=0}^{n-1}b_i\sin\theta\cos^i\theta,\end{aligned}$$

where

$$\tilde{b}_{j,i}=\sum_{\substack{0\leqslant k\leqslant n-j\\0\leqslant 2l\leqslant j-1}}^{2l+k=i}C_{n-j}^kC_{\frac{j-1}{2}}^l(-1)^{l+n-j-k},\qquad b_i=\sum_{j=1,\ j\text{ odd}}^{n}\tilde{b}_{j,i}C_n^j(\sqrt{3})^j,\quad 0\leqslant i\leqslant n-1.$$

It can be seen that

$$\tilde{b}_{j,n-1} = (-1)^{\frac{j-1}{2}}, \qquad b_{n-1} = \sum_{j=1,\ j \text{ odd}}^{n} (-1)^{\frac{j-1}{2}} C_n^j (\sqrt{3})^j. \tag{2.7}$$

Using the formula $2\sin m\theta\cos\theta = \sin(m+1)\theta + \sin(m-1)\theta$, one can get

$$\sin\theta\cos^i\theta = \sum_{j=1}^{i+1} \bar{b}_{i,j}\sin j\theta, \quad \text{where } \bar{b}_{i,i+1} = 2^{-i},\ \text{for } i \geqslant 0. \tag{2.8}$$

Hence, it follows that

$$\widetilde{I}_n(\theta) = 2^{-n+1}\sum_{i=0}^{n-1} b_i \sum_{j=1}^{i+1} \bar{b}_{i,j}\sin j\theta = 2^{-n+1}\sum_{j=1}^{n}\sum_{i=j-1}^{n-1} b_i\bar{b}_{i,j}\sin j\theta = \sum_{j=1}^{n} b_{n,j}\sin j\theta,$$

where

$$b_{n,j} = 2^{-n+1}\sum_{i=j-1}^{n-1} b_i\bar{b}_{i,j}, \quad 1 \leqslant j \leqslant n.$$

In particular, (2.7) and (2.8) together give

$$b_{n,n} = 2^{-n+1} b_{n-1}\bar{b}_{n-1,n} = 2^{-2n+2}\sum_{j=1,\ j \text{ odd}}^{n} C_n^j (\sqrt{3})^j (-1)^{\frac{j-1}{2}}.$$

Noting that

$$\begin{aligned} 2^{n+1}\mathrm{i}\sin\frac{n\pi}{3} &= 2^n\left(e^{\frac{n\pi}{3}\mathrm{i}} - e^{-\frac{n\pi}{3}\mathrm{i}}\right) = 2^n\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)^n - 2^n\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right)^n \\ &= (1+\mathrm{i}\sqrt{3})^n - (1-\mathrm{i}\sqrt{3})^n = \sum_{j=0}^{n} C_n^j(\sqrt{3})^j\mathrm{i}^j - \sum_{j=0}^{n} C_n^j(-\sqrt{3})^j\mathrm{i}^j \\ &= 2\sum_{j=1,\ j \text{ odd}}^{n} C_n^j(\sqrt{3})^j(-1)^{\frac{j-1}{2}}\mathrm{i}, \quad \text{where } \mathrm{i} = \sqrt{-1}, \end{aligned}$$

we have

$$\sum_{j=1,\ j \text{ odd}}^{n} C_n^j(\sqrt{3})^j(-1)^{\frac{j-1}{2}} = 2^n\sin\frac{n\pi}{3}.$$

Therefore, $b_{n,n} = 2^{-n+2}\sin\frac{n\pi}{3}$.

Obviously, in order to complete the proof we just need to show $b_{n,3m} = 0$ for any positive integer m.

In fact we have

$$\begin{aligned}
&\int_{-\pi}^{\pi} (-1+\sqrt{3}\sin\theta+\cos\theta)^n \sin(3m\theta)\,\mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \left(-1+2\sin\left(\theta+\frac{\pi}{6}\right)\right)^n \sin(3m\theta)\,\mathrm{d}\theta \quad \left(\text{let } u=\theta-\frac{5}{6}\pi\right) \\
&= \int_{-\frac{11}{6}\pi}^{\frac{\pi}{6}} (-1-2\sin u)^n \sin\left[3m\left(u+\frac{5}{6}\pi\right)\right]\mathrm{d}u \\
&= \int_{-\frac{11}{6}\pi}^{\frac{\pi}{6}} (-1-2\sin u)^n \sin\left[3m\left(u+\frac{\pi}{6}\right)\right]\mathrm{d}u \\
&= \int_{-\frac{7}{6}\pi}^{\frac{\pi}{6}} (-1-2\sin u)^n \sin\left[3m\left(u+\frac{\pi}{6}\right)\right]\mathrm{d}u + \int_{-\frac{11}{6}\pi}^{-\frac{7}{6}\pi} (-1-2\sin u)^n \sin\left[3m\left(u+\frac{\pi}{6}\right)\right]\mathrm{d}u \\
&= \int_{-\frac{7}{6}\pi}^{\frac{\pi}{6}} (-1-2\sin u)^n \sin\left[3m\left(u+\frac{\pi}{6}\right)\right]\mathrm{d}u + \int_{\frac{\pi}{6}}^{\frac{5}{6}\pi} (-1-2\sin u)^n \sin\left[3m\left(u+\frac{\pi}{6}\right)\right]\mathrm{d}u \\
&= \int_{-\frac{7}{6}\pi}^{\frac{5}{6}\pi} (-1-2\sin u)^n \sin\left[3m\left(u+\frac{\pi}{6}\right)\right]\mathrm{d}u \quad \left(\text{let } \theta=u+\frac{1}{6}\pi\right) \\
&= \int_{-\pi}^{\pi} \left(-1-2\sin\left(\theta-\frac{\pi}{6}\right)\right)^n \sin(3m\theta)\,\mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} (-1-\sqrt{3}\sin\theta+\cos\theta)^n \sin(3m\theta)\,\mathrm{d}\theta,
\end{aligned}$$

which implies

$$\int_{-\pi}^{\pi} \widetilde{I}_n(\theta)\sin(3m\theta)\,\mathrm{d}\theta = \int_{-\pi}^{\pi} \left[\left(\xi(-\theta)\right)^n - \left(\xi(\theta)\right)^n\right]\sin(3m\theta)\,\mathrm{d}\theta = 0.$$

Then by Fourier's formula, we have $b_{n,3m} = \frac{1}{\pi}\int_{-\pi}^{\pi} \widetilde{I}_n(\theta)\sin(3m\theta)\,\mathrm{d}\theta = 0$. The proof is completed. □

By Lemma 2 we have $b_{k,k} \neq 0$ for $k \neq 0 \pmod 3$. Hence, noting that different functions $\sin(k\theta)$ for $k \neq 0 \pmod 3$ are linearly independent, Lemma 2 implies that the functions $\widetilde{I}_i(\theta)$, $i \in S(n)$, are linearly independent for any positive integer $n \geqslant 1$. In other words, we have the following lemma.

Lemma 3. *For any integer $n \geqslant 1$, the $[\frac{2n-1}{3}]+1$ functions $I_i(x)$, $i \in S(n)$, are linearly independent.*

Then by Lemmas 1 and 3 we see the fact that the n functions $I_i(x)$, $i = 1, \dots, n$, can span a linear space of dimension $[\frac{2n-1}{3}]+1$.

Let $I_{n,m}(x) = I_{n,m}(\xi(\theta)) = \widetilde{I}_{n,m}(\theta)$. For the Fourier expansion of $\widetilde{I}_{n,m}(\theta)$ we have the following lemma.

Lemma 4. *For any positive integers n, m and $n > m$, the function $\widetilde{I}_{n,m}(\theta)$ has the following Fourier expansion*

$$\widetilde{I}_{n,m}(\theta) = \sum_{k=1}^{n+m} b_{n,m,k} \sin k\theta,$$

where $b_{n,m,k}$ are coefficients independent of θ with $b_{n,m,n+m} = 2^{-n-m+2} \sin \frac{n-m}{3}\pi$.

