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Bifurcation of ten small-amplitude limit cycles by perturbing a quadratic Hamiltonian system with cubic polynomials ☆

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Abstract

This paper contains two parts. In the first part, we shall study the Abelian integrals for Żołądek's example [13], in which the existence of 11 small-amplitude limit cycles around a singular point in a particular cubic vector field is claimed. We will show that the bases chosen in the proof of [13] are not independent, which leads to failure in drawing the conclusion of the existence of 11 limit cycles in this example. In the second part, we present a good combination of Melnikov function method and focus value (or normal form) computation method to study bifurcation of limit cycles. An example by perturbing a quadratic Hamiltonian system with cubic polynomials is presented to demonstrate the advantages of both methods, and 10 small-amplitude limit cycles bifurcating from a center are obtained by using up to 5th-order Melnikov functions.

MSC: 34C07; 34C23

Keywords: Bifurcation of limit cycles; Żołądek's example; Higher-order Melnikov function; Hamiltonian system; Focus value

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1. Introduction

The well-known Hilbert's 16th problem [1] has been studied for more than one century, and the research on this problem is still very active today. To be more specific, consider the following planar system:

$$\dot{x} = P_n(x, y), \qquad \dot{y} = Q_n(x, y), \tag{1}$$

where $P_n(x, y)$ and $Q_n(x, y)$ represent nth-degree polynomials in x and y. The second part of Hilbert's 16th problem is to find the upper bound, called Hilbert number $H(n)$, on the number of limit cycles that system (1) can have.

The progress in the solution of the problem is very slow. Even the simplest case $n = 2$ has not been completely solved, though in the early 1990's, Ilyashenko [2] and Écalle [3] independently proved that the number of limit cycles is finite for any given planar polynomial vector field. For general quadratic polynomial systems, the best result is $H(2) \geq 4$, obtained more than 30 years ago [4,5]. Recently, this result was also obtained for near-integrable quadratic systems [6]. However, whether $H(2) = 4$ is still open. For cubic polynomial systems, many results have been obtained on the lower bound of the Hilbert number. So far, the best result for cubic systems is $H(3) \geq 13$ [7,8]. Note that the 13 limit cycles obtained in [7,8] are distributed around several singular points. A comprehensive review on the study of Hilbert's 16th problem can be found in a survey article [9].

In order to help understand and attack Hilbert's 16th problem, the so-called weakened Hilbert's 16th problem was posed by Arnold [10]. The problem is to ask for the maximal number of isolated zeros of the Abelian integral or Melnikov function:

$$M(h, \delta) = \oint_{H(x,y)=h} Q(x, y)\, dx - P(x, y)\, dy, \tag{2}$$

where $H(x, y)$, $P(x, y)$ and $Q(x, y)$ are all real polynomials in x and y, and the level curves $H(x, y) = h$ represent at least a family of closed orbits for $h \in (h_1, h_2)$, and δ denotes the parameters (or coefficients) involved in P and Q. The weakened Hilbert's 16th problem itself is a very important and interesting problem, closely related to the study of limit cycles in the following near-Hamiltonian system [11]:

$$\dot{x} = H_y(x, y) + \varepsilon\, P(x,\ y), \qquad \dot{y} = -H_x(x, y) + \varepsilon\, Q(x,\ y), \tag{3}$$

where $0 < \varepsilon \ll 1$. Studying the bifurcation of limit cycles for such a system can be now transformed to investigating the zeros of the Melnikov function $M(h, \delta)$, given in (2).

When we focus on the maximum number of small-amplitude limit cycles, denoted by $M(n)$, bifurcating from an elementary center or an elementary focus, one of the best-known results is $M(2) = 3$, which was proved by Bautin in 1952 [12]. For $n = 3$, several results have been obtained (e.g. see [13–15]). Among them, in 1995 Żołądek [13] first constructed a rational Darboux integral to study existence of 11 small-amplitude limit cycles in cubic vector fields. This pioneer work later motivated many researches in this area to study bifurcation of limit cycles. The rational Darboux integral in [13] is given by

$$H_0 = \frac{f_1^5}{f_2^4} = \frac{(x^4 + 4x^2 + 4y)^5}{(x^5 + 5x^3 + 5xy + 5x/2 + a)^4}, \tag{4}$$

which generates the dynamical system in the form of

$$\begin{aligned}\dot{x} &= x^3 + xy + 5x/2 + a, \\ \dot{y} &= -ax^3 + 6x^2y - 3x^2 + 4y^2 + 2y - 2ax,\end{aligned} \tag{5}$$

with the integrating factor $M = 20 f_1^4 f_2^{-5}$.

It can be shown that for $a < -2^{5/4}$, system (5) has a center $C_0 = (-a/2, -a^2/4 - 1/2)$ and five (real or complex) critical points $(r, -r^2 - 5/2 - a/r)$, where r satisfies the polynomial equation $r^5 - 10r - 4a = 0$. In addition, system (5) has a saddle point and a non-elementary critical point at infinity. Let $h_0 = H_0(C_0) = -2/a$. Around C_0, there exists a family of periodic orbits given by $\{\gamma_h : H_0(x, y) = h,\ 0 < h - h_0 \ll 1\}$. γ_h approaches C_0 as $h \to h_0^+$.

Recently, Yu and Han [14] applied a different method to study system (5) and only got 9 small-amplitude limit cycles around the center C_0. This difference motivated us to reconsider system (5) carefully and finally find some not-easy-to-find mistakes in the proof of [13], which lead to failure in drawing the conclusion of existence of 11 limit cycles. In the next section, we shall present a detailed analysis on the Abelian integrals of system (5) and point out where the mistakes were made in [13]. However, we must emphasize that although some flaws were found in [13], the idea and methodology presented in this paper are still very valuable and useful. In fact, our work was motivated by this paper as well as that [17].

In the second part, we will present a good combination of higher-order Melnikov function method and focus value computation method to study the number of small-amplitude limit cycles. As a matter of fact, in proving existence of small-amplitude limit cycles, higher-order Melnikov functions are equivalent to higher ε-order focus values. These two methods have their advantages and disadvantages: Melnikov function method can be used to show the vanishing of a particular order Melnikov function, which is equivalent to show the vanishing of all this particular ε-order focus values, and this cannot be proved by using focus value computation since one cannot verify an infinite number of focus values. On the other hand, for higher-order Melnikov functions, it becomes extremely difficult to prove the independence of a choosing set of basis, while this is straightforward by using focus value computation. In particular, we present an example to demonstrate this good combination method and obtain 10 small-amplitude limit cycles by perturbing a quadratic Hamiltonian system with 3rd-degree polynomial functions.

In general, a perturbed quadratic Hamiltonian system can be described by

$$\begin{aligned}\dot{x} &= y + a_1xy + a_2y^2 + \varepsilon P(x, y, \varepsilon), \\ \dot{y} &= -x + x^2 - \frac{1}{2}a_1y^2 + \varepsilon Q(x, y, \varepsilon),\end{aligned} \tag{6}$$

where P and Q are nth-degree polynomials in x and y with coefficients depending analytically on the small parameter ε. When $\varepsilon = 0$, system (6) has a cubic Hamiltonian,

$$H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + \frac{1}{2}a_1xy^2 + \frac{1}{3}a_2y^3, \tag{7}$$

and its parameters a_1 and a_2 take values from the set,

$$\Omega = \left\{(a_1, a_2) \in \mathbb{R}^2 : -1 \le a_1 \le 2,\ 0 \le a_2 \le (1 - \frac{a_1}{2})\sqrt{1+a_1}\right\}.$$

The Hamiltonian given in (7) is actually the so-called normal form [16] for all quadratic Hamiltonian systems which have a center at the origin. There exists a family of closed ovals around the origin given by $\{\Gamma_h : H(x, y) = h, h \in (0, \frac{1}{6})\}$.

For any $h \in (0, \frac{1}{6})$ the displacement function $d(h, \epsilon)$ of system (6) has a representation

$$d(h, \varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \varepsilon^3 M_3(h) + \cdots, \tag{8}$$

where $M_i(h)$ is called the ith-order Melnikov function, particularly the higher-order Melnikov functions if $i \ge 2$. Then, we may determine the number of the limit cycles of system (6) emerging from the closed ovals $\{\Gamma_h\}$ by studying the zeros of the first non-vanishing Melnikov function $M_i(h)$ in $h \in (0, \frac{1}{6})$.

Suppose $M_1(h) \not\equiv 0$ in (8). Denote $Z(n)$ the sharp upper bound of the number of zeros of $M_1(h)$ for system (6), where $n = \max(\deg(P), \deg(Q))$. Gavrilov [18] proved $Z(2) = 2$ for the Hamiltonian H with four distinct critical values (in a complex domain). Horozov and Iliev [19] gave a linear estimate $Z(n) \le 5(n + 3)$. Also, some sharp upper bounds were obtained for certain particular cubic Hamiltonians, for example: $n - 1$ for the Bogdanov–Takens Hamiltonian, $H = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3$ (see [20]), and $[\frac{2}{3}(n-1)]$ (where the notation $[\,\cdot\,]$ denotes the maximal integer of the variable) for the Hamiltonian triangle, $H = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + xy^2$ (see [21]).