Proof. Suppose that $J(\theta) = \alpha(x)x|_{x=\xi(\theta)}$; it is easy to get that

$$J(\theta) = \frac{(-1+\sqrt{3}\sin\theta + \cos\theta)(-1-\sqrt{3}\sin\theta+\cos\theta)}{4} = \cos^2\theta - \frac{\cos\theta}{2} - \frac{1}{2}.$$

Further, we suppose that for positive integer m,

$$\left(\cos^2\theta - \frac{\cos\theta}{2} - \frac{1}{2}\right)^m = \sum_{i=0}^{2m} \tilde{e}_{mi} \cos^i\theta,$$

where

$$\tilde{e}_{mi} = \sum_{j=\max\{0,i-m\}}^{2j \leqslant i} \left(-\frac{1}{2}\right)^{m-j} C_m^j C_{m-j}^{i-2j}. \tag{2.9}$$

From Lemma 2, for $n > m$,

$$\begin{aligned}
\widetilde{I}_{n,m}(\theta) = J^m(\theta)\widetilde{I}_{n-m}(\theta) &= \left(\cos^2\theta - \frac{\cos\theta}{2} - \frac{1}{2}\right)^m \sum_{k\in S(n-m)} b_{n-m,k} \sin k\theta \\
&= \left(\sum_{i=0}^{2m} \tilde{e}_{mi}\cos^i\theta\right)\left(\sum_{k\in S(n-m)} b_{n-m,k}\sin k\theta\right) \\
&= \sum_{k\in S(n-m)} b_{n-m,k}\left(\sin k\theta \sum_{i=0}^{2m} \tilde{e}_{mi}\cos^i\theta\right).
\end{aligned}$$

By using the formula $2\sin m\theta\cos\theta = \sin(m+1)\theta + \sin(m-1)\theta$, one can get

$$\sin k\theta \sum_{i=0}^{2m} \tilde{e}_{mi}\cos^i\theta = \sum_{j=1}^{k+2m} \bar{e}_{k,m,j} \sin j\theta,$$

with

$$\bar{e}_{k,m,k+2m} = 2^{-2m}\tilde{e}_{m,2m} = 2^{-2m}, \tag{2.10}$$

since $\tilde{e}_{m,2m} = 1$ by (2.9). And then

$$\begin{aligned}
\widetilde{I}_{nm}(\theta) &= \sum_{k\in S(n-m)} b_{n-m,k} \sum_{j=1}^{k+2m} \bar{e}_{k,m,j} \sin j\theta \\
&= \sum_{j=1}^{n+m} \sum_{k=\max\{1,j-2m\}}^{n-m} b_{n-m,k}\bar{e}_{k,m,j} \sin j\theta \\
&= \sum_{j=1}^{n+m} b_{n,m,j} \sin j\theta
\end{aligned}$$

where

$$b_{n,m,j} = \sum_{k=\max\{1,j-2m\}}^{n-m} b_{n-m,k}\bar{e}_{k,m,j}.$$

In particular, by Lemma 2 and (2.10)

$$b_{n,m,n+m} = b_{n-m,n-m}\bar{e}_{n-m,m,n+m} = 2^{-n-m+2} \sin \frac{n-m}{3}\pi.$$

The proof is completed. □

Suppose that $J_n(x) = \alpha^n + x^n$ and $J_n(x) = J_n(\xi(\theta)) = \widetilde{J}_n(\theta)$. Let us prove the following lemma.

Lemma 5. *For any positive integer n, we have*

$$\widetilde{J}_n(\theta) = \sum_{i=0}^{n} c_{n,i} \cos i\theta,$$

where $c_{n,n} = 2^{-n+2} \cos \frac{n\pi}{3}$, *and* $\widetilde{J}_n(0) = \sum_{i=0}^{n} c_{n,i} = 0$ *since* $J_n(0) = 0$.

Proof. For any integer $n \geqslant 1$, we get

$$\begin{aligned}
\widetilde{J}_n(\theta) &= (-1 + \sqrt{3}\sin\theta + \cos\theta)^n/2^n + (-1 - \sqrt{3}\sin\theta + \cos\theta)^n/2^n \\
&= 2^{-n} \sum_{j=0}^{n} C_n^j\big[1 + (-1)^j\big](\sqrt{3}\sin\theta)^j(-1+\cos\theta)^{n-j} \\
&= 2^{-n+1} \sum_{j=0,\ j \text{ even}}^{n} C_n^j(\sqrt{3})^j\big(1-\cos^2\theta\big)^{\frac{j}{2}}(-1+\cos\theta)^{n-j}.
\end{aligned}$$

After expanding $(1-\cos^2\theta)^{\frac{j}{2}}(-1+\cos\theta)^{n-j}$ in the term of $\cos\theta$, we obtain

$$\begin{aligned}\widetilde{J}_n(\theta) &= 2^{-n+1}\sum_{j=0,\ j \text{ even}}^{n} C_n^j(\sqrt{3})^j\sum_{i=0}^{n}\tilde{b}_{j,i}\cos^i\theta \\ &= 2^{-n+1}\sum_{i=0}^{n}\sum_{j=0,\ j \text{ even}}^{n}\tilde{b}_{j,i}C_n^j(\sqrt{3})^j\cos^i\theta \\ &= 2^{-n+1}\sum_{i=0}^{n}b_i\cos^i\theta,\end{aligned}$$

where

$$\tilde{b}_{j,i} = \sum_{\substack{0\leqslant k\leqslant n-j\\ 0\leqslant 2l\leqslant j}}^{2l+k=i} C_{n-j}^k C_{\frac{j}{2}}^l(-1)^{l+n-j-k}, \qquad b_i = \sum_{j=0,\ j \text{ even}}^{n}\tilde{b}_{j,i}C_n^j(\sqrt{3})^j, \quad 0\leqslant i\leqslant n.$$

It can be seen that

$$\tilde{b}_{j,n} = (-1)^{\frac{j}{2}}, \qquad b_n = \sum_{j=0,\ j \text{ even}}^{n}(-1)^{\frac{j}{2}}C_n^j(\sqrt{3})^j. \tag{2.11}$$

Noting that for any integer m, we have $2\cos m\theta\cos\theta = \cos(m+1)\theta + \cos(m-1)\theta$, and hence

$$\cos^i\theta = \sum_{j=0}^{i}\bar{b}_{i,j}\cos j\theta, \quad \text{where } \bar{b}_{i,i} = 2^{-i+1}, \text{ for } i\geqslant 1. \tag{2.12}$$

Further, let $\bar{b}_{00} = 1$ and then we get

$$\widetilde{J}_n(\theta) = 2^{-n+1}\sum_{i=0}^{n}b_i\sum_{j=0}^{i}\bar{b}_{i,j}\cos j\theta = 2^{-n+1}\sum_{j=0}^{n}\sum_{i=j}^{n}b_i\bar{b}_{i,j}\cos j\theta = \sum_{j=0}^{n}c_{n,j}\cos j\theta,$$

where

$$c_{n,j} = 2^{-n+1}\sum_{i=j}^{n}b_i\bar{b}_{i,j}, \quad 0\leqslant j\leqslant n.$$

Particularly, from (2.11) and (2.12), we get

$$c_{n,n} = 2^{-n+1}b_n\bar{b}_{n,n} = 2^{-2n+2}\sum_{j=0,\ j \text{ even}}^{n}C_n^j(\sqrt{3})^j(-1)^{\frac{j}{2}}.$$

Noting that

$$2^{n+1}\cos\frac{n\pi}{3} = 2^n\left(e^{\frac{n\pi}{3}\mathrm{i}} + e^{-\frac{n\pi}{3}\mathrm{i}}\right) = 2^n\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)^n + 2^n\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right)^n$$
$$= (1+\sqrt{3}\mathrm{i})^n + (1-\sqrt{3}\mathrm{i})^n = \sum_{j=0}^{n} C_n^j(\sqrt{3})^j\mathrm{i}^j + \sum_{j=0}^{n} C_n^j(-\sqrt{3})^j\mathrm{i}^j$$
$$= 2\sum_{j=0,\ j\ \text{even}}^{n} C_n^j(\sqrt{3})^j(-1)^{\frac{j}{2}}, \quad \text{where } \mathrm{i} = \sqrt{-1},$$

we have that

$$\sum_{j=0,\ j\ \text{even}}^{n} C_n^j(\sqrt{3})^j(-1)^{\frac{j}{2}} = 2^n\cos\frac{n\pi}{3}$$

and hence $c_{n,n} = 2^{-n+2}\cos\frac{n\pi}{3}$. The proof is completed. □

From Lemmas 2 and 5, we get the following lemma.

Lemma 6. *For any integer $n \geqslant 1$, $I_i(x)$, $i \in S(n)$, and $J_i(x)$, $1 \leqslant i \leqslant n$, are linearly independent.*

3. Proof of the main results

In this section we will give a proof of Theorems 3, 4, 5 and 6 respectively.

Proof of Theorem 3. Firstly, we prove that Eq. (1.10) has at most $[\frac{2n-1}{3}]$ limit cycles near the origin by Theorem 1.

From (1.8) we have

$$q_n\big(\alpha(x)\big) - q_n(x) = \sum_{i=1}^{n} a_i I_i(x) = \sum_{i\geqslant 1} B_i x^i \equiv Q_n(x). \tag{3.1}$$

Then by Lemma 2 we have

$$Q_n(x) = \sum_{i=1}^{n} a_i\widetilde{I}_i(\theta) = \sum_{i=1}^{n} a_i \sum_{j\in S(i)} b_{i,j}\sin j\theta \equiv \widetilde{Q}_n(\theta).$$

Noting that $S(j) = \bigcup_{i=1}^{j}\widetilde{S}_i$, where $\widetilde{S}_i = \{i\}$ for $i \neq 0 \pmod 3$ and $\widetilde{S}_i = \emptyset$ for $i = 0 \pmod 3$, one has

$$\widetilde{Q}_n(\theta) = \sum_{j\in S(n)}\sum_{i=j}^{n} a_i b_{i,j}\sin j\theta = \sum_{j\in S(n)} c_j\sin j\theta,$$

where $c_j = \sum_{i=j}^{n} a_i b_{i,j}$. Then

$$\widetilde{Q}_n(\theta) = \sum_{j\in S(n)} c_j \sum_{i\geqslant 0}\frac{(-1)^i}{(2i+1)!} j^{2i+1}\theta^{2i+1} = \sum_{i\geqslant 0}\frac{(-1)^i}{(2i+1)!}\tilde{c}_{2i+1}\theta^{2i+1} \tag{3.2}$$

where

$$\tilde{c}_{2i+1} = \sum_{j \in S(n)} j^{2i+1} c_j, \quad i \geqslant 0, \tag{3.3}$$

since

$$\sin j\theta = \sum_{i \geqslant 0} \frac{(-1)^i}{(2i+1)!} j^{2i+1} \theta^{2i+1}.$$

From (3.3) we get that $\widetilde{C}_{l+1} = A_{l+1} C_{l+1}$, where $l = [\frac{2n-1}{3}]$ and

$$\widetilde{C}_{l+1} = \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_3 \\ \vdots \\ \tilde{c}_{2l+1} \end{pmatrix}, \qquad C_{l+1} = \begin{pmatrix} c_{k_0} \\ c_{k_1} \\ \vdots \\ c_{k_l} \end{pmatrix}, \qquad A_{l+1} = \begin{pmatrix} k_0 & k_1 & \cdots & k_l \\ k_0^3 & k_1^3 & \cdots & k_l^3 \\ \vdots & \vdots & \ddots & \vdots \\ k_0^{2l+1} & k_1^{2l+1} & \cdots & k_l^{2l+1} \end{pmatrix},$$

where $k_i \in S(n)$, $0 \leqslant i \leqslant l$ and $k_i < k_j$ if $i < j$. Because of

$$\det A_{l+1} = \prod_{i=0}^{l} k_i \prod_{0 \leqslant i < j \leqslant l} \left(k_j^2 - k_i^2\right) \neq 0,$$

we have $C_{l+1} = A_{l+1}^{-1} \widetilde{C}_{l+1}$. Hence by (3.3) we can obtain

$$\tilde{c}_{2j+1} = O\left(|\tilde{c}_1, \tilde{c}_3, \ldots, \tilde{c}_{2l+1}|\right) \quad \text{for } j \geqslant l+1. \tag{3.4}$$

Further, noting that $\theta = -\frac{2\sqrt{3}}{3} x + O(x^2)$, from (3.1) and (3.2) we have

$$B_{2j+1} = \frac{(2\sqrt{3})^{2j+1}(-1)^{j+1}}{3^{2j+1}(2j+1)!} \tilde{c}_{2j+1} + O\left(|\tilde{c}_1, \tilde{c}_3, \ldots, \tilde{c}_{2j-1}|\right), \quad \text{for } j \geqslant 0, \tag{3.5}$$

which gives

$$\tilde{c}_{2j+1} = \frac{3^{2j+1}(2j+1)!}{(2\sqrt{3})^{2j+1}(-1)^{j+1}} B_{2j+1} + O\left(|B_1, B_3, \ldots, B_{2j-1}|\right), \quad \text{for } 0 \leqslant j \leqslant l. \tag{3.6}$$

By (3.4), (3.5) and (3.6)

$$B_{2j+1} = O\left(|B_1, B_3, \ldots, B_{2l+1}|\right) \quad \text{for } j \geqslant l+1.$$

In particular, $Q_n(x) = 0$ when $B_{2j+1} = 0$, $0 \leqslant j \leqslant l$. It follows by Theorem 1 that Eq. (1.10) has at most l limit cycles near the origin.

Finally, we prove that l limit cycles can appear near the origin. For simplicity, take $a_{3j} = 0$, $1 \leqslant j \leqslant [\frac{n}{3}]$. In this case, from (3.1), we have

$$(B_1, B_3, \ldots, B_{2l+1})^T = S_l (a_{k_0}, a_{k_1}, \ldots, a_{k_l})^T,$$

where S_l is a constant matrix of order $l+1$. On the one hand, from Lemma 3, the functions $I_{k_0}(x), I_{k_1}(x), \ldots, I_{k_l}(x)$ are linearly independent, and hence from (3.1) it is easy to see that $a_{k_j} = 0$,

$0 \leqslant j \leqslant l$, if and only if $q_n(\alpha(x)) - q_n(x) \equiv 0$. On the other hand, from the above proof, we see that $B_{2j+1} = 0$, $0 \leqslant j \leqslant l$, if and only if $q_n(\alpha(x)) - q_n(x) \equiv 0$. Therefore, we have $\det S_l \neq 0$, and the conclusion follows by Theorem 1. The proof is completed. □

Proof of Theorem 4. Suppose $n \geqslant m$. It is obvious that the conclusion is true for Eq. (1.9) if and only if it is true for the following system

$$\dot{x} = v - \left(\frac{q_n(x)}{p_m(x)} - a_0\right), \qquad \dot{v} = -g(x) \tag{3.7}$$

at the origin, where $v = y - a_0$.

Suppose that $b_0 = 1$ and

$$F(x) = \frac{q_n(x)}{p_m(x)} - a_0.$$

Then

$$F\big(\alpha(x)\big) - F(x) = \frac{q_n(\alpha(x))}{p_m(\alpha(x))} - \frac{q_n(x)}{p_m(x)} = \frac{Q_{nm}(x)}{p_m(\alpha(x))p_m(x)}, \tag{3.8}$$

where by (1.8)

$$\begin{aligned}
Q_{nm}(x) &= q_n\big(\alpha(x)\big)p_m(x) - q_n(x)p_m\big(\alpha(x)\big) \\
&= \left(\sum_{i=0}^{n} a_i\alpha^i\right)\left(\sum_{i=0}^{m} b_i x^i\right) - \left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{i=0}^{m} b_i\alpha^i\right) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} a_i b_j\big(\alpha^i x^j - x^i\alpha^j\big) \\
&= \sum_{i=0}^{m}\sum_{j=0}^{m} a_i b_j\big(\alpha^i x^j - x^i\alpha^j\big) + \sum_{i=m+1}^{n}\sum_{j=0}^{m} a_i b_j\big(\alpha^i x^j - x^i\alpha^j\big) \\
&= \sum_{j=0}^{m}\sum_{i=j+1}^{m} (a_i b_j - a_j b_i)\big(\alpha^i x^j - x^i\alpha^j\big) + \sum_{i=m+1}^{n}\sum_{j=0}^{m} a_i b_j\big(\alpha^i x^j - x^i\alpha^j\big),
\end{aligned}$$

which can be written as

$$Q_{nm}(x) = \sum_{j=0}^{m}\sum_{i=j+1}^{n} c_{ij} I_{ij}(x) = \sum_{i\geqslant 1} B_i x^i, \tag{3.9}$$

where

$$c_{ij} = \begin{cases} a_i b_j - a_j b_i, & 1 \leqslant i \leqslant m, \\ a_i b_j, & m+1 \leqslant i \leqslant n. \end{cases}$$