Moreover, for the Bogdanov–Takens Hamiltonian, there are some results on the upper bound of the number of zeros of the first nonvanishing higher-order Melnikov function $M_k(h)$. Li and Zhang [22] got a sharp upper bound for $k = 2$: $2n - 2$ when n is even and $2n - 3$ when n is odd. Iliev [17] obtained a sharp upper bound $3n - 4$ for $k = 3$, by applying the Françoise's procedure [23] for computing higher-order Melnikov functions. The higher-order Melnikov functions can be also easily expressed via iterated integrals, which will be seen in the next section.

In this paper, we study the number of small-amplitude limit cycles in (6) bifurcating from the origin, using higher-order Melnikov functions. Hereafter we suppose P and Q are cubic polynomials in the form of

$$\begin{aligned} P(x, y, \varepsilon) &= \sum_{m=1}^{\infty} \varepsilon^{m-1} P_m(x, y) \quad \text{with } P_m(x, y) = \sum_{i+j=1}^{3} a_{ijm} x^i y^j, \\ Q(x, y, \varepsilon) &= \sum_{m=1}^{\infty} \varepsilon^{m-1} Q_m(x, y) \quad \text{with } Q_m(x, y) = \sum_{i+j=1}^{3} b_{ijm} x^i y^j. \end{aligned} \tag{9}$$

Our main result is given below, and its proof will be given in Section 4.

Theorem 1. *Let the polynomials P and Q in (6) be given by (9). Then for any $1 \le k \le 5$, there exist real values for $(a_1, a_2) \in \Omega$ such that system (6) can have $\lfloor \frac{4k}{3} \rfloor + 4$ small-amplitude limit cycles around the origin under proper cubic perturbations, when $M_k(h)$ is the first non-vanishing Melnikov function in (8).*

Remark 1. It follows from Theorem 1 that 10 small-amplitude limit cycles exist in the vicinity of the origin of system (6) when $k = 5$.

The rest of the paper is organized as follows. In the next section, we consider the example given in [13], and show that the bases chosen in the proof are not independent, leading to failure in drawing the conclusion of the existence of 11 limit cycles. In Section 3, we present some results for polynomial one-forms with respect to the Hamiltonian (7), which are needed for the proof of Theorem 1 in Section 4. Then, in Section 4 by choosing special forms for the polynomials P and Q without loss of generality, we prove Theorem 1. Finally, conclusion is drawn in Section 5.

2. Abelian integrals of system (5)

In this section, we consider system (5) and briefly describe the methodology used in [13]. Suppose the perturbed system of (5) is described by

$$\begin{aligned} \dot{x} &= M^{-1}H_{0y} + \varepsilon p(x, y, \varepsilon), \\ \dot{y} &= -M^{-1}H_{0x} + \varepsilon q(x, y, \varepsilon), \end{aligned} \tag{10}$$

where $p(x, y, \varepsilon)$ and $q(x, y, \varepsilon)$ are polynomials in x and y with coefficients depending analytically on the small parameter ε, and $\max(\deg(p), \deg(q)) \le 3$. Note that the non-perturbed system (10) (i.e. $\varepsilon = 0$) has a center at C_0.

Let S be a section transversal to the closed orbit $\{\gamma_h : H_0(x, y) = h,\ 0 < h - h_0 \ll 1\}$, with h as a parameter, we define the Poincaré map $\mathcal{P}(h, \varepsilon)$ of system (10), and thus the corresponding displacement function, $d(h, \varepsilon) = \mathcal{P}(h, \varepsilon) - h$, has the form

$$d(h, \varepsilon) = \varepsilon \int\limits_{L(h,\varepsilon)} M(q dx - p dy) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + O(\varepsilon^3), \tag{11}$$

where $L(h, \varepsilon)$ is a trajectory of the perturbed system (10). We can use the first non-vanishing Melnikov function $M_k(h)$ in (11) to investigate the number of the limit cycles around the center C_0. Generally, the zeros of $M_k(h)$ correspond to the limit cycles of system (10) around C_0.

Let $\varpi = q dx - p dy$, $\deg(\varpi) = \max(\deg(p), \deg(q))$. Then, the first-order Melnikov function $M_1(h)$ can be expressed in the form of

$$M_1(h) = \oint\limits_{\gamma_h} M\varpi|_{\varepsilon=0} = h \oint\limits_{\gamma_h} \frac{\varpi}{f_1 f_2}\bigg|_{\varepsilon=0}.$$

When $M_1(h) \equiv 0$, we may use an iterated integral to express the second-order Melnikov function $M_2(h)$. The first integral of system (10) can be approximated as $H_\varepsilon = H_0 - \varepsilon H_1$, where the function H_1 is defined by $H_1(B) = \int_A^B M\varpi|_{\varepsilon=0}$, evaluated along the orbit γ_h, with $A = \gamma_h \bigcap S$ and $B \in \gamma_h$. Thus, for system (10) we have the second-order Melnikov function, given by

$$M_2(h) = \frac{d}{d\varepsilon}\Big(\int\limits_{H_\varepsilon = h} M\varpi \Big)\bigg|_{\varepsilon=0}. \tag{12}$$

Suppose that the polynomials p and q are expanded as

$$p(x, y, \varepsilon) = p_1(x, y) + \varepsilon p_2(x, y) + O(\varepsilon^2),$$
$$q(x, y, \varepsilon) = q_1(x, y) + \varepsilon q_2(x, y) + O(\varepsilon^2).$$

Further, let $\varpi_i = q_i dx - p_i dy$, $i = 1, 2$. Then (12) can be rewritten as

$$M_2(h) = \frac{d}{d\varepsilon}\Big(\int_{H_\varepsilon = h} M\varpi_1\Big)\Big|_{\varepsilon=0} + \oint_{\gamma_h} M\varpi_2 = \oint_{\gamma_h} \frac{d(M\varpi_1)}{dH_0} H_1 + h\oint_{\gamma_h} \frac{\varpi_2}{f_1 f_2}, \tag{13}$$

where $\frac{d(M\varpi_1)}{dH_0}$ is the Gelfand–Leray form (see [26]).

In [13], the second-order Melnikov function $M_2(h)$ was used to study the small-amplitude limit cycles of system (10) bifurcating from the center C_0. More precisely, twelve Abelian integrals $I_{\omega_i}(h) = \oint_{\gamma_h} \omega_i/(f_1 f_2)$, $i = 1, \ldots, 12$, were chosen for (13), where one-forms ω_i are given as follows:

$$\begin{aligned}
&\omega_k = x^{k-1}dx,\ k = 1, 2, 3, 4, \quad \omega_5 = (18x^2 + 18y + 5)dx, \quad \omega_6 = xydx,\\
&\omega_7 = x^2ydx, \quad \omega_8 = xy^2dx, \quad \omega_9 = y^3dx, \quad \omega_{10} = xy^2dy, \quad \omega_{11} = y^3dy,\\
&\omega_{12} = y^2(5 - 3x^2)dx + xy(x^2 + 1)dy.
\end{aligned}$$

Then, by showing the independence of the integrals $I_{\omega_i}(h)$, $1 \le i \le 12$, it is claimed in [13] that 11 small-amplitude limit cycles can bifurcate from the center C_0 after suitable cubic perturbations.

Later, system (10) was re-investigated by using the method of focus values computation [14]. Based on the computation of ε-order and ε^2-order focus values, the authors of [14] showed that system (10) could have 9 small-amplitude limit cycles bifurcating from the center C_0. This obvious difference motivated us to study system (5), and finally to find that any vector of the linear space of integrals $I_\omega(h) = \oint_{\gamma_h} \omega/(f_1 f_2)$, $\deg(\omega) \le 3$, can be expressed as a linear combination of the ten integrals $I_{\omega_i}(h)$, $1 \le i \le 11$, $i \ne 4$. In the following, we show the details.