From Lemma 4, for $i > j$ we have

$$\widetilde{I}_{i,j}(\theta) = \sum_{k=1}^{i+j} b_{i,j,k} \sin k\theta, \tag{3.10}$$

where $b_{i,j,i+j} = 2^{-i-j+2} \sin \frac{i-j}{3}\pi$. Then from (3.9) and (3.10) we get

$$\begin{aligned} Q_{nm}(x) &= \sum_{j=0}^{m} \sum_{i=j+1}^{n} c_{ij} \widetilde{I}_{ij}(\theta) = \sum_{j=0}^{m} \sum_{i=j+1}^{n} c_{ij} \sum_{k=1}^{i+j} b_{i,j,k} \sin k\theta \\ &= \sum_{k=1}^{m+n} \sum_{i+j \geqslant k}^{i>j} c_{ij} b_{i,j,k} \sin k\theta = \sum_{k=1}^{m+n} c_k \sin k\theta \equiv \widetilde{Q}_{nm}(\theta), \end{aligned}$$

where $c_k = \sum_{i+j \geqslant k}^{i>j} c_{ij} b_{i,j,k}$, $1 \leqslant k \leqslant n+m$. Noting that if $n-m=0 \pmod 3$, then $c_{n+m} = c_{nm} b_{n,m,m+n} = 0$ since $b_{n,m,m+n} = 0$, $\widetilde{Q}_{nm}(\theta)$ can be rewritten as

$$\widetilde{Q}_{nm}(\theta) = \sum_{k=1}^{h+1} c_k \sin k\theta,$$

where

$$h = \left[\frac{4n+2m-4}{3}\right] - \left[\frac{n-m}{3}\right] = \begin{cases} m+n-1, & m-n \neq 0 \pmod 3, \\ m+n-2, & m-n = 0 \pmod 3. \end{cases}$$

Since

$$\sin k\theta = \sum_{i \geqslant 0} \frac{(-1)^i}{(2i+1)!} k^{2i+1} \theta^{2i+1},$$

we then obtain

$$\widetilde{Q}_{nm}(\theta) = \sum_{k=1}^{h+1} c_k \sum_{i \geqslant 0} \frac{(-1)^i}{(2i+1)!} k^{2i+1} \theta^{2i+1} = \sum_{i \geqslant 0} \frac{(-1)^i}{(2i+1)!} \tilde{c}_{2i+1} \theta^{2i+1}, \tag{3.11}$$

where

$$\tilde{c}_{2i+1} = \sum_{k=1}^{h+1} k^{2i+1} c_k, \quad i \geqslant 0. \tag{3.12}$$

From (3.12) it follows that $\widetilde{C}_{nm} = A_{nm} C_{nm}$, where

$$\widetilde{C}_{nm} = \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_3 \\ \vdots \\ \tilde{c}_{2h+1} \end{pmatrix}, \qquad C_{nm} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{h+1} \end{pmatrix}, \qquad A_{nm} = \begin{pmatrix} 1 & 2 & \cdots & h+1 \\ 1 & 2^3 & \cdots & (h+1)^3 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2^{2h+1} & \cdots & (h+1)^{2h+1} \end{pmatrix}.$$

Since

$$\det A_{nm} = (h+1)! \prod_{1 \leqslant i < j \leqslant h+1} \left(j^2 - i^2\right) \neq 0,$$

we have $C_{nm} = A_{nm}^{-1} \widetilde{C}_{nm}$ and hence by (3.12)

$$\tilde{c}_{2j+1} = \mathrm{O}\left(|\tilde{c}_1, \tilde{c}_3, \ldots, \tilde{c}_{2h+1}|\right) \quad \text{for } j \geqslant h+1. \tag{3.13}$$

Further, noting that $\theta = -\frac{2\sqrt{3}}{3} x + \mathrm{O}(x^2)$, from (3.9) and (3.11) we obtain

$$B_{2k+1} = \frac{(2\sqrt{3})^{2k+1}(-1)^{k+1}}{3^{2k+1}(2k+1)!} \tilde{c}_{2k+1} + \mathrm{O}\left(|\tilde{c}_1, \tilde{c}_3, \ldots, \tilde{c}_{2k-1}|\right), \quad \text{for } k \geqslant 0, \tag{3.14}$$

from which it follows that

$$\tilde{c}_{2k+1} = \frac{3^{2k+1}(2k+1)!}{(2\sqrt{3})^{2k+1}(-1)^{k+1}} B_{2k+1} + \mathrm{O}\left(|B_1, B_3, \ldots, B_{2k-1}|\right), \quad \text{for } 0 \leqslant k \leqslant h. \tag{3.15}$$

So by (3.13), (3.14) and (3.15)

$$B_{2j+1} = \mathrm{O}\left(|B_1, B_3, \ldots, B_{2h+1}|\right), \quad \text{for } j \geqslant h+1.$$

In particular, $\widetilde{C}_{nm} = 0$ and $Q_{nm}(x) = 0$ when $B_{2k+1} = 0$, $0 \leqslant k \leqslant h$. Thus, from (3.8) we can write

$$F\left(\alpha(x)\right) - F(x) = \frac{Q_{nm}(x)}{p_m(\alpha(x))p_m(x)} = \sum_{k=0}^{h} B_{2k+1} x^{2k+1}\left(1 + P_k(x)\right),$$

where $P_k(x) = \mathrm{O}(x) \in C^\omega$ for $|x|$ small. It follows by Theorem 1 that Eq. (3.7) has at most h limit cycles near the origin for $n \geqslant m$.

If $m > n$, then

$$\begin{aligned} Q_{nm}(x) &= q_n\left(\alpha(x)\right)p_m(x) - q_n(x)p_m\left(\alpha(x)\right) \\ &= -\left[p_m\left(\alpha(x)\right)q_n(x) - p_m(x)q_n\left(\alpha(x)\right)\right] \\ &= -\sum_{j=0}^{n} \sum_{i=j+1}^{m} \bar{c}_{ij} I_{ij}(x) = \sum_{i \geqslant 1} B_i x^i, \end{aligned}$$

where

$$\bar{c}_{ij} = \begin{cases} b_i a_j - b_j a_i, & 1 \leqslant i \leqslant n, \\ b_i a_j, & n+1 \leqslant i \leqslant m. \end{cases}$$

Therefore, from the proof above, it is obvious that the conclusion is also true for $m > n$. The proof is completed. □

Proof of Theorem 5. As before, we only need to show that the theorem is true for Eq. (3.7) at the origin. There are four cases to consider below.

Case 1. $n = m = 1$. In this case, from (2.2), (3.8) and (3.9) we have $Q_{11}(x) = (a_1 - a_0 b_1)(\alpha - x)$ and

$$F\big(\alpha(x)\big) - F(x) = \frac{Q_{11}(x)}{p_1(\alpha(x))p_1(x)} = -2(a_1 - a_0 b_1)x\big(1 + P_{10}(x)\big),$$

where $P_{10}(0) = 0$. So it is obvious that the conclusion is true.

Case 2. $n = m = 2$. In this case, from (3.9) we have

$$Q_{22}(x) = e_{21}(\alpha - x) + e_{22}\big(\alpha^2 - x^2\big) + e_{23}\big(\alpha^2 x - \alpha x^2\big) = \sum_{i\geqslant 1} B_i x^i, \tag{3.16}$$

where

$$e_{21} = a_1 - a_0 b_1, \qquad e_{22} = a_2 - a_0 b_2, \qquad e_{23} = a_2 b_1 - a_1 b_2. \tag{3.17}$$

Substituting (2.2) into (3.16) yields $\widetilde{B}_1 = R_1 C_1$, where

$$\widetilde{B}_1 = \begin{pmatrix} B_1 \\ B_3 \\ B_5 \end{pmatrix}, \qquad C_1 = \begin{pmatrix} e_{21} \\ e_{22} \\ e_{23} \end{pmatrix}, \qquad R_1 = \begin{pmatrix} -2 & 0 & 0 \\ -(\frac{2}{3})^2 & \frac{4}{3} & 2 \\ -4(\frac{2}{3})^4 & 6(\frac{2}{3})^3 & 4(\frac{2}{3})^2 \end{pmatrix}. \tag{3.18}$$

Because of $\operatorname{rank} R_1 = 3$, we have that $C_1 = 0$ if and only if $\widetilde{B}_1 = 0$. Hence from (3.8)

$$F\big(\alpha(x)\big) - F(x) = \frac{Q_{22}(x)}{p_2(\alpha(x))p_2(x)} = \sum_{j=0}^{2} B_{2j+1} x^{2j+1}\big(1 + P_{2j}(x)\big),$$

where $P_{2j}(0) = 0$. It follows by Theorem 1 that Eq. (3.7) has at most two limit cycles near the origin.

Next we prove that two limit cycles can appear near the origin. From (3.17) we have

$$\frac{\partial(e_{21}, e_{22}, e_{23})}{\partial(a_0, a_1, a_2, b_1, b_2)} = \begin{pmatrix} -b_1 & 1 & 0 & -a_0 & 0 \\ -b_2 & 0 & 1 & 0 & -a_0 \\ 0 & -b_2 & b_1 & a_2 & -a_1 \end{pmatrix}. \tag{3.19}$$

Further from (3.17), (3.18) and (3.19) we have $B_1 = B_3 = 0$, $B_5 = -2(\frac{2}{3})^2$, and $e_{21} = 0$, $e_{22} = -\frac{3}{2}$, $e_{23} = 1$ when $a_0 = a_1 = b_2 = 0$, $a_2 = -\frac{3}{2}$, $b_1 = -\frac{2}{3}$, and

$$\det\frac{\partial(B_1, B_3)}{\partial(a_1, a_2)} = \det\frac{\partial(B_1, B_3)}{\partial(e_{21}, e_{22})}\det\frac{\partial(e_{21}, e_{22})}{\partial(a_1, a_2)} = \begin{vmatrix} -2 & 0 \\ -(\frac{2}{3})^2 & \frac{4}{3} \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0,$$

which ensures that two limit cycles appear near the origin.

Case 3. $n = m = 3$. In this case, noting that $\alpha^2 - x^2 = -\frac{2}{3}(\alpha^3 - x^3)$, from (3.9) we have

$$Q_{33}(x) = e_{31}(\alpha - x) + e_{32}\big(\alpha^2 x - \alpha x^2\big) + e_{33}\big(\alpha^3 - x^3\big) + e_{34}\big(\alpha^3 x - \alpha x^3\big) + e_{35}\big(\alpha^3 x^2 - \alpha^2 x^3\big),$$

where

$$e_{31}=a_1-a_0b_1,\qquad e_{33}=(a_3-a_0b_3)-\frac{2}{3}(a_2-a_0b_2), \tag{3.20}$$

$$e_{32}=a_2b_1-a_1b_2,\qquad e_{34}=a_3b_1-a_1b_3,\qquad e_{35}=a_3b_2-a_2b_3. \tag{3.21}$$

As before, we can obtain $\widetilde{B}_2=R_2C_2$, where

$$\widetilde{B}_2=\begin{pmatrix}B_1\\B_3\\B_5\\B_7\\B_9\end{pmatrix},\qquad C_2=\begin{pmatrix}e_{31}\\e_{32}\\e_{33}\\e_{34}\\e_{35}\end{pmatrix},$$

$$R_2=\begin{pmatrix}-2&0&0&0&0\\-(\frac{2}{3})^2&2&-2&0&0\\-4(\frac{2}{3})^4&4(\frac{2}{3})^2&-6(\frac{2}{3})^2&-\frac{4}{3}&-2\\-21(\frac{2}{3})^6&17(\frac{2}{3})^4&-30(\frac{2}{3})^4&-11(\frac{2}{3})^3&-9(\frac{2}{3})^2\\-127(\frac{2}{3})^8&93(\frac{2}{3})^6&-178(\frac{2}{3})^6&-63(\frac{2}{3})^5&-43(\frac{2}{3})^4\end{pmatrix}. \tag{3.22}$$

In the same way as in Case 2, because of $\operatorname{rank}R_2=5$, we have that $C_2=0$ if and only if $\widetilde{B}_2=0$, and that (3.8) can be rewritten as

$$F\big(\alpha(x)\big)-F(x)=\frac{Q_{33}(x)}{p_3(\alpha(x))p_3(x)}=\sum_{j=0}^{4}B_{2j+1}x^{2j+1}\big(1+P_{3j}(x)\big),$$

where $P_{3j}(0)=0$, which ensures that Eq. (3.7) has at most four limit cycles near the origin by Theorem 1.

Next we prove that four limit cycles can appear near the origin. By (3.20) and (3.21) we have

$$\frac{\partial(e_{31},e_{32},e_{33},e_{34},e_{35})}{\partial(a_0,a_1,a_2,a_3,b_1,b_2,b_3)}=\begin{pmatrix}-b_1&1&0&0&-a_0&0&0\\0&-b_2&b_1&0&a_2&-a_1&0\\-b_3+\frac{2}{3}b_2&0&-\frac{2}{3}&1&0&\frac{2}{3}a_0&-a_0\\0&-b_3&0&b_1&a_3&0&-a_1\\0&0&-b_3&b_2&0&a_3&-a_2\end{pmatrix}. \tag{3.23}$$

When $a_0=a_1=0$, $a_2=\frac{1}{2}$, $a_3=-\frac{2}{3}$, $b_1=-2$, $b_2=\frac{2}{3}$, $b_3=0$, by (3.22) and (3.23) we have

$$\det\frac{\partial(B_1,B_3,B_5,B_7)}{\partial(a_1,a_2,a_3,b_1)}=\det\frac{\partial(B_1,B_3,B_5,B_7)}{\partial(e_{31},e_{32},e_{33},e_{34})}\det\frac{\partial(e_{31},e_{32},e_{33},e_{34})}{\partial(a_1,a_2,a_3,b_1)}\neq 0,$$

and $B_1=B_3=B_5=B_7=0$, $B_9=2(\frac{2}{3})^6$, $e_{31}=0$, $e_{32}=-1$, $e_{33}=-1$, $e_{34}=\frac{4}{3}$, $e_{35}=-\frac{4}{9}$. It implies the existence of four limit cycles near the origin.

Case 4. $n=m=4$. Multiplying the equation $\alpha^2-x^2=-\frac{2}{3}(\alpha^3-x^3)$ by αx and $\alpha+x$, respectively, we get

$$\alpha^3x-\alpha x^3=-\frac{2}{3}\big(\alpha^4x-\alpha x^4\big),$$

$$\big(\alpha^3-x^3\big)+\big(\alpha^2x-\alpha x^2\big)=-\frac{2}{3}\big[\big(\alpha^4-x^4\big)+\big(\alpha^3x-\alpha x^3\big)\big],$$

and then by (3.9) we have

$$Q_{44}(x) = e_{41}(\alpha - x) + e_{42}\big(\alpha^2 x - \alpha x^2\big) + e_{43}\big(\alpha^3 - x^3\big) + e_{44}\big(\alpha^3 x^2 - \alpha^2 x^3\big)$$
$$+ e_{45}\big(\alpha^4 - x^4\big) + e_{46}\big(\alpha^4 x^2 - \alpha^2 x^4\big) + e_{47}\big(\alpha^4 x^3 - \alpha^3 x^4\big),$$