Nine one-forms η_j, $1 \le j \le 9$, were obtained in [13] satisfying $I_{\eta_j}(h) = 0$, where

$$\begin{aligned}
\eta_1 &= (x^3 + 2x)dx + dy,\\
\eta_2 &= (-3ax^2 + 12xy - 6x - 2a)dx - (3x^2 + y + \tfrac{5}{2})dy,\\
\eta_3 &= (6x^2 + 8y + 2)dx - xdy,\\
\eta_4 &= (-3ax^3 + 12x^2y - 6x^2 - 2ax)dx - (2x^3 - a)dy,\\
\eta_5 &= (ax^3 + 3x^2 + 4y^2 + 2ax)dx - xydy,\\
\eta_6 &= (-ax^3 + 6x^2y - 3x^2 + 4y^2 + 2y - 2ax)dx - (x^3 + xy + \tfrac{5}{2}x + a)dy,\\
\eta_7 &= (3ax^2y - 12xy^2 + 6xy + 2ay)dx - (3x^2y - ax^3 - 3x^2 + 3y^2 - \tfrac{1}{2}y - 2ax)dy,\\
\eta_8 &= (-5x^3 - 7xy + \tfrac{1}{2}x + a)dx + x^2dy,\\
\eta_9 &= (\tfrac{21}{2}xy - 7xy^2 + ay)dx + (2x^2y - \tfrac{3}{2}x^2 + ax + y)dy.
\end{aligned}$$

We find another one-form η_{10}, given by

$$\eta_{10} = \big[-\tfrac{29}{3}ax^3 - \tfrac{8}{3}y^3 - (2a^2 - \tfrac{5}{2})x^2 - 9axy + 6y^2 + \tfrac{13}{6}ax + a^2\big]dx + xy^2dy,$$

which can be shown to satisfy $I_{\eta_{10}}(h) = 0$. To achieve this, consider the Darboux integral, $H_\varepsilon = \frac{(f_1+\varepsilon g_1)^5}{(f_2+\varepsilon g_2)^4}$, with

$$g_1 = \tfrac{2}{3}x^4 + \tfrac{8}{3}ax^3 + \tfrac{4}{3}x^2y + \tfrac{2}{3}x^2 - \tfrac{4}{3}y^2 - 4ax + 4y,$$
$$g_2 = \tfrac{10}{3}ax^4 + \tfrac{5}{3}yx^3 - \tfrac{5}{2}x^3 - \tfrac{5}{3}xy^2 - \tfrac{5}{3}ax^2 + \tfrac{10}{3}xy - \tfrac{5}{2}x + \tfrac{10}{3}ay - a,$$

which yields the following system,

$$\begin{aligned}
\dot{x} = {} & M^{-1}H_{0y} - \varepsilon xy^2 + \varepsilon^2[-\tfrac{2}{9}x^7 + \tfrac{2}{9}ax^6 + \tfrac{5}{9}x^5y - \tfrac{3}{2}x^5 - \tfrac{4}{9}ax^4y - \tfrac{1}{3}x^3y^2 \\
& + \tfrac{17}{9}ax^4 + \tfrac{8}{3}x^3y + \tfrac{2}{9}xy^3 - (\tfrac{34}{9} + \tfrac{16}{9}a^2)x^3 - \tfrac{4}{3}ax^2y - \tfrac{1}{3}xy^2 + \tfrac{2}{9}ax^2 \\
& + \tfrac{7}{3}xy - \tfrac{4}{3}ay^2 - (\tfrac{5}{2} - \tfrac{8}{3}a^2)x + \tfrac{4}{3}ay - a], \\
\dot{y} = {} & - M^{-1}H_{0x} + \varepsilon[-\tfrac{29}{3}ax^3 - \tfrac{8}{3}y^3 - (2a^2 - \tfrac{5}{2})x^2 - 9axy + 6y^2 + \tfrac{13}{6}ax + a^2] \\
& + \varepsilon^2[-\tfrac{4}{9}ax^7 - \tfrac{4}{9}x^6y + (\tfrac{4}{9}a^2 + \tfrac{2}{3})x^6 + \tfrac{2}{3}ax^5y + \tfrac{10}{9}x^4y^2 + \tfrac{7}{3}ax^5 - 2x^4y \\
& - \tfrac{10}{9}ax^3y^2 - \tfrac{2}{3}x^2y^3 + (\tfrac{7}{6} - \tfrac{52}{9}a^2)x^4 + \tfrac{13}{9}ax^3y + 5x^2y^2 + \tfrac{4}{9}y^4 + \tfrac{143}{18}ax^3 \\
& - \tfrac{17}{3}x^2y - \tfrac{20}{3}a^2x^2y - \tfrac{5}{3}axy^2 - \tfrac{20}{9}y^3 + (\tfrac{1}{2} + 3a^2)x^2 - \tfrac{22}{9}axy + \tfrac{10}{3}y^2 \\
& - \tfrac{1}{6}ax - (2 - \tfrac{10}{3}a^2)y - a^2].
\end{aligned}$$

The above system has a center near C_0 when $a < -2^{5/4}$ and $|\varepsilon| \ll 1$. Thus, all the Melnikov functions of the above system vanish, and $M_1(h) = hI_{\eta_{10}}(h) \equiv 0$, implying that $I_{\eta_{10}}(h) = 0$.

Next, a direct calculation using η_j, $1 \le j \le 9$, yields

$$\begin{aligned}
& \tfrac{1}{2}(a\eta_1 - \eta_4) = (2ax^3 - 6x^2y + 3x^2 + 2ax)dx + x^3dy \triangleq \bar{\eta}_4, \\
& \tfrac{1}{2}(5\eta_1 + 2\eta_2 + 6\eta_8) = (-\tfrac{25}{2}x^3 - 3ax^2 - 9xy + \tfrac{1}{2}x + a)dx - ydy \triangleq \bar{\eta}_2, \\
& \tfrac{1}{4}(5\eta_1 + 2\eta_2 + 2a\eta_3 + 9\eta_8 + 2\eta_9) \\
& \quad = (-10x^3 - \tfrac{7}{2}xy^2 + \tfrac{3}{2}ax^2 - \tfrac{9}{2}xy + \tfrac{5}{8}x + \tfrac{9}{2}ay + \tfrac{9}{4}a)dx + x^2ydy \triangleq \bar{\eta}_9, \\
& \tfrac{1}{12}[2(a^2 - 10)\eta_1 - 8\eta_2 - 14a\eta_3 - 2a\eta_4 - 4\eta_7 - 21\eta_8 - 6\eta_9] = [(\tfrac{85}{12} + \tfrac{2}{3}a^2)x^3 \\
& -3ax^2y + \tfrac{15}{2}xy^2 - 4ax^2 - 3xy + (\tfrac{2}{3}a^2 - \tfrac{5}{24})x - \tfrac{21}{2}ay - \tfrac{11}{4}a]dx + y^2dy \triangleq \bar{\eta}_7, \\
& \tfrac{1}{2}(3a\eta_1 - 5\eta_3 - \eta_4) - \eta_5 + \eta_6 = (ax^3 - 18x^2 - 18y - 5)dx = a\omega_4 - \omega_5 \triangleq \bar{\eta}_6.
\end{aligned}$$

Now, suppose we have $\frac{\omega}{\partial x} = f$ and $\frac{\omega}{\partial y} = g$ for any one-form $\omega = fdx + gdy$. By noticing that

$$\begin{aligned}
& \frac{\eta_1}{\partial y} = 1, \quad \frac{\bar{\eta}_2}{\partial y} = -y, \quad \frac{\eta_3}{\partial y} = -x, \quad \frac{\bar{\eta}_4}{\partial y} = x^3, \quad \frac{\eta_5}{\partial y} = -xy, \quad \frac{\bar{\eta}_7}{\partial y} = y^2, \\
& \frac{\eta_8}{\partial y} = x^2, \quad \frac{\bar{\eta}_9}{\partial y} = x^2y, \quad \frac{\eta_{10}}{\partial y} = xy^2, \quad \frac{\bar{\eta}_6}{\partial y} = 0, \quad \frac{\bar{\eta}_6}{\partial x} \neq 0,
\end{aligned}$$

we can see that η_1, $\bar{\eta}_2$, η_3, $\bar{\eta}_4$, η_5, $\bar{\eta}_6$. $\bar{\eta}_7$, η_8, $\bar{\eta}_9$ and η_{10} are linearly independent. Thus, η_{10} does not lie in the span of η_j, $1 \le j \le 9$, and so it follows from $I_{\eta_j}(h) = 0$, $j = 1, \ldots, 10$, that the dimension of the linear space of integrals $I_\omega(h)$, $\deg(\omega) \le 3$ is at most 10. Therefore, the independence of integrals $I_{\omega_i}(h)$, $1 \le i \le 11$, proved in [13] does not hold, and there are at most 10 independent integrals $I_\omega(h)$ with $\deg(\omega) \le 3$. The basis can be chosen as $I_{\omega_j}(h)$, $1 \le j \le 11$, $j \ne 5$, since $I_{\bar\eta_6}(h) = \alpha I_{\omega_4}(h) - I_{\omega_5}(h) = 0$, and thus we can remove $I_{\omega_5}(h)$ from the basis given in [13].

Regarding ω_{12}, the authors have obtained a one-form $\bar\omega$, $\deg(\bar\omega) = 3$, based on focus value computation such that the corresponding focus values for $\widetilde{\omega}_{12} = \omega_{12} + \bar\omega$ vanish up to a sufficiently high order. This, together with the above result that $I_{\omega_5}(h)$ can be removed from the basis, implies that using ω_j, $1 \le j \le 12$, can only yield 9 limit cycles.

3. Cubic Hamiltonian with cubic perturbations

In order to prove Theorem 1, we need some preliminary results for cubic Hamiltonian given in (7) with cubic perturbations. Using the idea and methodology of Żołądek [13] and [17], we have the following results summarized in Lemmas 2–5.

Let $\omega_{ij} = x^i y^j dx$ and $\sigma_{ij} = x^i y^j dy$.

Lemma 2. *For the cubic Hamiltonian given in (7), the following identities hold.*

$$(a)\ \sigma_{ij} = \tfrac{1}{j+1} d(x^i y^{j+1}) - \tfrac{i}{j+1}\omega_{i-1,j+1};$$

$$\begin{aligned}(b)\ \omega_{ij} = {} & \omega_{i-1,j} + \tfrac{j-2i+4}{2j+4} a_1\omega_{i-2,j+2} - \tfrac{i-2}{j+2}\omega_{i-3,j+2} - \tfrac{i-2}{j+3} a_2\omega_{i-3,j+3} \\ & - x^{i-2}y^j dH + d\big(\tfrac{1}{j+2}x^{i-2}y^{j+2} + \tfrac{a_1}{j+2}x^{i-1}y^{j+2} + \tfrac{a_2}{j+3}x^{i-2}y^{j+3}\big),\ i \ge 2;\end{aligned}$$

$$\begin{aligned}(c)\ \omega_{0,j} = {} & \tfrac{3j}{a_2(j+1)}\big[H\omega_{0,j-3} - \tfrac{1}{6}\omega_{1,j-3} - \tfrac{a_1(j-3)+6j-2}{12(j-1)}\omega_{0,j-1} - \tfrac{a_1(j+1)}{3(j-1)}\omega_{1,j-1} \\ & + r_{0,j}(x,y)dH + dR_{0,j}(x,y)\big],\ j \ge 3;\end{aligned}$$

$$\begin{aligned}(d)\ \omega_{1,j} = {} & \tfrac{3j}{a_2(j+2)}\big[H\omega_{1,j-3} - \tfrac{(j+2)a_1^2}{6(j+1)}\omega_{0,j+1} + \tfrac{a_2}{6j}\omega_{0,j} - \tfrac{a_1(5j+3)+6j+2}{12(j-1)}\omega_{1,j-1} \\ & - \tfrac{a_1 j - 3a_1 - 2}{12(j-1)}\omega_{0,j-1} - \tfrac{1}{6}\omega_{1,j-3} + r_{1,j}(x,y)dH + dR_{1,j}(x,y)\big],\ j \ge 3;\end{aligned}$$

where $r_{i,j}(x,y)$ and $R_{i,j}(x,y)$ are polynomials in x and y with degrees $i+j-2$ and $i+j+1$, respectively.

Proof. A direct calculation using integration by parts results in the formula (a). From the Hamiltonian, we have the equation $\frac{1}{3}x^3 = \frac{1}{2}(x^2+y^2) + \frac{1}{2}a_1xy^2 + \frac{1}{3}a_2y^3 - H$, giving the relation,

$$x^2dx = xdx + ydy + \frac{a_1}{2}y^2dx + a_1xydy + a_2y^2dy - dH,$$

which in turn yields

$$\omega_{i,j} = \omega_{i-1,j} + \sigma_{i-2,j+1} + \frac{a_1}{2}\omega_{i-2,j+2} + a_1\sigma_{i-1,j+1} + a_2\sigma_{i-2,j+2} - x^{i-2}y^j dH,\ i \ge 2. \quad (14)$$

Then, combining (14) with the formula (a) we obtain the formula (b).

Similarly, the equation, $\frac{1}{3}a_2y^3 = H - \frac{1}{2}(x^2+y^2) + \frac{1}{3}x^3 - \frac{1}{2}a_1xy^2$, generates

$$\frac{1}{3}a_2\omega_{i,j} = H\omega_{i,j-3} - \frac{1}{2}\omega_{i+2,j-3} - \frac{1}{2}\omega_{i,j-1} + \frac{1}{3}\omega_{i+3,j-3} - \frac{1}{2}a_1\omega_{i+1,j-1}, \; j \geq 3. \tag{15}$$

Finally, the formulas (c) and (d) follow the formula (b) and (15). □

From Lemma 2, we know that any polynomial one-form ω, $\deg(\omega) = m$, can be expressed in the form of

$$\omega = r(x, y)dH + dR(x, y) + \sum_{i=0,1}\sum_{j=0}^{m-i}\alpha_{i,j}\omega_{i,j}.$$

The next lemma shows that there also exist some relationships among the one-forms $\omega_{i,j}$, $i = 0, 1$.

Lemma 3. *For any non-negative integer, $m \bmod 3 \neq 2$, there exist $\beta_{i,j,m}$, $\widetilde{r}_m(x, y)$ and $\widetilde{R}_m(x, y)$ satisfying the following identity,*

$$\sum_{i=0,1}\sum_{j=0}^{m-i}\beta_{i,j,m}\omega_{i,j} = \widetilde{r}_m(x, y)dH + d\widetilde{R}_m(x, y), \tag{16}$$

where $\widetilde{R}_m(x, y)$ and $\widetilde{r}_m(x, y)$ are polynomials in x and y of degrees $m + 1$ and $m - 1$, respectively; and $\beta_{i,j,m}$ are polynomials in a_1 and a_2, with $\beta_{0,0,0} = \beta_{1,0,1} = 1$, $\beta_{0,1,1} = 0$, and

$$\begin{aligned}
\beta_{0,m+3,m+3} =& \frac{m+4}{3(m+3)}(a_2\beta_{0,m,m} + \frac{a_1^2}{2}\beta_{1,m-1,m}), \\
\beta_{1,m+2,m+3} =& \frac{m+4}{3(m+2)}(a_1\beta_{0,m,m} + a_2\beta_{1,m-1,m}),
\end{aligned} \tag{17}$$

if $\beta_{1,-1,0}$ is defined as $\beta_{1,-1,0} = 0$.

Proof. We use the method of mathematical induction to prove this lemma. It is easy to see that the conclusion is true for $m = 0, 1$. Now, suppose (16) holds for $m \bmod 3 \neq 2$. Then, we prove that (16) also holds for $m + 3$. Multiplying (16) by H on both sides yields

$$\sum_{i=0,1}\sum_{j=0}^{m-i}\beta_{i,j,m}H\omega_{i,j} = H\widetilde{r}_m dH + Hd\widetilde{R}_m. \tag{18}$$

The right-hand side of (18) can be rewritten as

$$H\widetilde{r}_m dH + Hd\widetilde{R}_m = (H\widetilde{r}_m - \widetilde{R}_m)dH + d(H\widetilde{R}_m). \tag{19}$$

For the left-hand side of (18), it follows from the formulas (c) and (d) in Lemma 2 that

$$\begin{aligned} &H\omega_{i,j} = \xi_{i,j+3} + \eta_{i,j+3}, \ i+j<m, \\ &H\omega_{0,m} = \frac{a_2(m+4)}{3(m+3)}\omega_{0,m+3} + \frac{a_1(m+4)}{3(m+2)}\omega_{1,m+2} + \eta_{0,m+3}, \\ &H\omega_{1,m-1} = \frac{a_1^2(m+4)}{6(m+3)}\omega_{0,m+3} + \frac{a_2(m+4)}{3(m+2)}\omega_{1,m+2} + \eta_{1,m+2}, \ m>0, \end{aligned} \tag{20}$$

where $\eta_{i,j} = r_{i,j}dH + dR_{i,j}$, and $\xi_{i,j}$ is a one-form with $\deg(\xi_{i,j}) \le i+j$. Then, substituting (20) into the left-hand side of (18) yields

$$\begin{aligned} \sum_{i=0,1}\sum_{j=0}^{m-i}\beta_{i,j}H\omega_{i,j} = &\frac{m+4}{3(m+3)}\left(a_2\beta_{0,m,m} + \frac{a_1^2}{2}\beta_{1,m-1,m}\right)\omega_{0,m+3} \\ &+ \frac{m+4}{3(m+2)}(a_1\beta_{0,m,m} + a_2\beta_{1,m-1,m})\omega_{1,m+2} \\ &+ \sum_{i=0,1}\sum_{j=0}^{m-i}\beta_{i,j}(\xi_{i,j+3} + \eta_{i,j+3}). \end{aligned} \tag{21}$$

Finally, combining (21) with (18) and (19) shows that the conclusion is also true for $m+3$.

The proof of the lemma is complete. □

Noting that $\beta_{0,0,0} = \beta_{1,0,1} = 1$, $\beta_{1,-1,0} = \beta_{0,1,1} = 0$, we know from (17) that $\beta_{k,m-k,m}$ in Lemma 3 are polynomials in a_1 and a_2 with positive coefficients for $m \bmod 3 = k$, $k<2$. Thus, it follows from (16) that $\omega_{k,m-k}$, $m \bmod 3 = k < 2$, can be expressed in terms of other one-forms $\omega_{i,j}$, $i+j \le m$ and $r_m dH + dR_m$. This gives the following lemma.