where

$$e_{41} = a_1 - a_0 b_1, \qquad e_{42} = (a_2 b_1 - a_1 b_2) + \frac{9}{4}(a_4 b_1 - a_1 b_4) - \frac{3}{2}(a_3 b_1 - a_1 b_3), \tag{3.24}$$

$$e_{43} = (a_3 - a_0 b_3) - \frac{2}{3}(a_2 - a_0 b_2) + \frac{9}{4}(a_4 b_1 - a_1 b_4) - \frac{3}{2}(a_3 b_1 - a_1 b_3), \tag{3.25}$$

$$e_{44} = a_3 b_2 - a_2 b_3, \qquad e_{45} = (a_4 - a_0 b_4) + \frac{3}{2}(a_4 b_1 - a_1 b_4) - (a_3 b_1 - a_1 b_3), \tag{3.26}$$

$$e_{46} = a_4 b_2 - a_2 b_4, \qquad e_{47} = a_4 b_3 - a_3 b_4. \tag{3.27}$$

Also as before, we have that $\widetilde{B}_3 = R_3 C_3$, where

$$\widetilde{B}_3 = \begin{pmatrix} B_1 \\ B_3 \\ B_5 \\ B_7 \\ B_9 \\ B_{11} \\ B_{13} \end{pmatrix}, \qquad C_3 = \begin{pmatrix} e_{41} \\ e_{42} \\ e_{43} \\ e_{44} \\ e_{45} \\ e_{46} \\ e_{47} \end{pmatrix},$$

$$R_3 = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(\frac{2}{3})^2 & 2 & -2 & 0 & 0 & 0 & 0 \\ -4(\frac{2}{3})^4 & 4(\frac{2}{3})^2 & -6(\frac{2}{3})^2 & -2 & \frac{8}{3} & 0 & 0 \\ -21(\frac{2}{3})^6 & 17(\frac{2}{3})^4 & -30(\frac{2}{3})^4 & -9(\frac{2}{3})^2 & 24(\frac{2}{3})^3 & \frac{4}{3} & 2 \\ -127(\frac{2}{3})^8 & 93(\frac{2}{3})^6 & -178(\frac{2}{3})^6 & -43(\frac{2}{3})^4 & 148(\frac{2}{3})^5 & 18(\frac{2}{3})^3 & 16(\frac{2}{3})^2 \\ -835(\frac{2}{3})^{10} & 577(\frac{2}{3})^8 & -1158(\frac{2}{3})^8 & -250(\frac{2}{3})^6 & 980(\frac{2}{3})^7 & 118(\frac{2}{3})^5 & 89(\frac{2}{3})^4 \\ -5798(\frac{2}{3})^{12} & 3858(\frac{2}{3})^{10} & -7986(\frac{2}{3})^{10} & -1608(\frac{2}{3})^8 & 6828(\frac{2}{3})^9 & 802(\frac{2}{3})^7 & 556(\frac{2}{3})^6 \end{pmatrix}.$$

Because of $\operatorname{rank} R_3 = 7$, we get that $C_3 = 0$ if and only if $\widetilde{B}_3 = 0$. Therefore,

$$F\big(\alpha(x)\big) - F(x) = \frac{Q_{44}(x)}{p_4(\alpha(x))p_4(x)} = \sum_{j=0}^{6} B_{2j+1} x^{2j+1}\big(1 + P_{4j}(x)\big),$$

where $P_{4j}(0) = 0$. It follows by Theorem 1 that Eq. (3.7) has at most six limit cycles near the origin.

Next we prove that six limit cycles can appear near the origin. From (3.24)–(3.27) we have

$$\frac{\partial(e_{41}, e_{42}, e_{43}, e_{44}, e_{45}, e_{46})}{\partial(a_0, a_1, a_2, a_4, b_1, b_2)}$$
$$= \begin{pmatrix} -b_1 & 1 & 0 & 0 & -a_0 & 0 \\ 0 & -b_2 - \frac{9}{4}b_4 + \frac{3}{2}b_3 & b_1 & \frac{9}{4}b_1 & a_2 + \frac{9}{4}a_4 - \frac{3}{2}a_3 & -a_1 \\ -b_3 + \frac{2}{3}b_2 & -\frac{9}{4}b_4 + \frac{3}{2}b_3 & -\frac{2}{3} & \frac{9}{4}b_1 & \frac{9}{4}a_4 - \frac{3}{2}a_3 & \frac{2}{3}a_0 \\ 0 & 0 & -b_3 & 0 & 0 & a_3 \\ -b_4 & -\frac{3}{2}b_4 + b_3 & 0 & 1 + \frac{3}{2}b_1 & \frac{3}{2}a_4 - a_3 & 0 \\ 0 & 0 & -b_4 & b_2 & 0 & a_4 \end{pmatrix}.$$

When $a_0 = a_1 = 0$, $a_2 = \frac{81}{52}$, $a_3 = -\frac{54}{13}$, $a_4 = \frac{33}{13}$, $b_1 = -\frac{10}{3}$, $b_2 = \frac{104}{33}$, $b_3 = -\frac{208}{297}$, $b_4 = 0$, we have

$$\begin{aligned}
&\det \frac{\partial(B_1, B_3, B_5, B_7, B_9, B_{11})}{\partial(a_0, a_1, a_2, a_4, b_1, b_2)} \\
&\quad = \det \frac{\partial(B_1, B_3, B_5, B_7, B_9, B_{11})}{\partial(e_{41}, e_{42}, e_{43}, e_{44}, e_{45}, e_{46})} \det \frac{\partial(e_{41}, e_{42}, e_{43}, e_{44}, e_{45}, e_{46})}{\partial(a_0, a_1, a_2, a_4, b_1, b_2)} \neq 0,
\end{aligned}$$

and $B_1 = B_3 = B_5 = B_7 = B_9 = B_{11} = 0$, $B_{13} = 18(\frac{2}{3})^{10}$, and $e_{41} = 0$, $e_{42} = e_{43} = -45$, $e_{44} = -12$, $e_{45} = -24$, $e_{46} = 8$, $e_{47} = -\frac{16}{9}$. Thus, as before, six limit cycles can appear near the origin.

The proof is completed. □

Proof of Theorem 6. Suppose $m \geqslant n$. Firstly, we prove that Eq. (1.11) has at most $[\frac{3m+2n-1}{3}]$ limit cycles near the origin.

For $0 < x \ll 1$, from (1.12) we have

$$F\big(\alpha(x)\big) - F(x) = \sum_{i=1}^{m} a_i^- \alpha^i(x) - \sum_{i=1}^{n} a_i^+ x^i = \sum_{i=1}^{m} \big(\tilde{a}_i I_i(x) + \bar{a}_i J_i(x)\big) = \sum_{i \geqslant 1} B_i x^i \equiv Q(x), \quad (3.28)$$

where

$$\tilde{a}_i = \begin{cases} \frac{a_i^- + a_i^+}{2}, & 1 \leqslant i \leqslant n, \\ \frac{a_i^-}{2}, & n+1 < i \leqslant m, \end{cases} \qquad \bar{a}_i = \begin{cases} \frac{a_i^- - a_i^+}{2}, & 1 \leqslant i \leqslant n, \\ \frac{a_i^-}{2}, & n+1 < i \leqslant m, \end{cases} \quad (3.29)$$

and further from Lemmas 2 and 6 we have

$$Q(x) = \sum_{i=1}^{m} \big(\tilde{a}_i \widetilde{I}_i(\theta) + \bar{a}_i \widetilde{J}_i(\theta)\big) = \sum_{i=1}^{m} \left(\tilde{a}_i \sum_{j \in S(i)} b_{i,j} \sin j\theta + \bar{a}_i \sum_{j=0}^{i} c_{i,j} \cos j\theta\right) \equiv \widetilde{Q}(\theta).$$

Noting that $S(j) = \bigcup_{i=1}^{j} \widetilde{S}_i$, where $\widetilde{S}_i = \{i\}$ for $i \neq 0 \pmod 3$ and $\widetilde{S}_i = \emptyset$ for $i = 0 \pmod 3$, one has

$$\begin{aligned}
\widetilde{Q}(\theta) &= \sum_{j \in S(m)} \sum_{i=j}^{m} \tilde{a}_i b_{i,j} \sin j\theta + \sum_{j=0}^{m} \sum_{i=j}^{m} \bar{a}_i c_{i,j} \cos j\theta \\
&= \sum_{j \in S(m)} \tilde{c}_j \sin j\theta + \sum_{j=0}^{m} \bar{c}_j \cos j\theta, \qquad (3.30)
\end{aligned}$$

where

$$\tilde{c}_j = \sum_{i=j}^{m} \tilde{a}_i b_{i,j}, \quad j \in S(m); \qquad \bar{c}_j = \sum_{i=j}^{m} \bar{a}_i c_{i,j}, \quad 0 \leqslant j \leqslant m, \quad (3.31)$$

since $S(i) \subset S(n)$ if $i \leqslant n$.