Lemma 4. *Any polynomial one-form ω of degree m can be expressed as*

$$\omega = r(x,y)dH + dR(x,y) + \sum_{i=0,1}\sum_{j \bmod 3 \neq 0}^{1\le j\le m-i}\alpha_{ij}\omega_{ij}, \tag{22}$$

where $R(x,y)$ and $r(x,y)$ are polynomials in x and y of degrees $m+1$ and $m-1$, respectively.

Now, we use (22) to obtain

$$M(h) = \oint_{\Gamma_h}\omega = \sum_{i=0,1}\sum_{j \bmod 3 \neq 0}^{1\le j\le m-i}\alpha_{ij}\oint_{\Gamma_h}\omega_{ij}, \tag{23}$$

which implies that any Melnikov function $M(h) = \oint_{\Gamma_h}\omega$, $\deg(\omega) = m$, can be expressed as a linear combination of integrals $I_{ij}(h) = \oint_{\Gamma_h}\omega_{ij}$, $i = 0, 1$, $j \bmod 3 \neq 0$. A reasonable expectation is that the integrals $I_{i,j}(h)$ form a basis for the linear space of Melnikov functions $M(h) = \oint_{\Gamma_h}\omega$. Actually, it will be seen in the next section that the space of Melnikov functions $M(h)$ could be Chebyshev with accuracy at least 2. So the number of limit cycles in system (6) is not determined by the number of elements in the basis. Further, the coefficients $\alpha_{i,j}$ in (23) could become very complicated when $M(h)$ is a higher-order Melnikov function of system (6). In this case, it is really not easy to prove the independence of $\alpha_{i,j}$s, which is the second big obstacle in the use of the independence of the integrals $I_{i,j}(h)$ to determine the number of limit cycles.

To overcome the above mentioned difficulty, we turn to an alternative, which decreases the complexity in computing $M(h)$ by (22), but it still does not solve the problem of independence of basis. Let $\omega_j = Q_j(x, y)dx - P_j(x, y)dy$. Then, for higher-order Melnikov functions of system (6), we have the following result.

Lemma 5. *(Cf. [17,23].) Let (9) hold. Assume that in system (6) for some $k \geq 2$, Melnikov function $M_m(h) = \oint_{\Gamma_h} \Phi_m \equiv 0$, $1 \leq m \leq k-1$, and Φ_m can be expressed as*

$$\Phi_m = r_m dH + dR_m. \tag{24}$$

Then,

$$M_k(h) = \oint\limits_{\Gamma_h} \Big(\omega_k + \sum_{i+j=k} r_i \omega_j\Big),$$

$$r_m dH + dR_m = \omega_m + \sum_{i+j=m} r_i \omega_j, \quad 1 \leq m \leq k-1. \tag{25}$$

Proof. We prove this lemma by using the method of mathematical induction. First, write system (6) in the Pfaffian form,

$$dH - \varepsilon\omega_1 - \varepsilon^2\omega_2 - \cdots = 0. \tag{26}$$

Multiplying (26) by $1 + \varepsilon r_1 + \ldots + \varepsilon^{k-1} r_{k-1}$ and combing the like terms yield

$$\begin{aligned} &dH + \varepsilon(r_1 dH - \omega_1) + \varepsilon^2(r_2 dH - r_1\omega_1 - \omega_2) + \cdots \\ &\quad + \varepsilon^k(-r_{k-1}\omega_1 - \cdots - r_1\omega_{k-1} - \omega_k) + O(\varepsilon^{k+1}) = 0, \end{aligned}$$

which, by using (25), can be written as

$$dH - \varepsilon dR_1 - \cdots - \varepsilon^{k-1} dR_{k-1} - \varepsilon^k(r_{k-1}\omega_1 + \cdots + r_1\omega_{k-1} + \omega_k) + O(\varepsilon^{k+1}) = 0.$$

Then, we integrate the above equation along the phase curve γ from point A to point B, which are used to define the first return map. Note that

$$d(h, \varepsilon) = \int\limits_\gamma dH = H(B) - H(A) = O(|A - B|)$$

and

$$\Big|\int\limits_\gamma (\varepsilon dR_1 + \varepsilon^2 dR_2 + \cdots + \varepsilon^{k-1} dR_{k-1})\Big| = \varepsilon\, O(|A - B|).$$

In addition, it follows from (8) that $d(h,\varepsilon) = O(\varepsilon^k)$. Therefore, $|A - B| = O(\varepsilon^k)$ and we finally obtain

$$d(h,\varepsilon) = \varepsilon^k \int_\gamma (r_{k-1}\omega_1 + \cdots + r_1\omega_{k-1} + \omega_k) + O(\varepsilon^{k+1}),$$

which yields

$$M_k(h) = \oint_{\Gamma_h} \Big(\omega_k + \sum_{i+j=k} r_i \omega_j\Big).$$

The proof is finished. □

Remark 2. For the generic (system) parameters $(a_1, a_2) \in \Omega$, system (6) satisfies Françoise's $*$-property [24]: for any polynomial one-form ω, if $\oint_{\Gamma_h} \omega \equiv 0$, then $\omega = r dH + dR$ for some polynomials r and R. So the only condition which is needed in Lemma 5 is $M_m(h) \equiv 0$ when generic Hamiltonians are considered.

Remark 3. For some cubic Hamiltonians, the Françoise's $*$-property does not hold (see [25]). In other words, in such systems we could have polynomial one-forms ω satisfying $\oint_{\Gamma_h} \omega \equiv 0$, but ω cannot be expressed in the form of $\omega = r dH + dR$, where r and R are some polynomials. Therefore, it is required that Φ_m should not contain such "bad" one-forms for Melnikov function $M_m(h) = \oint_{\Gamma_h} \Phi_m \equiv 0$ in Lemma 5.

4. Proof of Theorem 1

Now with the results obtained in the previous section, we are ready to prove Theorem 1.

Proof. We return to system (6) with $P(x, y)$ and $Q(x, y)$ defined in (9), and want to use higher-order Melnikov functions to prove the existence of 10 small-amplitude limit cycles around the origin.

Due to the difficulty in the proof of independence of basis, we use the computation of focus values to prove the theorem. However, the computation becomes very demanding or almost impossible for computing higher-order focus values if all the coefficients are retained in the computation, and in fact many terms are not necessarily needed. Thus, before computing the focus values of system (6), without loss of generality, we want to simplify this system by choosing a group of coefficients a_{ijm}, b_{ijm} in the polynomials $P(x, y)$ and $Q(x, y)$, which does not reduce the number of limit cycles bifurcating from the origin.

In the following, we shall show how to choose a group of coefficients which are necessary for the first non-vanishing Melnikov function $M_k(h)$ in (8). Based on the results presented in the previous section (in particular, Lemmas 2, 4 and 5), we provide an algorithm as follows.

Consider $M_1(h)$ in system (6), we know $M_1(h) = \oint_{\Gamma_h} \omega_1$. Using Lemma 4, we have

$$\omega_1 = Q_1 dx - P_1 dy = \sum_{i=0}^{1} \sum_{j=1}^{2} \alpha_{ij1} x^i y^j dx + r_1 dH + dR_1, \tag{27}$$

with $r_1 = -(b_{211} + 3a_{301})y$. Then,

$$M_1(h) = \oint_{\Gamma_h} \left(\alpha_{011} y dx + \alpha_{111} xy dx + \alpha_{021} y^2 dx + \alpha_{121} xy^2 dx\right).$$

It is seen that $M_1(h)$ depends on α_{ij1}, $i = 0, 1,\ j = 1, 2$. So only four coefficients in the polynomials $P_1(x, y)$ and $Q_1(x, y)$ are needed in order to keep α_{ij1}, $i = 0, 1,\ j = 1, 2$ being independent without decreasing the number of zeros of $M_1(h)$. We choose these four coefficients as b_{ij1}, $i = 0, 1,\ j = 1, 2$. (Certainly, the choice is not unique.) Then, we have polynomials

$$P_1(x, y) = 0, \quad Q_1(x, y) = b_{011}x + b_{111}xy + b_{021}y^2 + b_{121}xy^2. \tag{28}$$

Next, let us consider $M_2(h)$ when $M_1(h) = \oint_{\Gamma_h} r_1 dH + dR_1 \equiv 0$, i.e., all $\alpha_{ij1} = 0$ in (27). Lemma 5 gives $M_2(h) = \oint_{\Gamma_h} \widetilde{\omega}_2$, where $\widetilde{\omega}_2 = \omega_2 + r_1\omega_1$. Thus, by using Lemma 4, we obtain

$$\widetilde{\omega}_2 = \sum_{i=0}^{1}\sum_{j=1}^{2} \alpha_{ij2} x^i y^j dx + \alpha_{042} y^4 dx + r_2 dH + dR_2,$$

which shows that $M_2(h)$ depends on α_{ij2}, $i = 0, 1,\ j = 1, 2$ and α_{042}. Obviously, the coefficient α_{042} is derived from $r_1\omega_1$ by Lemma 4 because the one-form y^4dx of degree 4 comes from $r_1\omega_1$. For ε-order perturbations, b_{ij1}, $i = 0, 1,\ j = 1, 2$ are needed to get all $\alpha_{ij1} = 0$ in (27). For r_1 we may simply take $b_{211} = 1$ and $a_{301} = 0$, yielding $r_1 = -y$. We also see that the one-form y^4dx can be derived from x^3ydx by using the formula (b) in Lemma 2. Hence, we may choose b_{301} for α_{042} so that $b_{301}x^3ydx$ could appear in $r_1\omega_1$. For α_{ij2}, $i = 0, 1,\ j = 1, 2$, by an argument similar to that for $M_1(h)$, we choose b_{012}, b_{112}, b_{022} and b_{122}. Hence, we obtain the following polynomials,

$$\begin{aligned} &P_1(x, y) = 0, \quad Q_1(x, y) = b_{011}x + b_{111}xy + b_{021}y^2 + b_{121}xy^2 + b_{301}x^3 + x^2y, \\ &P_2(x, y) = 0, \quad Q_2(x, y) = b_{012}x + b_{112}xy + b_{022}y^2 + b_{122}xy^2. \end{aligned} \tag{29}$$

Following the above procedure, we can choose the coefficients for $M_3(h)$, and so on. In the following, we list the polynomials for $M_k(h)$ up to $k = 5$ (the detailed arguments are omitted here for brevity):

$$\begin{aligned} &P_j(x, y) = a_{21j}x^2y + a_{12j}xy^2,\ j = 1, 2, 3, \quad P_4(x, y) = P_5(x, y) = 0, \\ &Q_1(x, y) = b_{011}y + b_{111}xy + b_{021}y^2 + b_{121}xy^2 + b_{301}x^3 + b_{031}y^3 + b_{211}x^2y, \\ &Q_2(x, y) = b_{012}y + b_{112}xy + b_{022}y^2 + b_{122}xy^2 + b_{302}x^3 + b_{032}y^3, \\ &Q_3(x, y) = b_{013}y + b_{113}xy + b_{023}y^2 + b_{123}xy^2 + b_{303}x^3, \\ &Q_4(x, y) = b_{014}y + b_{114}xy + b_{024}y^2 + b_{124}xy^2 + b_{304}x^3, \\ &Q_5(x, y) = b_{015}y + b_{115}xy + b_{025}y^2 + b_{125}xy^2. \end{aligned} \tag{30}$$

Here, the difficult part is to compute the functions r_i, $i=1,2,3,4$ in $\widetilde{\omega}_i$. Proving the independence of the basis for each k is even more difficulty. Thus, we turn to focus value computation which can be easily used to show the independence of the basis (i.e. the focus values).

Having determined the coefficients we need in P and Q of system (6), we now use the computation of focus values to prove the existence of 10 small-amplitude limit cycles. We compute the focus values up to ε^5 order as follows:

$$V=\sum_{i=0}^{5}\varepsilon^i V_i, \quad \text{where} \quad V_i=\{v_{i0}, v_{i1}, v_{i2}, \cdots\}. \tag{31}$$

We call v_{ij} the jth ε^i-order focus value of system (6), and note that $v_{0j}=0,\ j=0,1,2,\ldots$ since at $\varepsilon=0$ system (6) is a Hamiltonian system. The computation of V_i is equivalent to the computation of ith-order Melnikov function $M_i(h)$. But the computation of focus values is much easier than that of the higher-order Melnikov functions. The disadvantage of the focus value computation is that conditions obtained from the first few focus values are hard to be used to prove vanishing of an infinite number of focus values. But this can be easily verified by the above formulas $\widetilde{\omega}_i$.

The focus values v_{ij} can be obtained by using many different symbolic programs (e.g., the Maple program developed in [27]). Firstly, note that $v_{i0}=\frac{1}{2}b_{01i}, i=1,2,\ldots$. In order to execute the Maple program, set $b_{01i}=0$, $i=1,2,\ldots$. In addition, set $b_{211}=1$. Now, we start from V_1 and obtain

$$v_{11}=\tfrac{1}{8}(a_{121}+3b_{031}+b_{111}-\tfrac{1}{2}a_1b_{111}-2a_2b_{021}+1).$$

Setting $v_{11}=0$ yields $b_{031}=\frac{1}{3}(\frac{1}{2}a_1b_{111}+2a_2b_{021}-a_{121}-b_{111}-1)$. Further, setting $v_{12}=0$ results in

$$b_{121}=a_1b_{021}-a_{211}+\tfrac{1}{4a_2(5a_1-2)}(3a_1^2+20a_2^2+4a_1-20)(b_{111}+1).$$

Then, we have

$$\begin{aligned}
v_{13}&=\tfrac{35}{3072(5a_1-2)}\,(b_{111}+1)(a_1^3-3a_1^2+4-4a_2^2)F_{11},\\
v_{14}&=\tfrac{-7}{73\,728(+5a_1-2)}\,(b_{111}+1)(a_1^3-3a_1^2+4-4a_2^2)F_{12},\\
v_{15}&=\tfrac{-7}{84\,934\,656(+5a_1-2)}\,(b_{111}+1)(a_1^3-3a_1^2+4-4a_2^2)F_{13},
\end{aligned}$$

where

$$\begin{aligned}
F_{11}&=3a_1^2+12a_1-4-4a_2^2,\\
F_{12}&=27a_1^4-90a_1^3-1308a_1^2+1608a_1-256+(420a_1^2+1608a_1-1376-256a_2^2)a_2^2,\\
F_{13}&=19\,683a_1^6+343\,116a_1^5-124\,524a_1^4-6\,168\,672a_1^3+7\,612\,368a_1^2+1\,585\,344a_1\\
&\quad-1\,071\,424+4[3(140\,715a_1^4+622\,536a_1^3+39\,880a_1^2-1\,689\,568a_1+421\,808)\\
&\quad-(404\,508a_1^2-396\,336a_1+267\,856a_2^2-1\,265\,424)a_2^2]a_2^2.
\end{aligned}$$

It is easy to see that setting $b_{111}=-1$ results in $v_{13}=v_{14}=v_{15}=\cdots=0$, as discussed above. In order to obtain maximal number of small-amplitude limit cycles bifurcating from the origin, we have to use the coefficients a_1 and a_2 to solve $F_{11} = F_{12} = 0$ (i.e., $v_{13} = v_{14} = 0$). If the solution of $F_{11} = F_{12} = 0$ yields $F_{13} \neq 0$, i.e., we have parameter values such that $v_{10} = v_{11} = \cdots = v_{14} = 0$, but $v_{15} \neq 0$, then we obtain 5 small-amplitude limit cycles by properly perturbing b_{011}, b_{031}, b_{021}, a_1 and a_2, respectively. To show this, we use the Groebner basis reduction procedure to reduce F_{12} and F_{13} to

$$\begin{aligned}\tilde{F}_{12} &= F_{12}|_{F_{11}=0} = 18(a_1+2)(11a_1^3 + 46a_1^2 - 84a_1 + 24),\\ \tilde{F}_{13} &= F_{13}|_{F_{11}=\tilde{F}_{12}=0} = -\tfrac{179\,712}{121}(a_1+2)(3073a_1^2 - 5272a_1 + 1500) \neq 0.\end{aligned}$$

Then, solving the system of two equations, $F_{11} = F_{12} = 0$ (or $F_{11} = \tilde{F}_{12} = 0$) we obtain the solutions for a_1 as follows:

$$\begin{aligned}&a_1 = a_{11}^i, \quad a_2 = a_{21}^i = \pm\tfrac{1}{2}\sqrt{3(a_{11}^i)^2 + 12a_{11}^i - 4}, \quad i = 1,2,3, \quad \text{for which}\\ &a_{11}^1 = -5.61185383\cdots, \quad a_{11}^2 = 0.36507058\cdots, \quad a_{11}^3 = 1.06496506\cdots,\end{aligned} \tag{32}$$

where the second number '1' in the subscripts of a_{11}^i and a_{21}^i denotes the solutions corresponding to the first-order Melnikov function, i.e, $k = 1$. Note that $a_1 = -2$ is not a solution of $F_{11} = 0$. Further, we obtain

$$\det\left[\tfrac{\partial(F_{11},F_{12})}{\partial(a_1,a_2)}\right]_{F_{11}=\tilde{F}_{12}=0} = 576a_2(a_1+1)(11a_1^2 + 40a_1 - 36) \neq 0,$$

since none of the factors in the above equations are included in F_{11} and $\tilde{F}_{12}$.