Since $\sum_{j=0}^{m}\bar{c}_j=0$ we have

$$\widetilde{Q}(\theta)=\sum_{j\in S(m)}\tilde{c}_j\sum_{i\geqslant 0}\frac{(-1)^i}{(2i+1)!}j^{2i+1}\theta^{2i+1}+\sum_{j=0}^{m}\bar{c}_j\left(1+\sum_{i\geqslant 1}\frac{(-1)^i}{(2i)!}j^{2i}\theta^{2i}\right)$$
$$=\sum_{i\geqslant 0}\frac{(-1)^i}{(2i+1)!}c_{2i+1}\theta^{2i+1}+\sum_{i\geqslant 1}\frac{(-1)^i}{(2i)!}c_{2i}\theta^{2i}\equiv\widetilde{Q}_1(\theta)+\widetilde{Q}_2(\theta),$$

by using the fact that

$$\sin j\theta=\sum_{i\geqslant 0}\frac{(-1)^i}{(2i+1)!}j^{2i+1}\theta^{2i+1},\qquad \cos j\theta=\sum_{i\geqslant 0}\frac{(-1)^i}{(2i)!}j^{2i}\theta^{2i},$$

where

$$c_{2i+1}=\sum_{j\in S(m)}j^{2i+1}\tilde{c}_j,\quad i\geqslant 0;\qquad c_{2i}=\sum_{j=1}^{m}j^{2i}\bar{c}_j,\quad i\geqslant 1. \tag{3.32}$$

Write $S(m)$ as $S(m)=\{k_i\mid 0\leqslant i\leqslant h,\ \text{and}\ k_i<k_j\ \text{for}\ i<j\}$, where $h=[\frac{2m-1}{3}]$. From (3.32) we get $C_1=R_1\widetilde{C}$ and $C_2=R_2\overline{C}$, where

$$C_1=\begin{pmatrix}c_1\\c_3\\\vdots\\c_{2h+1}\end{pmatrix},\qquad \widetilde{C}=\begin{pmatrix}\tilde{c}_{k_0}\\\tilde{c}_{k_1}\\\vdots\\\tilde{c}_{k_h}\end{pmatrix},\qquad R_1=\begin{pmatrix}k_0&k_1&\cdots&k_h\\k_0^3&k_1^3&\cdots&k_h^3\\\vdots&\vdots&\ddots&\vdots\\k_0^{2h+1}&k_1^{2h+1}&\cdots&k_h^{2h+1}\end{pmatrix}, \tag{3.33}$$

$$C_2=\begin{pmatrix}c_2\\c_4\\\vdots\\c_{2m}\end{pmatrix},\qquad \overline{C}=\begin{pmatrix}\bar{c}_1\\\bar{c}_2\\\vdots\\\bar{c}_m\end{pmatrix},\qquad R_2=\begin{pmatrix}1&2^2&\cdots&m^2\\1&2^4&\cdots&m^4\\\vdots&\vdots&\ddots&\vdots\\1&2^{2m}&\cdots&m^{2m}\end{pmatrix}. \tag{3.34}$$

Because of

$$\det R_1=\prod_{s=0}^{h}k_s\prod_{0\leqslant i<j\leqslant h}\left(k_j^2-k_i^2\right)\neq 0,$$

$C_1=0$ if and only if $\widetilde{C}=0$, and hence $C_1=0$ if and only if $\widetilde{Q}_1(\theta)=0$. In other words,

$$c_{2j+1}=O\left(|c_1,c_3,\ldots,c_{2h+1}|\right)\quad\text{for } j\geqslant h+1. \tag{3.35}$$

In the same way, we obtain that

$$c_{2j}=O\left(|c_2,c_4,\ldots,c_{2m}|\right)\quad\text{for } j\geqslant m+1. \tag{3.36}$$

From (3.29) and (3.31), there exist matrices $\widetilde{R}$ and $\bar{R}$ such that $\widetilde{C}^* = \widetilde{R}A$ and $\bar{C}^* = \bar{R}A$, where

$$\widetilde{C}^* = \begin{pmatrix} \tilde{c}_{k_{l+1}} \\ \tilde{c}_{k_{l+2}} \\ \vdots \\ \tilde{c}_{k_h} \end{pmatrix}, \qquad \bar{C}^* = \begin{pmatrix} \bar{c}_{n+1} \\ \bar{c}_{n+2} \\ \vdots \\ \bar{c}_m \end{pmatrix}, \qquad A = \begin{pmatrix} \bar{a}_{n+1} \\ \bar{a}_{n+2} \\ \vdots \\ \bar{a}_m \end{pmatrix}, \tag{3.37}$$

where $l = [\frac{2n-1}{3}]$.

From (3.33), (3.34) and (3.37) we know that there exits an $(m+h+1) \times (m+l+1)$ matrix S such that $\binom{c_1}{c_2} = SC^*$, where $C^* = (\tilde{c}_{k_0}, \ldots, \tilde{c}_{k_l}, \bar{c}_1, \ldots, \bar{c}_n, A^T)^T$. Write S in the form

$$S = \left(S_1^T, S_3^T, \ldots, S_{2h+1}^T, S_2^T, S_4^T, \ldots, S_{2m}^T\right)^T,$$

then S_i is the row vector of S and $c_i = S_i C^*$. Suppose $s = \operatorname{rank} S$. There exists a subset $\{S_{r_i} \mid 1 \leqslant i \leqslant s,$ and $r_i < r_j$ for $i < j\}$ of $\{S_1, S_3, \ldots, S_{2h+1}, S_2, S_4, \ldots, S_{2m}\}$ such that the vectors in the subset are linearly independent and for any other vector S_j we have $S_j = \sum_{r_i=r_1}^{\beta(j)} \alpha_{j,i} S_{r_i}$, where $\beta(j) = \max\{r_i \mid r_i < j,\ 1 \leqslant i \leqslant s\}$ and $\alpha_{j,i}$ are real numbers and not all zero. Then

$$c_j = S_j C^* = \sum_{r_i=r_1}^{\beta(j)} \alpha_{j,i} S_{r_i} C^* = \sum_{r_i=r_1}^{\beta(j)} \alpha_{j,i}\left(S_{r_i} C^*\right) = \sum_{r_i=r_1}^{\beta_(j)} \alpha_{j,i} c_{r_i},$$

in other words,

$$c_j = \mathrm{O}\left(|c_{r_1}, c_{r_2}, \ldots, c_{\beta(j)}|\right).$$

From (3.35) and (3.36) we obtain that

$$c_j = \mathrm{O}\left(|c_{r_1}, c_{r_2}, \ldots, c_{\beta(j)}|\right) \quad \text{for } j \neq r_i,\ 1 \leqslant i \leqslant s. \tag{3.38}$$

Therefore, $c_{r_i} = 0$, $1 \leqslant i \leqslant s$, if and only if $\widetilde{Q}(\theta) = 0$.