Summarizing the above results we can conclude that based on the ε^1-order focus values (equivalently based on the first-order Melnikov function $M_1(h)$) we obtain 5 small-amplitude limit cycles around the origin.

Now let $b_{111} = -1$, then $b_{121} = a_1 b_{021} - 1$ and $b_{031} = -\frac{1}{3}(a_{121} + \frac{1}{2}a_1 - 2a_2 b_{021})$, under which all ε^1-order focus values vanish, or equivalently, the first-order Melnikov function $M_1(h) \equiv 0$. Note here that a_1 and a_2 are not used in making $M_1(h) \equiv 0$. Then, one uses the ε^2-order focus values to solve the polynomial equations $v_{21} = v_{22} = v_{23} = 0$, yielding the solutions for b_{032}, b_{122} and b_{112}. Under these solutions, we further obtain

$$\begin{aligned}v_{24} &= -\frac{1}{36\,864(3a_1^2+12a_1-4-4a_2^2)}F_{20}F_{21},\\ v_{25} &= \frac{1}{31\,850\,496(3a_1^2+12a_1-4-4a_2^2)}F_{20}F_{22},\\ v_{26} &= \frac{11}{107\,297\,229\,312(3a_1^2+12a_1-4-4a_2^2)}F_{20}F_{23},\end{aligned}$$

for which we have applied the Groebner basis reduction procedure to obtain

$$\begin{aligned}F_{20} &= \left[2(3a_1^3 - 4a_2^2)b_{021} - 3(a_1^3 - 4a_2^2)b_{301} - 6a_1^2 a_{211} + 4a_2 a_{121} - 4a_1 a_2 b_{211}\right]b_{211}\\ &\quad + 12a_1(a_1 a_{121} - a_2 a_{211})b_{021},\\ F_{21} &= 81a_1^4 - 648a_1^3 - 648a_1^2 + 1632a_1 - 880 - (504a_1^2 - 1632a_1 - 1696 + 880a_2^2)a_2^2,\end{aligned}$$

$$
\begin{aligned}
\tilde{F}_{22} =\ & F_{22}|_{F_{21}=0} \\
=\ & 1408\big[243a_1^3 - 522a_1^2 + 5172a_1 + 6664 + (1053a_1^2 - 2424a_1 \\
& - 5572 + 1300a_2^2)a_2^2\big]a_2^2 - 50\,688(63a_1^3 + 56a_1^2 - 148a_1 + 80), \\
\tilde{F}_{23} =\ & F_{23}|_{F_{21}=\tilde{F}_{22}=0} \\
=\ & 72(675\,121\,644a_1^3 + 475\,639\,745a_1^2 - 1\,491\,227\,668a_1 + 849\,702\,020) \\
& + \big\{3\,893\,155\,245a_1^3 + 22\,056\,197\,796a_1^2 - 131\,201\,934\,348a_1 - 117\,343\,356\,608 \\
& + 20\big[303\,274\,623a_1 + 3\,083\,354\,476 - 26(55\,458a_1 - 130\,879)a_2^2\big]a_2^2\big\}a_2^2 \neq 0.
\end{aligned}
$$

Similarly, we obtain the following solutions satisfying $F_{21} = \tilde{F}_{22} = 0$:

$$
\begin{aligned}
& a_1 = a_{12}^i, \quad i = 1, 2, \ldots, 7, \\
& a_2 = a_{22}^i = \sqrt{\tfrac{10\,179a_1^6 - 81\,864a_1^5 - 179\,172a_1^4 + 204\,992a_1^3 - 32\,496a_1^2 - 124\,032a_1 + 66\,880}{4(5109a_1^4 + 12\,076a_1^3 - 75\,936a_1^2 - 167\,664a_1 + 48\,944)}}, \ (a_1 = a_{12}^i), \\
& \text{where} \\
& a_{12}^1 = -2.43192492\cdots, \ a_{12}^2 = 0.12148877\cdots, \ a_{12}^3 = 0.23963547\cdots, \\
& a_{12}^4 = 0.89471272\cdots, \ a_{12}^5 = 1.60031174\cdots, \ a_{12}^6 = 7.33752703\cdots, \\
& a_{12}^7 = 10.40950390\cdots.
\end{aligned}
\tag{33}
$$

In addition, we can show that for the above solutions the following determinant is non-zero,

$$
\begin{aligned}
& \det\left[\tfrac{\partial(F_{21}, F_{22})}{\partial(a_1, a_2)}\right]_{F_{21}=\tilde{F}_{22}=0} \\
=\ & \tfrac{360\,448}{351}a_2\big\{36(1\,571\,445a_1^3 + 860\,083a_1^2 - 3\,207\,848a_1 + 1\,911\,580) \\
& + a_2^2\big[4\,977\,612a_1^3 + 24\,045\,705a_1^2 - 138\,196\,596a_1 - 132\,836\,684 \\
& + 20a_2^2\big(-119\,877a_1 + 2\,945\,227 + 169a_2^2(459a_1 + 1799)\big)\big]\big\} \neq 0.
\end{aligned}
$$

The above results show that we have parameter values such that $v_{20} = v_{21} = \cdots = v_{25} = 0$, but $v_{26} \neq 0$. Then, taking proper perturbations on the coefficients b_{012}, b_{032}, b_{122}, b_{112}, a_1 and a_2 yields 6 small-amplitude limit cycles around the origin of system (6) when the ε^2-order focus values (or the second-order Melnikov function $M_2(h)$) are used.

In order to get more limit cycles, we let $F_{20} = 0$ and solve this equation for b_{301}, yielding all the ε^2-order focus values $v_{2j} = 0$. Under these conditions, we then use the ε^3-order focus values v_{3j} to determine the number of small-amplitude limit cycles. Similarly, we may linearly solve the polynomial equations $v_{31} = v_{32} = v_{33} = v_{34} = 0$ for the coefficients b_{023}, b_{123}, b_{113} and b_{302}. After this, no coefficients can be solved linearly. So we solve a_{211} from the equation, $v_{35} = 0$, which is quadratic about a_{211}, to obtain two solutions $a_{211}^{\pm}$. We choose $a_{211} = a_{211}^{+}$ and then v_{36}, v_{37} and v_{38} are simplified to

$$
v_{36} = -624\,F_{30}\,F_{31}, \quad v_{37} = -1248\,F_{30}\,F_{32}, \quad v_{38} = -208\,F_{30}\,F_{33},
$$

where F_{30} is a lengthy irrational function, and we further apply the Groebner reduction procedure to F_{32} and F_{33} to obtain

$$\begin{aligned}
F_{31} =& 405a_1^4 + 6264a_1^3 + 6264a_1^2 - 5664a_1 + 1360 - 8(99a_1^2 + 708a_1 + 524 - 170a_2^2)a_2^2,\\
\tilde{F}_{32} =& F_{32}|_{F_{31}=0}\\
=& 4(261\,117a_1^3 + 307\,422a_1^2 - 260\,532a_1 + 60\,680) - \big[9(1035a_1^3 + 13\,266a_1^2\\
&+ 111\,492a_1 + 84\,376 + 5(513a_1^2 - 4824a_1 - 57\,156 + 2660a_2^2)a_2^2\big]a_2^2,\\
\tilde{F}_{33} =& F_{32}|_{F_{31}=\tilde{F}_{32}=0}\\
=& 4(152\,348\,063\,679a_1^3 + 175\,217\,936\,814a_1^2 - 151\,386\,504\,684a_1\\
&+ 35\,757\,329\,960) + \big\{7\,428\,338\,685a_1^3 - 38\,896\,637\,238a_1^2\\
&- 568\,264\,627\,476a_1 - 439\,876\,872\,808 - 20\big[714\,254\,595a_1 - 6\,998\,804\,702\\
&- 380(11\,970a_1 + 132\,193)a_2^2\big]a_2^2\big\}a_2^2 \neq 0.
\end{aligned}$$

Solving $F_{31} = \tilde{F}_{32} = 0$ yields

$$\begin{aligned}
a_1 &= a_{13} = 0.01871627\cdots,\\
a_2 &= a_{23} = \pm\sqrt{\frac{99a_{13}^2+708a_{13}+524-12\sqrt{1104+8496a_{13}+504a_{13}^2-2724a_{13}^3-171a_{13}^4}}{340}}.
\end{aligned} \tag{34}$$

Further, we have $\det\big[\partial(F_{31}, F_{32})/\partial(a_1, a_2)\big] = -0.1124026367\cdots \times 10^{10} \neq 0$ at $(a_1, a_2) = (a_{13}, a_{23})$. This, together with the above results, suggests that we may have parameter values such that $v_{3i} = 0$, $i = 0, 1, 2, \ldots, 7$, $v_{38} \neq 0$, and so the system could have 8 small-amplitude limit cycles, by properly applying perturbations on the coefficients, b_{013}, b_{023}, b_{123}, b_{113}, b_{302}, a_{211}, a_1 and a_2.

Now, we want all ε^3-order focus values to vanish (i.e., $M_3(h) \equiv 0$). This can be achieved by solving the coefficient a_{121} from a polynomial equation. Having obtained the conditions for which all the ε^1-, ε^2- and ε^3-order focus values vanish, we now use the ε^4-order focus values to linearly solve for b_{024}, b_{124}, b_{114}, b_{303}, a_{212} and a_{122} one by one from the equations $v_{41} = v_{42} = v_{43} = v_{44} = v_{45} = v_{46} = 0$. Then, the higher-order focus values are given by

$$v_{47} = \frac{13}{1\,179\,648}F_{40}F_{41},\quad v_{48} = \frac{-13}{127\,401\,984}F_{40}F_{42},\quad v_{49} = \frac{13}{244\,611\,809\,280}F_{40}F_{43},$$

where F_{40} is a common factor, and F_{41}, F_{42} and F_{43} are polynomials in a_1 and a_2. Similarly, we obtain the solutions of a_1 and a_2 for $F_{41} = F_{42} = 0$, but $F_{43} \neq 0$, given as follows:

$$\begin{aligned}
&a_1 = a_{14}^i,\ a_2 = \pm a_{24}^i = \pm a_2(a_{14}^i),\quad i = 1, 2, \ldots, 6,\quad \text{where}\\
&a_{14}^1 = -4.58252393\cdots,\ \ a_{14}^2 = -1.72294798\cdots,\ \ a_{14}^3 = -0.21827689\cdots,\\
&a_{14}^4 = -0.09420293\cdots,\ \ a_{14}^5 = 0.14811742\cdots,\ \ a_{14}^6 = 1.45012903\cdots,
\end{aligned} \tag{35}$$

and $a_2(.)$ denotes a rational function of the variable, which satisfy $F_{43} \neq 0$ and $\det[\partial(F_{41}, F_{42})\,/\,\partial(a_1, a_2)]_{F_{41}=\tilde{F}_{42}=0} \neq 0$. This suggests that with the ε^4-order focus values, we can obtain 9 small-amplitude limit cycles by properly perturbing the coefficients, $b_{014}, b_{024}, b_{124}, b_{114}, b_{303}, a_{212}, a_{122}, a_1$ and a_2.

Finally, in order to have all the ε^4-order focus values to become zero, we let $b_{021} = -\frac{2a_2}{a_1^2}$. Then, we obtain the simplified conditions, under which all the ε^1-, ε^2-, ε^3-, ε^4-order focus values vanish. Then, we use the ε^5-order focus values to find 10 small-amplitude limit cycles. Linearly solving the seven polynomial equations, $v_{51} = v_{52} = \cdots = v_{57} = 0$ one by one for the seven coefficients, $b_{025}, b_{125}, b_{115}, b_{304}, a_{213}, a_{123}$ and b_{022}. Then, v_{58}, v_{59} and v_{510} are given in terms of a_1 and a_2:

$$v_{58} = \frac{187}{6\,193\,152\,000} F_{50} F_{51}, \quad v_{59} = \frac{-187}{990\,904\,320\,000} F_{50} F_{52}, \quad v_{510} = \frac{17}{11\,890\,851\,840\,000} F_{50} F_{53},$$

where the common factor F_{50} is a rational function of a_1 and a_2, and F_{5i}, $i = 1, 2, 3$ are polynomials in a_1 and a_2, with degrees 6, 7 and 8 with respect to a_2^2, respectively. It can be shown that there are in a total 12 real solutions for (a_1, a_2) such that $F_{51} = F_{52} = 0$, but $F_{53} \neq 0$, given as follows:

$$\begin{aligned} &a_1 = a_{15}^i, \; a_2 = \pm a_{25}^i = \pm a_2(a_{15}^i) \quad i = 1, 2, \ldots, 6, \quad \text{where} \\ &a_{15}^1 = -2.39560267\cdots, \; a_{15}^2 = -1.53681619\cdots, \; a_{15}^3 = -0.38249860\cdots, \\ &a_{15}^4 = -0.19575710\cdots, \; a_{15}^5 = 0.05960015\cdots, \; a_{15}^6 = 0.29402249\cdots, \end{aligned} \tag{36}$$

and $a_2(.)$ denotes a rational function of the variable, which satisfy $F_{53} \neq 0$ and $\det[\partial(F_{51}, F_{52}) / \partial(a_1, a_2)]_{F_{51}=F_{52}=0} \neq 0$, implying that we can apply perturbations on the 10 parameters, $b_{015}, b_{025}, b_{125}, b_{115}, b_{304}, a_{213}, a_{123}, b_{022}, a_1$ and a_2 to obtain 10 small-amplitude limit cycles around the origin.

Finally, we need to check the critical values given in equations (32), (33), (34), (35) and (36) which are properly distributed in the bifurcation diagram in terms of the parameters a_1 and a_2 with the Hamiltonian function $H(x, y)$ given in (7). (See Fig. 1 in [16] for the Hamiltonian function $H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{3}x^3 + axy^2 + \frac{1}{3}by^3$, in terms of the parameters a and b.) For convenience, we define the following points in the a_1–a_2 plane:

$$\begin{aligned} k = 1:\; &P_1 = (\;\;0.3650705869\ldots, \; 0.4417795388\ldots) \\ k = 2:\; &P_2 = (\;\;0.1214887712\ldots, \; 0.6855794168\ldots) \\ &P_3 = (\;\;0.8947127237\ldots, \; 0.3648137316\ldots) \\ k = 3:\; &P_4 = (\;\;0.0187162703\ldots, \; 0.5708409903\ldots) \\ k = 4:\; &P_5 = (-0.0942029335\ldots, \; 0.6741464973\ldots) \\ &P_6 = (\;\;0.1481174260\ldots, \; 0.2303270018\ldots) \\ k = 5:\; &P_7 = (-0.1957571086\ldots, \; 0.7336772199\ldots) \\ &P_8 = (\;\;0.0596001501\ldots, \; 0.4237619510\ldots), \end{aligned}$$

where the number k denotes the order of Melnikov functions. Note that all of these points satisfy the conditions $-1 \le a_1 \le 2$ and $0 \le a_2 \le (1 - a_1/2)\sqrt{1 + a_1}$, that is, they are inside the curve defined by

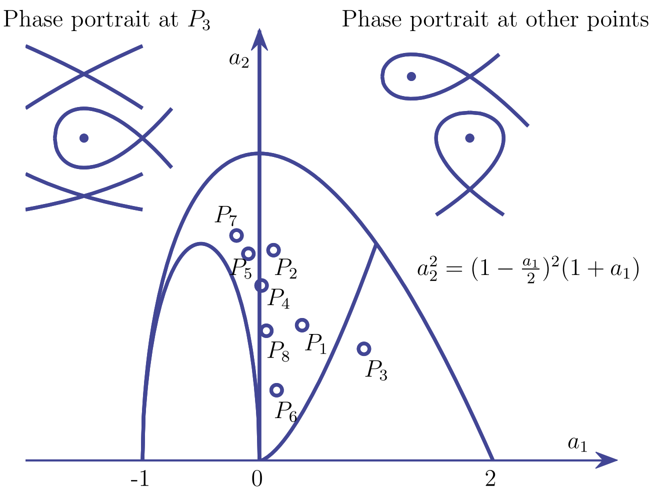

Fig. 1. Distribution of points P_i and their corresponding phase portraits.

$$a_2^2 = \left(1 - \frac{a_1}{2}\right)^2 (1 + a_1),$$

as shown in Fig. 1. But it should be noted that there are other points outside the curve (not shown in this figure) which are also solutions. For each k, there exist proper Hamiltonian functions for which the conclusion in Theorem 1 holds. It has been seen from our solution procedures that $a_2 = 0$ is not allowed, and none of the above cases is degenerate. In particular, the degenerate case, defined by $a_1^3 = 2a_2^2$, does not belong to our parameter values. The corresponding phase portraits for the eight sets of parameter values (8 points P_i) are also sketched in Fig. 1.

The above results indeed show that by using the kth-order Melnikov function M_k, we can obtain $\lfloor \frac{4k}{3} \rfloor + 4$ number small-amplitude limit cycles bifurcating from the origin of system (6).

The proof for Theorem 1 is complete. $\square$

5. Conclusion

In this paper, we have shown that the bases chosen in the proof of [13] are not independent, leading to the conclusion of the existence of 11 limit cycles in this example being not true. Further, with an example, we have demonstrated a good method combining both advantages of the Melnikov function method and the focus value computation method in studying bifurcation of limit cycles. In particular, we perturb a quadratic Hamiltonian system with cubic polynomials to obtain 10 small-amplitude limit cycles by using up to 5th-order Melnikov functions. This illustrates the usefulness of the combination method, and it is expected that this method can be applied to investigate other polynomial systems to obtain more limit cycles.

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