Noting $\theta = -\frac{2\sqrt{3}}{3}x + \mathrm{O}(x^2) = -\frac{2\sqrt{3}}{3}x(1 + \mathrm{O}(x))$, $Q(x)$ can be rewritten in the following form

$$Q(x) = \sum_{i \geqslant 1} \frac{(-1)^{[\frac{i}{2}]}}{i!} c_i \left(-\frac{2\sqrt{3}}{3} x\left(1 + \mathrm{O}(x)\right)\right)^i = \sum_{i \geqslant 1} \frac{(-2\sqrt{3})^i (-1)^{[\frac{i}{2}]}}{3^i i!} c_i x^i \left(1 + P_i(x)\right)$$

where $P_i(x) \in C^\infty$, $P_i(x) = \mathrm{O}(x)$. Comparing the like powers of x between the above equation and (3.28), we get

$$B_i = \frac{(-2\sqrt{3})^i (-1)^{[\frac{i}{2}]}}{3^i i!} c_i + \mathrm{O}\left(|c_1, c_2, \ldots, c_{i-1}|\right), \quad \text{for } i \geqslant 1.$$

Then from (3.38) we have

$$B_j = \mathrm{O}\left(|c_{r_1}, c_{r_2}, \ldots, c_{\beta(j)}|\right), \quad \text{for } j \neq r_i,\ 1 \leqslant i \leqslant s, \tag{3.39}$$

and

$$B_{r_i} = \frac{(-2\sqrt{3})^{r_i}(-1)^{[\frac{r_i}{2}]}}{3^{r_i} r_i!} c_{r_i} + \mathrm{O}\big(|c_{r_1}, c_{r_2}, \ldots, c_{r_{i-1}}|\big), \quad 1 \leqslant i \leqslant s,$$

and then

$$c_{r_i} = \frac{3^{r_i} r_i!}{(-2\sqrt{3})^{r_i}(-1)^{[\frac{i}{2}]}} B_{r_i} + \mathrm{O}\big(|B_{r_1}, B_{r_2}, \ldots, B_{r_{i-1}}|\big). \tag{3.40}$$

Formulas (3.39) and (3.40) give

$$B_j = \mathrm{O}\big(|B_{r_1}, B_{r_2}, \ldots, B_{\beta(j)}|\big), \quad \text{for } j \neq r_i,\ 1 \leqslant i \leqslant s. \tag{3.41}$$

The displacement function $d(r,a)$ of Eq. (1.11) has the form $d(r,a) = \sum_{i \geqslant 1} d_i(a) r^i$ for small r. From Theorem 2, in the same way as above, we get

$$d_j = \mathrm{O}\big(|d_{r_1}, d_{r_2}, \ldots, d_{\beta(j)}|\big), \quad \text{for } j \neq r_i,\ 1 \leqslant i \leqslant s.$$

Therefore, $d(r,a) = 0$ when $d_{r_i} = 0$, $1 \leqslant i \leqslant s$. Thus $d(r,a)$ can be rewritten in the following form

$$d(r,a) = \sum_{i=1}^{s} d_{r_i} r^{r_i} \widetilde{P}_{r_i}(r, a, d_{r_1}, d_{r_2}, \ldots, d_{r_s}),$$

where $\widetilde{P}_i \in C^\infty$, $\widetilde{P}_i(r, a, d_{r_1}, d_{r_2}, \ldots, d_{r_s}) = 1 + \mathrm{O}(r)$. Then following the proof idea of Theorem 1.3 of Han [8], we can prove that the function $d(r,a)$ has at most $s-1$ positive zeros in r near $r = 0$.

Next we prove $s = m + l + 1$. Obviously, $s \leqslant m + l + 1$. It is easy to get $(c_{r_1}, c_{r_2}, \ldots, c_{r_s})^T = S^* C^*$, where S^* is an $s \times (m+l+1)$ matrix and $S^* = (S_{r_1}^T, S_{r_2}^T, \ldots, S_{r_s}^T)^T$. Let $S_i^- = S(i) - S(n)$ for $i > n$. From (3.30), $\widetilde{Q}(\theta)$ can be rewritten in the form

$$\begin{aligned}
\widetilde{Q}(\theta) &= \sum_{j \in S(m)} \sum_{i=j}^{m} \tilde{a}_i b_{i,j} \sin j\theta + \sum_{j=0}^{m} \sum_{i=j}^{m} \bar{a}_i c_{i,j} \cos j\theta \\
&= \sum_{j \in S(n)} \tilde{c}_j \sin j\theta + \sum_{j=0}^{n} \bar{c}_j \cos j\theta + \sum_{j \in S_m^-} \sum_{i=j}^{m} \frac{a_i^-}{2} b_{i,j} \sin j\theta + \sum_{j=n+1}^{m} \sum_{i=j}^{m} \frac{a_i^-}{2} c_{i,j} \cos j\theta \\
&= \sum_{j \in S(n)} \tilde{c}_j \sin j\theta + \sum_{j=0}^{n} \bar{c}_j \cos j\theta + \sum_{i=n+1}^{m} a_i^- \left(\sum_{j \in S_i^-} \frac{b_{i,j}}{2} \sin j\theta + \sum_{j=n+1}^{i} \frac{c_{i,j}}{2} \cos j\theta \right) \\
&= \sum_{j \in S(n)} \tilde{c}_j \sin j\theta + \sum_{j=0}^{n} \bar{c}_j \cos j\theta + \sum_{i=n+1}^{m} a_i^- H_i(\theta),
\end{aligned}$$

where

$$H_i(\theta) = \sum_{j \in S_i^-} \frac{b_{i,j}}{2} \sin j\theta + \sum_{j=n+1}^{i} \frac{c_{i,j}}{2} \cos j\theta.$$

From Lemmas 2 and 5, we get $|b_{i,i}| + |c_{i,i}| = 2^{-i+2}(|\sin \frac{i\pi}{3}| + |\cos \frac{i\pi}{3}|) \neq 0$, and then $\sin j\theta$, $j \in S(n)$, $\cos j\theta$, $1 \leqslant j \leqslant n$, and $H_i(\theta)$, $n+1 \leqslant i \leqslant m$, are linearly independent. Therefore, $C^* = 0$ if and only if

$\widetilde{Q}(\theta) = 0$. Besides, we also know that $c_{r_i} = 0$, $1 \leqslant i \leqslant s$, if and only if $\widetilde{Q}(\theta) = 0$. Therefore, $C^* = 0$ if and only if $c_{r_i} = 0$, $1 \leqslant i \leqslant s$, implying $s = m + l + 1$.

Finally, we prove that $l + m$ limit cycles can appear near the origin. For simplicity, take $a_{3j}^- = -a_{3j}^+$, $1 \leqslant j \leqslant [\frac{n}{3}]$, which gives $\tilde{a}_{3j} = 0$ by (3.29). From (3.28), we get

$$Q(x) = \sum_{i \in S(n)} \tilde{a}_i I_i(x) + \sum_{i=1}^{n} \bar{a}_i J_i(x) + \sum_{i=n+1}^{m} \bar{a}_i \big(I_i(x) + J_i(x)\big) = \sum_{i \geqslant 1} B_i x^i,$$

from which we get

$$(B_{r_1}, B_{r_2}, \ldots, B_{r_{l+m}})^T = R(\tilde{a}_{k_0}, \tilde{a}_{k_1}, \ldots, \tilde{a}_{k_l}, \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m)^T,$$

where R is a constant matrix of order $l + m + 1$. On the one hand, from Lemma 6, we know that $I_{k_i}(x)$, $k_i \in S(n)$, $J_i(x)$, $1 \leqslant i \leqslant n$, and $I_i(x) + J_i(x)$, $n + 1 \leqslant i \leqslant m$, are linearly independent, and hence $\tilde{a}_{k_i} = 0$, $0 \leqslant i \leqslant l$, and $\bar{a}_i = 0$, $1 \leqslant i \leqslant m$, if and only if $Q(x) = 0$. On the other hand, from (3.41) we get that $B_{r_i} = 0$, $1 \leqslant i \leqslant l + m + 1$, if and only if $Q(x) = 0$. Therefore, we have $\det R \neq 0$. From (3.29), it is easy to see

$$(\tilde{a}_{k_0}, \tilde{a}_{k_1}, \ldots, \tilde{a}_{k_l}, \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m)^T = D\big(a_{k_0}^+, a_{k_1}^+, \ldots, a_{k_l}^+, a_1^-, a_2^-, \ldots, a_m^-\big)^T,$$

and $\det D \neq 0$. Then following the proof idea of Theorem 1.3 of Han [8], we obtain that (1.11) has $l + m$ limit cycles near the origin. Then the conclusion follows by Theorem 1. The proof is completed. □

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